Phys 110B Disc 21Oct20

Griffiths 10.6

a) Show it's always possible to choose $\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0 \partial_t \phi = 0$

b) Show that $\phi = 0$ is always possible

c) Argue that the Coulomb gauge is always possible

a) Suppose $\vec{\nabla}\cdot\vec{A} \neq \mu_0\epsilon_0 \partial_t \phi$

$$\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0 \partial_t \phi = \psi$$

Consider gauge transformation generalized by λ

$$\vec{\nabla}\cdot\vec{A}' + \mu_0\epsilon_0 \partial_t \phi' = 0 = \vec{\nabla}\cdot\vec{A} + \nabla^2\lambda + \mu_0\epsilon_0\partial_t\phi - \mu_0\epsilon_0\partial_t^2\phi$$

$$\Rightarrow (\nabla^2 + \mu_0\epsilon_0\partial_t^2)\lambda = \psi$$

forced wave equation ✓ L-L gauge

$$\Box\chi = 0$$

b) $\vec{A}, \phi$

$$\chi(\vec{x}, t) = \int_0^t dt' \, \phi(\vec{x}, t')$$

$$\phi' = \phi - \partial_t \chi = 0$$

✓ Temporal Gauge

c) $\vec{\nabla}\cdot(\vec{A}+\vec{\nabla}\chi)=0$

$\nabla^2\chi = -\vec{\nabla}\cdot\vec{A}$ ✓ Coulomb Gauge

d) $\vec{A}=\vec{0}$ ⟵ 3 conditions

$\Rightarrow \vec{\nabla}\times\vec{A}=\vec{0} \Rightarrow \vec{B}=\vec{0}$ X

gauge → 1 condition

Griffiths 10.7

$\rho(\vec{x},t) = \delta^3(\vec{x})\,q(t)$ $\qquad \vec{r}=\vec{x}$

$\vec{J}(\vec{x},t) = \frac{-1}{4\pi}\,i(t)\,\frac{\hat{r}}{r^2}$

a) verify charge conservation holds

b) Find a $\phi, \vec{A}$ in the Coulomb Gauge

c) Find the corresponding $\vec{E},\vec{B}$ fields

d) Verify they satisfy Maxwell's Equations

a) $\vec{\nabla}\cdot\vec{J}(\vec{x},t) = \frac{-\dot{q}}{4\pi}\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \frac{-\dot{q}}{4\pi}4\pi\delta^3(\vec{r}) = -\dot{q}\,\delta^3(\vec{r}) = \frac{-\partial\rho}{\partial t}$

b) $\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}dV = \frac{q(t)}{4\pi\epsilon_0}\int\frac{\delta^3(\vec{r}')}{|\vec{r}-\vec{r}'|}dV = \frac{q(t)}{4\pi\epsilon_0 r}$

By Symmetry, $\vec{B}=\vec{0} \Rightarrow \vec{\nabla}\times\vec{A}=\vec{0}$

$\vec{\nabla}\cdot\vec{A}=\vec{0}$

$\vec{A}\to\vec{0}$ as $|\vec{r}|\to\infty$

$\vec{A}(\vec{r},t)=\vec{0}$

c) $\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A} = -\vec{\nabla}\phi = \frac{1}{4\pi\epsilon_0}\frac{q(t)}{r^2}\hat{r}$

$\vec{B} = \vec{0}$

d) $\vec{\nabla}\cdot\vec{E} = \frac{q(t)}{4\pi\epsilon_0}\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{q(t)}{4\pi\epsilon_0}4\pi\delta^3(\vec{r}) = \frac{q(t)}{\epsilon_0}\delta(\vec{r}) = \frac{\rho(\vec{r},t)}{\epsilon_0}$ ✓

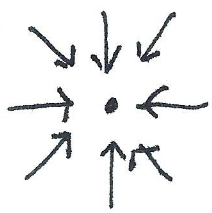
$\vec{\nabla}\cdot\vec{B} = 0$ ✓

$\vec{\nabla}\times\vec{E} = \vec{0} = -\frac{\partial \vec{B}}{\partial t}$ ✓

by symmetry radial symmetry has no curl

$\vec{0} = \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0\epsilon_0 \partial_t \vec{E}$

$= \mu_0\left(\frac{-1}{4\pi}\frac{\dot{q}}{r^2}\hat{r}\right) + \mu_0\epsilon_0\left(\frac{\dot{q}}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\right) = \vec{0}$ ✓



Griffiths 10.13

Suppose $\vec{J}(\vec{r})$ is independent of time

a) Find the most general form of $\rho(\vec{r},t)$

b) Find $\vec{E}(\vec{r},t)$

a) $\vec{\nabla}\cdot\vec{J} + \partial_t \rho = 0$

$\partial_t \rho = -\vec{\nabla}\cdot\vec{J}$ - independent of time, ρ not necessarily time independent

$\rho(\vec{r},t) = \rho(\vec{r},0) + \dot{\rho}(\vec{r},0)t$

ρ can at most be linear in time

b) $\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\left(\frac{(\rho(\vec{r}\,',t'))_{ret}}{R^2} + \frac{(\partial_t \rho(\vec{r},t'))_{ret}}{cR} + \frac{(\partial_t \vec{J})_{ret}}{c^2 R}\right) d^3\vec{r}\,'$

$\searrow 0$

$$= \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\left(\frac{\rho(\vec{r}\,',0) + t_r\dot{\rho}(\vec{r},0)}{R^2} + \frac{\dot{\rho}(\vec{r},0)}{cR}\right)\hat{R}$$

$$t_r = t - \frac{R}{c}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\left(\frac{\rho(\vec{r}\,',0)}{R^2} + \frac{\dot{\rho}(\vec{r}\,',0)}{R^2} - \frac{\dot{\rho}(\vec{r}\,',0)}{R^2}\frac{R}{c} + \frac{\dot{\rho}(\vec{r}\,',0)}{cR}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\,\frac{(\rho(\vec{r}\,',0) + \dot{\rho}(\vec{r}\,',0)t)}{R^2}\hat{R}$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\left(\frac{\rho(\vec{r}\,',t)}{R^2}\hat{R}\right)$$

Coulomb Field due to instantaneous $\rho(\vec{r},t)$