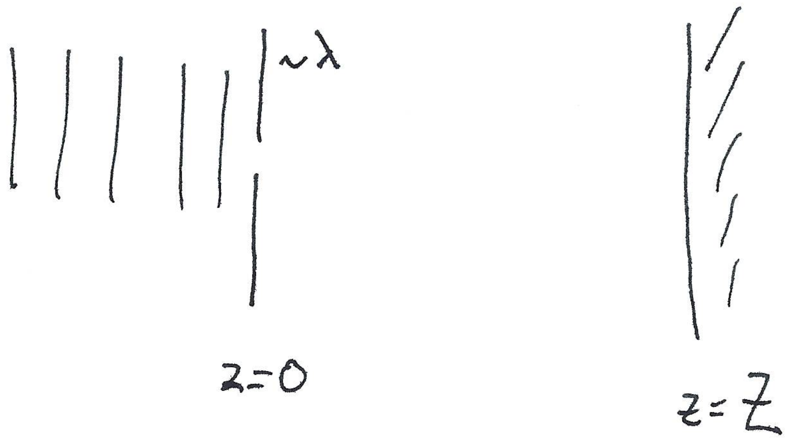
Phys 110B 23 Nov 20

Fraunhofer Diffraction "far-field" diffraction



$$F = \frac{d^2}{\lambda Z} \ll 1$$

$$Z \gg Z_R$$

rays from aperture are approximately parallel

$$\Delta l \approx \sqrt{\left(x+\frac{s}{2}\right)^2 + Z^2} - \sqrt{\left(x-\frac{s}{2}\right)^2 + Z^2}$$

difference in path length

$$Z \gg s, Z_R \qquad \Delta l \approx \frac{sx}{\sqrt{x^2+Z^2}}$$

x, y aperture coordinates

$$x, y \ll Z$$

taylor expand

$$R = \sqrt{(X-x)^2 + (Y-y)^2 + Z^2} \approx R_0 + \Delta R$$

$$R_0 = \sqrt{X^2 + Y^2 + Z^2}$$

$$\Delta R = \frac{-xX - yY}{R} + \dots$$

in propagating from the aperture $(x,y,0)$ to (X,Y,Z)

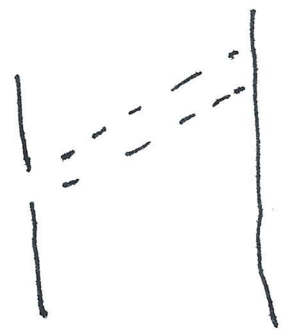
pick up a phase $ik(R+\Delta R)$

$$\psi(X,Y,Z,t) \propto \frac{e^{ikR}}{R} \int dx\,dy\, \psi(x,y,0)\, e^{-i\left(\frac{x}{R}X - i\frac{y}{R}Y\right)k}$$

fall off in field strength — superposition over aperture — amplitude in aperture

$\propto$ Fourier Transform of $\psi(x,y,0)$

evaluated at $k_x = \frac{x}{R}k \quad k_y - \frac{y}{R}k$

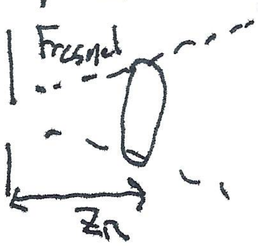


plane waves that contribute must be traveling along line of site

Fraunhofer pattern $\propto$ Fourier Transform of the aperture

$Z \gg Z_R \leftarrow$ Rayleigh distance

where spot size due to diffraction is comparable to spot size due to the aperture



diffraction spreading $\gg$ initial spot size

Fraunhofer

Diffraction via Fourier Optics

in aperture $f(x,y,0)e^{-i\omega t}$

downstream $(z>0)$ superposition of free-space modes

$$e^{i\vec{k}\cdot\vec{r}-i\omega t} \qquad \omega^2 = \frac{c^2}{n^2}|\vec{k}|^2$$

anticipating

$$\psi(\vec{r},t) = \int dk_x\, dk_y\, C(k_x,k_y)\, e^{i\vec{k}\cdot\vec{r}-i\omega t}$$

field downstream

$$\frac{\omega^2}{c^2} = k^2 = k_x^2 + k_y^2 + k_z^2$$

not independent

$$k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

if $\mathrm{Im}(k_z)=0$ then $\mathrm{Re}(k_z)>0$

otherwise $\mathrm{Im}\, k_z > 0$

incoming plane waves from the aperture at $z=\infty$

(boundary conditions "at ∞")

boundary conditions at $z=0$

$$\psi(x,y,0,t) = \begin{cases} 0 & \text{outside aperture} \\ f(x,y)e^{-i\omega t} & \text{inside} \end{cases}$$

$$f(x,y) = \frac{1}{\sqrt{2\pi}} \int dk_x\, dk_y\, C(k_x, k_y)\, e^{ik_x x + ik_y y}$$

$$C(k_x, k_y) = \frac{1}{\sqrt{2\pi}} \int dx\, dy\, f(x,y)\, e^{-ik_x x - ik_y y}$$

$$k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

$$\psi(\vec{r}, t) = \frac{1}{\sqrt{2\pi}} \int dk_x\, dk_y\, C(k_x, k_y)\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

initial superposition of plane waves

propagates it downstream

"Shadow Regime" if z is sufficiently small

$$k_z z = \left(\sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}\right) z \approx z\frac{\omega}{c} - z\frac{c}{2\omega}(k_x^2 + k_y^2) + \dots$$

taylor expand

$$z\frac{c}{2\omega}\frac{1}{d^2}$$

small if d is finite and $z \ll \frac{\omega d^2}{c}$

can keep only the leading term

$$\cancel{\psi(\vec{r}, t) \propto \int dk_x\, dk_y\, C(k_x, k_y)\, e^{ik_x x + ik_y y}}$$

$$\psi(\vec{r}, t) \propto \int dk_x\, dk_y\, C(k_x, k_y)\, e^{ik_x x + ik_y y + \frac{iz\omega}{c} - i\omega t}$$

$$\approx f(x,y)\, e^{i\frac{\omega}{c}(z - ct)}$$

no diffraction

important diffraction effects appear when

$$\frac{zc}{\omega}(k_x^2+k_y^2) \sim 1$$

Fresnel Regime

intermediate z regime

or $z \sim \frac{\omega}{c} d^2 \sim kd^2 \sim z_R$

keep linear and quadratic terms

Fraunhofer (very large z) regime $\quad z \gg \frac{\omega}{c} d^2$

$$z \gg z_R$$

expect to see near plane wave traveling from the aperture to observation point

$$\psi(\vec{r},t) \propto \int dk_x\, dk_y\, C(k_x,k_y)\, e^{i\vec{k}\cdot\vec{r}-i\omega t}$$

$\vec{k}$ parallel to line of sight

observation point

$$(k_x, k_y, k_z(k_x,k_y)) \propto (X, Y, Z)$$

$$k_z = \sqrt{\frac{\omega^2}{c^2}-k_x^2-k_y^2}$$

(assuming aperture is located near origin)

amplitude $\propto \frac{1}{R} \quad R \simeq \sqrt{X^2+Y^2+Z^2}$

$$|\psi|^2 \propto \frac{|C(k_x,k_y)|^2}{R^2}$$

$$\Rightarrow \frac{k_x}{X} = \frac{k_y}{Y} = \frac{k_z}{Z} = \frac{\omega/c}{R}$$

$$k_x = k\frac{X}{R} \quad k_y = \frac{kY}{R} \quad k = \frac{\omega}{c}$$

$$k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

$$\psi(\vec{r},t) \propto \int dx\,dy\, f(x,y)\, e^{-ik_x x - ik_y y}$$

$$\propto \frac{e^{ikR}}{R} \int dx\,dy\, f(x,y)\, e^{-i\left(x\frac{X}{R} + y\frac{Y}{R}\right)k}$$

"interference" argument $\longleftrightarrow$ Fourier Transform argument

"Stationary Phase Approximation"

$$\psi(X,Y,Z,t) \propto \int dk_x dk_y\, \underbrace{C(k_x,k_y)}_{\text{slowly varying}}\, e^{\overbrace{i\vec{k}\cdot\vec{r} - i\omega t}^{\text{rapidly varying phase}}}$$

integrand tends to destructively interfere except at special values of $\vec{k}$ where phase is "stationary"

$$\frac{\partial}{\partial k_x}\left(X k_x + Y k_y + Z k_z(k_x, k_y)\right) = 0$$

$$\Rightarrow \frac{X}{k_x} = \frac{Z}{k_z(k_x, k_y)}$$

$$\frac{\partial}{\partial k_y}\left(X k_x + Y k_y + Z k_z(k_x, k_y)\right) = 0$$

$$\frac{Y}{k_y} = \frac{Z}{k_z(k_x, k_y)}$$

At next order of "asymptotic" stationary phase expansion pick up an extra factor of $\frac{Z}{R^2} = \frac{\cos\theta}{R}$ ← obliquity factor

↖ fall off of radiative fields

compare

Fraunhofer

Kirchhoff $\frac{1}{2}(1 + \cos\theta) \approx \cos\theta$

small correction

Stratton-Chu Vector Diffraction Theory