Phys 110B 02Nov20

Multipole Radiation in Hertz Regime $(d \ll \lambda)$

$\omega > 0$

Lorentz-Lorenz gauge $\quad \phi = -i\frac{c^2}{\omega}\vec{\nabla}\cdot\vec{A}$

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{A} = -\mu_0\vec{J}$$

$$\vec{B} = \vec{\nabla}\times A$$

$$\vec{E} = -\vec{\nabla}\phi - i\omega\vec{A} = \ldots = i\frac{c^2}{\omega^2}(\vec{\nabla}\times\vec{B}) = \frac{ic^2}{\omega}(\vec{\nabla}\times\vec{\nabla}\times\vec{A})$$

$$R = |\vec{x} - \vec{x}'| = r(1+\eta)^{1/2}$$

$$r = |\vec{x}| \qquad \eta = -2\frac{\vec{r}\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2 + \ldots$$

$$\frac{1}{R} \approx \frac{1}{r}\left(1 + \frac{1}{2}\eta + \ldots\right)$$

$$e^{ikR} = e^{ikr}e^{ik(R-r)} \approx e^{ikr}e^{\frac{1}{2}kr\eta} = \left(1 + \tfrac{1}{2}ikr\eta\right)e^{ikr}$$

$$\frac{e^{ikR}}{R} \sim \frac{e^{ikr}}{r}\left(1 - \tfrac{1}{2}(1 - ikr)\eta + \ldots\right)$$

$$\approx \frac{e^{ikr}}{r}\left(1 + (1 - ikr)\frac{\hat{r}\cdot\vec{r}'}{r} + \ldots\right)$$

$$\vec{A}(\vec{x},\omega) = \frac{\mu_0}{4\pi}e^{-i\omega t}\int d^3\vec{x}'\,\frac{e^{ikR}}{R}J(\vec{x}',\omega)$$

$$\approx \frac{\mu_0}{4\pi}\frac{e^{ikr-i\omega t}}{r}\int d^3\vec{x}'\,\vec{J}(\vec{x}',\omega)$$

$\vec{x} = \vec{r}$

$\vec{x}' = \vec{r}'$

$$+ \frac{\mu_0}{4\pi}e^{ikr-i\omega t}\frac{(1-ikr)}{r^2}\int(\hat{r}\cdot\vec{x}')\vec{J}(\vec{x}')\,d^3\vec{x}' + \ldots$$

$$\vec{A}(\vec{x};\omega) = \frac{\mu_0}{4\pi} \frac{e^{ikr-i\omega t}}{r} \int d^3\vec{x}' \, \vec{J}(\vec{x}')$$ leading order term for the far field

$$\int d^3\vec{x}' \, \vec{J}(\vec{x}';\omega) = -\int d^3\vec{x}' \, \vec{x}' \underbrace{\left(\vec{\nabla}' \cdot \vec{J}\right)}^{-\frac{\partial \rho}{\partial t}}$$

integration by parts in reverse

$$= -i\omega \int d^3\vec{x}' \, \vec{x}' \rho(\vec{x}';\omega)$$

$\int d^3\vec{x}' \, \vec{x}' \rho = \vec{p}$ = electric dipole moment (at frequency ω)

$$\vec{A}(\vec{x};\omega) = \frac{\mu_0}{4\pi}(-i\omega) \frac{e^{ikr}}{r} \vec{p}(\omega)$$

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{r} \times \vec{p}) \frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right) = \vec{H}(\vec{r};\omega)$$

$$\vec{E}(\vec{r};\omega) = \dots = \frac{1}{4\pi\epsilon_0}\left(k^2(\hat{r} \times \vec{p}) \times \hat{r} \frac{e^{ikr}}{r} + \left[3\hat{r}(\hat{r} \cdot \vec{p}) - \vec{p}\right]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr} \right)$$

as $kr \to \infty$

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{r} \times \vec{p}) \frac{e^{ikr}}{r}$$

$$\vec{E} = Z_0 \vec{H} \times \hat{r}$$ $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$ = impedance of free space

Power radiated?

angular contributions $$\frac{dP}{d\Omega} = \frac{1}{2} \mathrm{Re}\left[(\vec{E} \times \vec{H}^*) \cdot \hat{r} r^2\right]$$

radiated power at ω per unit solid angle $$= \frac{c^2 Z_0 k^4}{32\pi^2} \left|(\hat{r} \times \vec{p}) \times \hat{r}\right|^2$$

& polarization

dot product inside can yield power in a direction if all vector components of $\vec{p}$ have the same phase

then $$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |\vec{p}|^2 \sin^2\theta$$

total Power (at frequency ω)

$$P = \ldots = \frac{c^2 Z_0 k^4}{12\pi} |\vec{p}|^2$$

looking at next order term

$$\vec{A}(\vec{x};\omega) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right) \int d^3\vec{x}' (\hat{r}\cdot\vec{x}') \vec{J}(\vec{x}';\omega)$$

split integral into symmetric (in $\vec{x}'$ and $\vec{J}(\vec{x}')$)

and anti-symmetric parts

$$(\hat{r}\cdot\vec{x}')\vec{J}(\vec{x}') = \frac{1}{2}\left((\hat{r}\cdot\vec{x}')\vec{J} + (\hat{r}\cdot\vec{J})\vec{x}'\right) + \frac{1}{2}(\vec{x}'\times\vec{J})\times\hat{r}$$

$$\frac{1}{2}(\vec{x}'\times\vec{J}) = \vec{M} \qquad \int d^3\vec{x}' \vec{M}(\vec{x}') = \vec{m}$$

magnetization — magnetic dipole

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} ik (\hat{r}\times\vec{m}) \frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$

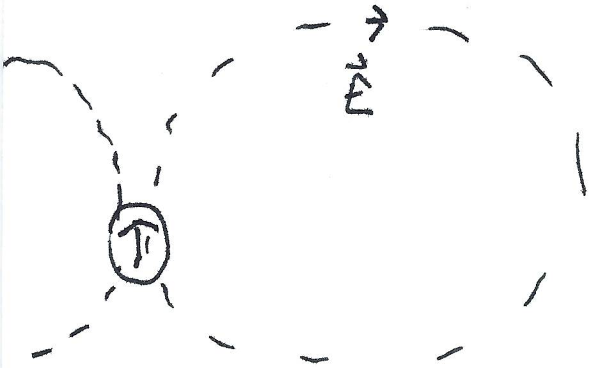
corresponding far-fields

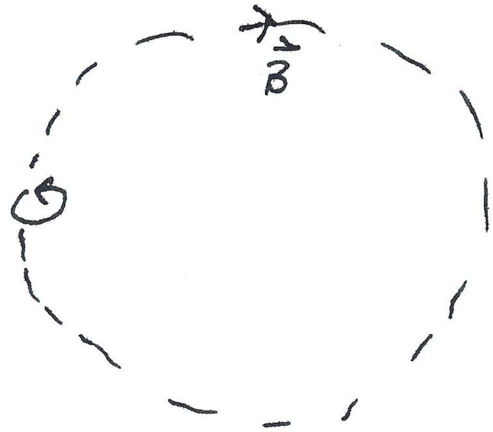
$$\vec{H} = \frac{1}{4\pi}\left(k^2 (\hat{r}\times\vec{m})\times\hat{r} \frac{e^{ikr}}{r}\right) + \ldots$$

$$\vec{E} = -\frac{Z_0}{4\pi} k^2 (\hat{r}\times\vec{m}) \frac{e^{ikr}}{r} + \ldots$$

to go from electric dipole radiation to magnetic dipole radiation $\vec{E} \to Z_0\vec{H}$ $\quad Z_0\vec{H} \to -\vec{E}$ $\quad \vec{p} \to \frac{\vec{m}}{c}$

Electromagnetic duality.





Hertzian multipoles intrinsically non-relativistic

$$d \ll \lambda$$

$$kd \ll 1$$

$$ckd \ll c$$

$$\omega d \ll c$$

$$|\vec{v}| \ll c$$

electric dipole

E1 = electric dipole E2 = electric quadrupole

M1 = magnetic dipole

E1 contribution $P = \dfrac{Z_0 c^2 k^4}{12\pi} |\vec{p}|^2$

$m = I \cdot \text{area}$

$$p \sim qd \sim \frac{I}{\omega} d$$

$\text{area} \sim d^2$

$$P_{E1} \cong \frac{Z_0 I^2 (kd)^2}{48\pi}$$

$kd \ll 1$

if $\vec{p} \neq 0$ (or not abnormally small)

then $P_{E1} \gg P_{M1}$

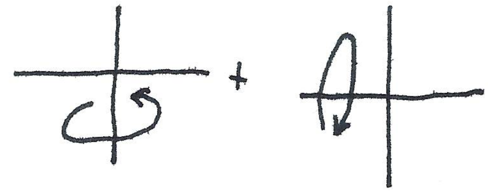
$$P_{M1} = \frac{Z_0 c^2 k^4}{12\pi} \left|\frac{\vec{m}}{c}\right|^2 \approx \frac{Z_0 I^2 (kd)^4}{12\pi}$$

if $\vec{p} = 0$, then leading order radiation is due to $\vec{m}$

current loop

or rotating ring of charge

revolving point charge



Superposition of 2 dipoles ($\perp$ oriented oscillating in phase)

$\vec{p}(\omega) = 0$

$\vec{m}(\omega) \neq 0$

$\frac{1}{2}\left((\hat{r}\cdot\vec{x}')\vec{J} + (\hat{r}\cdot\vec{J})\vec{x}' \right)$ contribution

E2 electric quadrupole radiation