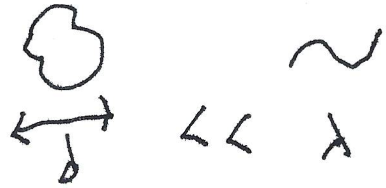
Phys 110B 04 Nov 20

Hertzian Multipoles



$\ll \lambda$

far-field expansion in terms of multipoles

$$\int \frac{1}{2}\left((\hat{R}\cdot\vec{x}')\vec{J} + (\hat{R}\cdot\vec{J})\vec{x}'\right) d^3\vec{x}' = -\frac{i\omega}{2}\int \vec{x}'(\hat{R}\cdot\vec{x}')\rho(\vec{x}')\, d^3\vec{x}'$$

$$J \sim e^{-i\omega t}$$

E2 contribution $\quad A(\vec{x}) = \dfrac{-\mu_0 c k^2}{8\pi}\dfrac{e^{ikr}}{r}\left(1 - \dfrac{1}{ikr}\right)\int \vec{x}'(\hat{R}\cdot\vec{x}')\rho(\vec{x}';\omega)\, d^3\vec{x}'$

far-field, $\quad \vec{H}(\vec{x}) = \dfrac{ik\hat{R}\times\vec{A}}{\mu_0}$

$$\vec{E} = \frac{ikZ_0(\hat{R}\times\vec{A})\times\hat{R}}{\mu_0}$$

$$\vec{H} = \frac{-ick^2}{8\pi}\frac{e^{ikr}}{r}\int d^3\vec{x}'\,(\hat{R}\times\vec{x}')(\hat{R}\cdot\vec{x}')\rho(\vec{x}';\omega)$$

$\propto$ electric quadrupole moment

$$Q_{\alpha\beta} = \int (3x_\alpha x_\beta - |\vec{x}'|^2\delta_{\alpha\beta})\rho(\vec{x}')\, d^3\vec{x}'$$

Q is a symmetric 3×3 matrix

$$\hat{R}\times\int \vec{x}'(\hat{R}\cdot\vec{x}')\rho(\vec{x}';\omega)\, d^3\vec{x}' = \cdots = \frac{1}{3}\hat{R}\times(Q\hat{R})$$

$$\vec{H}(\vec{x}) = \frac{-ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{R}\times Q\hat{R}$$

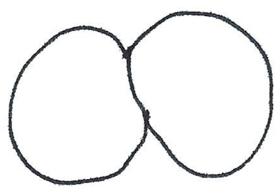
far field

Poynting vector $\Rrightarrow$

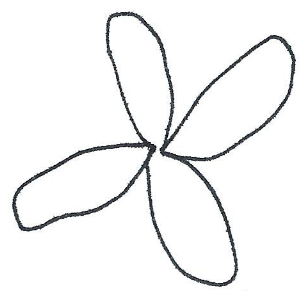
$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2} k^6 \left| \hat{R} \times Q\hat{R} \times \hat{R} \right|^2 \; ; \quad \hat{R} = \hat{r} \text{ line of sight from the origin to the observation point}$$

$$\Rrightarrow P = \frac{c^2 Z_0 k^6}{1440\pi} \sum_{\alpha,\beta} |Q_{\alpha\beta}|^2 \quad \text{power radiated at } \omega$$

$$Q \sim q d^2 \sim \frac{I}{\omega} d^2 \qquad P \sim k^4 I^2 d^4$$



dipole shape



quadrupole shape

Each new term ~~pair~~ brings in a pair of poles, one ~~for~~ magnetic and the other electric

Vector spherical Harmonics (orthonormal basis of vector fields) } an aside more in grad school

$$\lim_{r \to \infty} \iint \vec{S} \cdot \hat{r} \, da = P$$

$\vec{S}$ has non-radial components $\frac{\vec{E} \times \vec{B}^*}{\mu_0}$

Are either $\perp$ to $\hat{r}$ and/or are imaginary

$\frac{1}{2} \mathrm{Re} \frac{\vec{E} \times \vec{B}^*}{\mu_0}$ = power

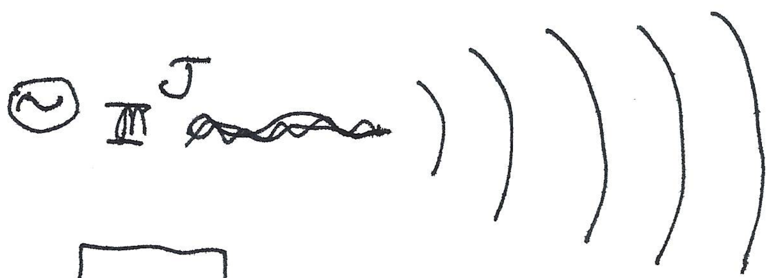
real components - radiation "resistive" components

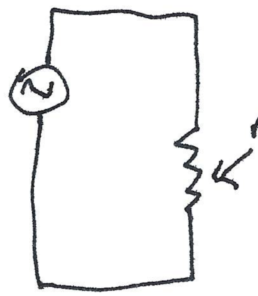
imaginary components → reactive components represent reactive sloshing of the fields but don't reach out away from the source radiated power $P_{rad} \sim Z_0 \cdot \text{const} \cdot I^2$

impedance of free space (Ohms)

radiation resistance of the source

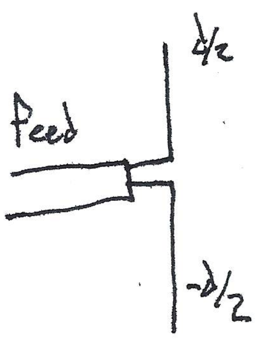
$P_{resistor} = Re(Z)\, I^2$





Hertzian $d \ll \lambda \Rightarrow |v| \ll c$

Antennas typically $d \lesssim \lambda$



if $d \ll \lambda$ antenna looks like a Hertzian dipole

$d \sim \frac{\lambda}{2}$ or $d \simeq \lambda$

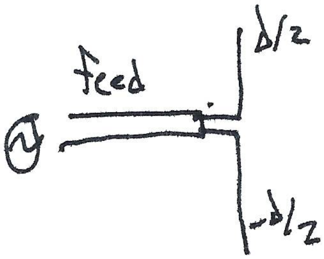
Antenna theory is vast and complicated

dipole antenna

thin, dipole antenna made from a very good conductor
with a small gap

Also assuming ~~antenna is thin enough so that~~ that the damping
due to the radiation is small

$\Rightarrow$ prescribed, sinusoidal function of time



standing wave pattern

very thin antenna

direction of antenna

$$\vec{J}(\vec{x};\omega) = I \sin\left(\frac{kd}{2} - k|z|\right) \delta(x)\delta(y)\, \hat{z}\, e^{-i\omega t}$$

in far zone,

$$\vec{A}(\vec{x};\omega) = \hat{z}\, \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} I \int_{-d/2}^{d/2} \sin\left(\frac{kd}{2} - k|z|\right) e^{-ikz'\cos\theta'}\, dz$$

$$\vec{A}(\vec{x};\omega) = \tilde{z}\, \frac{\mu_0}{4\pi} 2I \frac{e^{ikr}}{r} \left(\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta} \right)$$

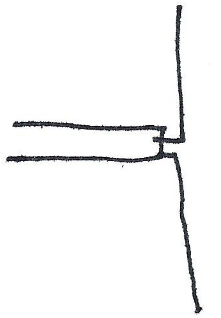
$$\vec{H} = \frac{ik\,\hat{r}\times\vec{A}}{\mu_0} \qquad |\vec{H}| = \frac{k\sin\theta\, |A_z|}{\mu_0}$$

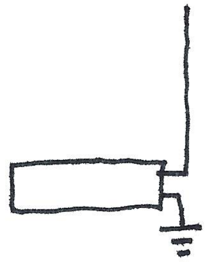
$$\Rightarrow \frac{dP}{d\Omega} = \cdots = \frac{Z_0}{8\pi^2} I^2 \left(\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right)^2$$

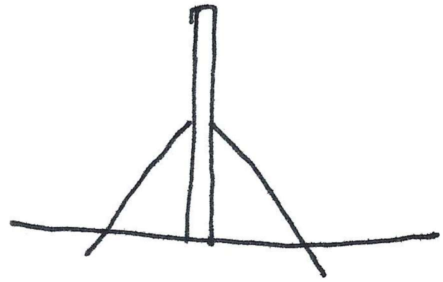
$kd \ll 1$ reduces to Hertzian dipole case

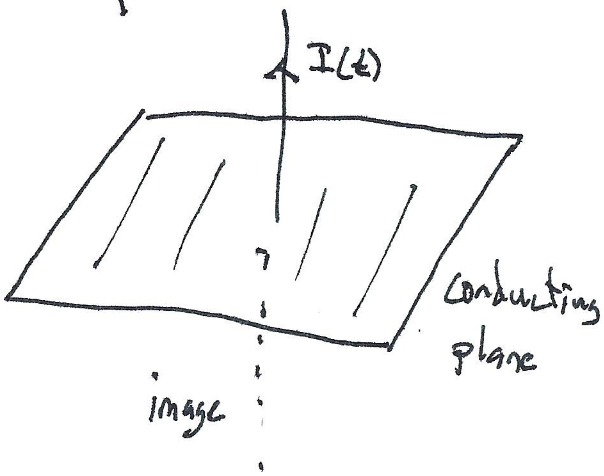
$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi} \begin{cases} \dfrac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} & \text{if } kd = \pi \\ \dfrac{4\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} & \text{if } kd = 2\pi \end{cases}$$

not only answers but the most popular





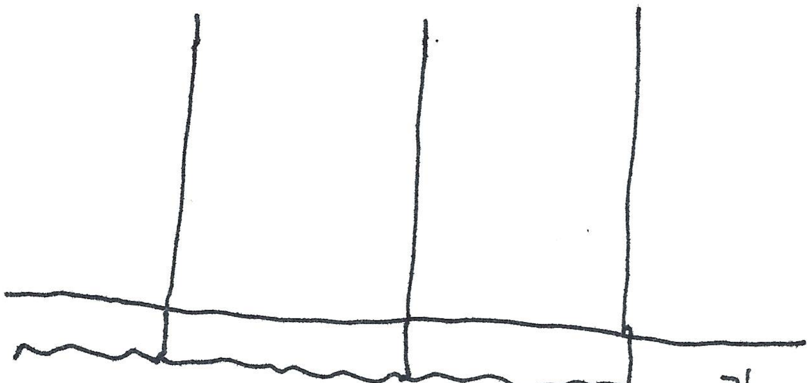




ground is pretty good conductor

Can see the ~~mirror in~~ image of the current too,

why need 3 or more antennas?

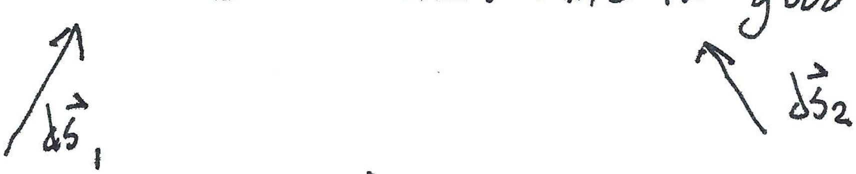


Phase delay the signal

constructive + destructive interference

Can make them more directional

Reciprocity good emitters make for good receivers



emf induced in $d\vec{s}_1$ by current in $d\vec{s}_2$

$$d\mathcal{E} = \ldots = \frac{i\mu_0}{4\pi}\frac{[I]}{r}\left[(\hat{r}\times d\vec{s}_1)\cdot(\hat{r}\times d\vec{s}_2)\right]$$

symmetric under interchanging the roles of wire 1 and 2