Phys 110B 04Dec20

Ray Optics

Lagrangian Approximation → Fermat's Principle of "Least time"
stationary phase

$$\int k\,ds \propto \int n\,ds = \int n \frac{ds}{d\sigma} d\sigma = \int n(\vec{x}) \left[\frac{d\vec{x}}{d\sigma}\cdot\frac{d\vec{x}}{d\sigma}\right]^{1/2} d\sigma$$

σ is some parameter

$\int k\,ds$ to be stationary $\quad \delta \int k\,ds = 0$

$$\vec{x}\,' = \frac{d\vec{x}}{d\sigma}$$

⇒ Euler-Lagrange Equations

$$\frac{d}{d\sigma}\left[\frac{\partial}{\partial \vec{x}\,'}\left(n(\vec{x})\sqrt{\vec{x}\,'\cdot\vec{x}\,'}\right)\right] - \frac{\partial}{\partial \vec{x}}\left[n(\vec{x})\sqrt{\vec{x}\,'\cdot\vec{x}\,'}\right] = 0$$

$$\frac{d}{d\sigma}\left[n\,\frac{1}{2}(\vec{x}\,'\cdot\vec{x}\,')^{-1/2}\cdot 2\vec{x}\,'\right] - \vec{\nabla} n \left|\frac{d\vec{x}}{d\sigma}\right| = 0$$

$$\frac{d}{d\sigma}\left[n(\vec{x}) \left|\frac{d\vec{x}}{d\sigma}\right| \frac{d\vec{x}}{d\sigma}\right] = \left|\frac{d\vec{x}}{d\sigma}\right| \vec{\nabla} n(\vec{x})$$

$$\left|\frac{d\vec{x}}{d\sigma}\right| = \frac{ds}{d\sigma} \Leftarrow \text{arclength}$$

$$\frac{d}{ds}\left[n\,\frac{d\vec{x}}{ds}\right] = \vec{\nabla} n$$

$\vec{\nabla} n$ ← gradient of refractive index

$\frac{d\vec{x}}{ds}$ ↑ tangent vector to ray

generalized Snell's Law

Hamiltonian Ray Optics

Local plane wave approximation

$$\cancel{\vec{A}}\ \cancel{\vec{E}} \approx \underbrace{\hat{\epsilon}(\vec{x},t) A(\vec{x},t)}_{\text{slowly varying}} e^{i\underbrace{\Theta(\vec{x},t)}_{\text{rapidly varying}}} + c.c. \qquad = \vec{A}$$

plane wave $\Theta = \vec{k}\cdot\vec{x} - \omega t$

more generally $\underbrace{\vec{k}(\vec{x},t)}_{\text{local wavevector slowly varying}} = \frac{\partial}{\partial \vec{x}} \Theta(\vec{x},t)$, $\underbrace{\omega(\vec{x},t)}_{\text{slowly varying}} = -\frac{\partial}{\partial t}\Theta(\vec{x},t)$

$$d\theta = \vec{k}\cdot d\vec{x} - \omega dt$$

$$\cancel{d}\Theta = \int d\theta = \int \vec{k}\cdot d\vec{x} - \omega dt$$

enforce a local dispersion relation

LHI case: $D(\omega,\vec{k}) = 0$

slowly varying case $D(\underbrace{-\frac{\partial\Theta}{\partial t}}_{\omega}, \underbrace{\frac{\partial\Theta}{\partial\vec{x}}}_{\vec{k}}; \vec{x}, t) = 0$

in principle, this is solvable by method of characteristics

characteristics solve Hamilton's Equations

ODE's along a "ray" or "photon" trajectory

$$\frac{d\vec{x}}{d\sigma} = \frac{\partial D}{\partial \vec{k}}$$

$$\frac{d\vec{k}}{d\sigma} = \frac{-\partial D}{\partial \vec{x}}$$

$$\frac{d\omega}{d\sigma} = \frac{\partial D}{\partial t}$$

$$\frac{dt}{d\sigma} = \frac{-\partial D}{\partial \omega}$$

can re-parameterize in terms of time

$$D(\vec{k},\omega;\vec{x},t) = 0$$

implicitly determine $\Omega(\vec{k};\vec{x},t) = \omega$

e.g. $$\frac{dx}{dt} = \frac{\frac{dx}{d\sigma}}{\frac{dt}{d\sigma}} = \frac{\frac{\partial D}{\partial k_x}}{\frac{-\partial D}{\partial \omega}} = \frac{\partial \omega}{\partial k_x} \quad \frac{\partial \Omega}{\partial k_x}$$

Hamilton's Ray Equations

local group velocity $$\frac{d\vec{x}}{dt} = \frac{\partial \Omega}{\partial \vec{k}} = \vec{v}_g$$

$\cancel{\omega = \Omega}$

$$\omega(\vec{x},t) = \Omega(\vec{k}(\vec{x},t),\vec{x},t)$$

$$\frac{d\vec{k}}{dt} = -\frac{\partial \Omega}{\partial \vec{x}}$$ Snell's Law

$$\frac{d\omega}{dt} = \frac{\partial \Omega}{\partial t}$$ time dependence of medium

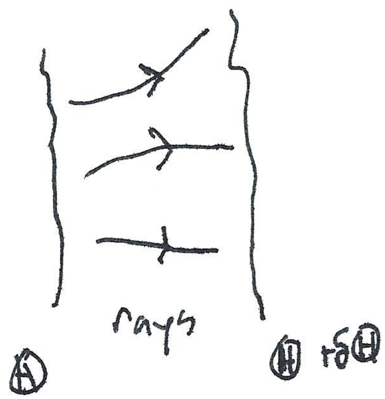
$$E = \hbar\omega = \cancel{H}\ \cancel{H}\ \mathcal{H}$$

$$\vec{p} = \hbar\vec{k}$$

$$\frac{d\vec{x}}{dt} = \frac{\partial \mathcal{H}}{\partial \vec{p}} \qquad \frac{d\vec{p}}{dt} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \qquad \frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}$$

Classical Hamiltonian EoM for a "particle" with Hamiltonian $\mathcal{H}$

EoM for a "classical" ~~point~~ photon



e.g. special case $\Omega = \frac{c}{n(\vec{x})}\sqrt{\vec{k}\cdot\vec{k}}$

↑ spatially dependent index of refraction

$$\Rightarrow \frac{d\vec{x}}{dt} = \frac{c}{n}\hat{k}$$

$$\frac{d\vec{k}}{dt} = \frac{c}{n^2}\vec{\nabla} n\,|\vec{k}|$$

$$\frac{d}{ds}\left(n\frac{d\vec{x}}{dt}\right) = \frac{d}{ds}(n\hat{k}) = \frac{d}{ds}\left(\frac{n}{k}\vec{k}\right) = \frac{d}{ds}\left(\frac{c}{\omega}\vec{k}\right)$$

$$\frac{c}{\omega}\frac{d}{ds}\vec{k} = \vec{\nabla} n$$

Conservation Laws "Noether's Theorem"

if Ω is $\vec{x}$ independent then $\frac{d}{dt}\vec{k} = 0$ $\vec{k}$ is conserved

if Ω is t independent then $\frac{d\omega}{dt} = 0$ ω is conserved

symmetry of Ω $\longleftrightarrow$ "conservation law" in ray motion

ray trajectories $\rightarrow$ phase accumulation

amplitude?

Amplitude transport Equation

$$\frac{\partial}{\partial t} J + \vec{\nabla} \cdot [J \vec{v}_g] = 0$$

continuity equation for J J is locally conserved

$$J \propto \frac{|\vec{E}|^2}{\omega} = \text{"wave. Action"}$$

$$\propto \frac{\text{energy density}}{\omega}$$

$\mathcal{E}$ per photon $\sim \hbar\omega$

$J \propto$ photon density

$$\frac{\partial}{\partial t} J + \vec{\nabla}[J \vec{v}_g] = 0$$

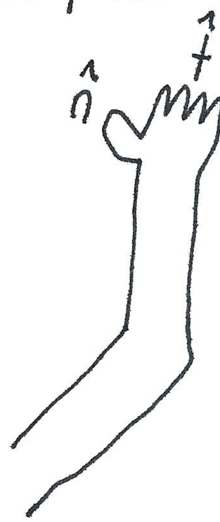
photons are conserved (locally)

Polarization?

polarization undergoes "parallel transport"

minimal change required to maintain normalization
and angle with ray

e.g. polarization is rotated as little as possible to
remain orthogonal



$\hat{t}$ tangent

$\hat{n}$ normal

$$\frac{d\hat{n}}{\Delta t} = \frac{d}{\Delta t}\hat{t}\times(\hat{n}\times\hat{t})$$

"geometric phase"