Phys 110B 21Oct20

Covariant Formulation of Green Function for wave equation

$$\Box G(\underset{\sim}{x},\underset{\sim}{x}') = \delta^4(\underset{\sim}{x}-\underset{\sim}{x}')$$

$$G = \frac{-1}{(2\pi)^4}\int d^4k \, \frac{e^{-i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}}{\underset{\sim}{k}\cdot\underset{\sim}{k}} \longleftarrow \text{manifestly covariant}$$

$$G_r(\underset{\sim}{x},\underset{\sim}{x}') = \Theta(x^0-x^{0\prime})\,\delta(x^0-x^{0\prime}-R)/4\pi R$$

$$G_a(\underset{\sim}{x},\underset{\sim}{x}') = \Theta(x^{0\prime}-x^0)\,\delta(x^0-x^{0\prime}+R)/4\pi R$$

$$\delta((\underset{\sim}{x}-\underset{\sim}{x}')^2) = \delta(-(x^0-x^{0\prime})^2 + |\vec{x}-\vec{x}'|^2)$$

$$= \delta((x^0-x^{0\prime})^2 - |\vec{x}-\vec{x}'|^2) = \delta((x^0-x^{0\prime})^2 - R^2)$$

$$= \delta((x^0-x^{0\prime}-R)(x^0-x^{0\prime}+R))$$

$$= \frac{\delta(x^0-x^{0\prime}-R) + \delta(x^0-x^{0\prime}+R)}{2R}$$

$$G_r(\underset{\sim}{x},\underset{\sim}{x}') = \frac{1}{2\pi}\Theta(x^0-x^{0\prime})\,\delta((\underset{\sim}{x}-\underset{\sim}{x}')^2)$$

$$G_a(\underset{\sim}{x},\underset{\sim}{x}') = \frac{1}{2\pi}\Theta(x^{0\prime}-x^0)\,\delta((\underset{\sim}{x}-\underset{\sim}{x}')^2)$$

↑ Not invariant under all Lorentz transformations

Invariant under all orthochronous. (No time reversal)

$$J^\alpha(\underset{\sim}{X}) = q_c \int d\tau\, U^\alpha(\tau)\, \delta^4(\underset{\sim}{X} - \underset{\sim}{r}(\tau))$$

for each point charge

worldline

$$A^\mu(\underset{\sim}{x}) \int d^4\underset{\sim}{x}'\, G(\underset{\sim}{x} - \underset{\sim}{x}')\, J^\mu(\underset{\sim}{x}')$$

$\vec{A}, \phi$ "generated" by $\rho, \vec{J}$ Lorentz-gauge

Hertz Vectors / Polarization Potentials

$\vec{\Pi}_e, \vec{\Pi}_m$ sourced by $\vec{P}, \vec{M}$

$$\vec{J}_m = \vec{\nabla} \times \vec{M} \qquad \vec{K}_m = \vec{M} \times \hat{n}$$

$$\rho = -\vec{\nabla} \cdot \vec{P} \qquad \sigma_e = \vec{P} \cdot \hat{n} \qquad \vec{J}_e = \partial_t \vec{P}$$

$$\vec{D} = \epsilon \vec{E} + \vec{P} \qquad \vec{B} = \mu \vec{H} + \mu_0 \vec{M}$$

external source

external source

L-L gauge

$$\mu\epsilon\, \partial_t^2 \vec{A} - \nabla^2 \vec{A} = \mu\, \partial_t \vec{P} + \mu_0\, \vec{\nabla} \times \vec{M}$$

$$\mu\epsilon\, \partial_t^2 \phi - \nabla^2 \phi = -\frac{1}{\epsilon} \vec{\nabla} \cdot \vec{P}$$

$$\vec{\nabla} \cdot \vec{A} + \mu\epsilon\, \partial_t \phi = 0$$

Let $\vec{A} = \mu \partial_t \vec{\Pi}_e + \mu_0 \vec{\nabla} \times \vec{\Pi}_m$

$\phi = -\frac{1}{\epsilon} \vec{\nabla} \cdot \vec{\Pi}_e$

Automatically satisfy the L-L gauge condition

$$\vec{\nabla} \cdot \left(\nabla^2 \vec{\Pi}_e - \mu\epsilon \partial_t^2 \vec{\Pi}_e + \vec{P} \right) = 0$$

$$\mu \partial_t \left(\nabla^2 \vec{\Pi}_e - \mu\epsilon \partial_t^2 \vec{\Pi}_e + \vec{P} \right) + \mu_0 \vec{\nabla} \times \left(\nabla^2 \vec{\Pi}_m - \mu\epsilon \partial_t^2 \vec{\Pi}_m + \vec{M} \right) = \vec{0}$$

$$\mu\epsilon \partial_t^2 \vec{\Pi}_e - \nabla^2 \vec{\Pi}_e = \vec{P} - \frac{\mu_0}{\mu} \vec{\nabla} \times \vec{V}$$ some vector field

$$\mu\epsilon \partial_t^2 \vec{\Pi}_m - \nabla^2 \vec{\Pi}_m = \vec{M} + \partial_t \vec{V} + \vec{\nabla} \partial_t \xi$$ some scalar field

$\vec{A} \to \vec{A} + \vec{\nabla} \chi$ $\Rightarrow$ stay in L-L gauge as long as

$\phi \to \phi - \partial_t \chi$ $\qquad \Box \chi = 0$

iF we use $\chi = -\mu \partial_t g$ then...

~~Then~~ can gauge transform away $\vec{V}, \xi$

$$\mu\epsilon \partial_t^2 \vec{\Pi}_e - \nabla^2 \vec{\Pi}_e = \vec{P} \qquad *1$$

$$\mu\epsilon \partial_t^2 \vec{\Pi}_m - \nabla^2 \vec{\Pi}_m = \vec{M}$$

$$\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A} = \frac{1}{\epsilon} \vec{\nabla}(\vec{\nabla} \cdot \vec{\Pi}_e) - \mu \partial_t^2 \vec{\Pi}_e - \mu_0 \vec{\nabla} \times \partial_t \vec{\Pi}_m$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \mu \vec{\nabla} \times \partial_t \vec{\Pi}_e + \mu_0 \vec{\nabla} \times \vec{\nabla} \times \vec{\Pi}_m$$

Gauge structure of the Hertz vectors:

~~in $\vec{\Pi}_e \to$~~

$\vec{\Pi}_e \to \vec{\Pi}_e + \mu_0 \vec{\nabla} \times \vec{G} - \vec{\nabla} g$

$\vec{\Pi}_m \to \vec{\Pi}_m - \mu \partial_t \vec{G}$

if $(\mu \epsilon \partial_t^2 - \nabla^2) G = 0$

$(\mu \epsilon \partial_t^2 - \nabla^2) g = 0$

then $\vec{\Pi}_e, \vec{\Pi}_m$ satisfy *1

$\vec{E}, \vec{B}$ = 6 components

$\phi, \vec{A}$ = 4 components − 1 gauge constraint = 3 independent components

$\vec{\Pi}_e, \vec{\Pi}_m$ = 6 components

sourced by $\vec{P}, \vec{M}$

From Potentials to Fields

Jeffimenko Expressions

$$\phi = \frac{1}{4\pi\epsilon_0} \int d^3\vec{x}' \frac{1}{R} (\rho(\vec{x}', t'))_{ret}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int d^3\vec{x}' \frac{1}{R} (\vec{J}(\vec{x}', t'))_{ret}$$

$R = |\vec{x} - \vec{x}'|$

$c(t - t') = |\vec{x} - \vec{x}'| = R$

$\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}$ $\qquad$ $\vec{B} = \vec{\nabla} \times \vec{A}$

explicit ~~$R(\vec{x})$~~

depend on $\vec{x}$ through $\frac{1}{R} = \frac{1}{|\vec{x} - \vec{x}'|}$

but also through retarded time implicitly

$$\vec{E}(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{x}\,' \left\{ \frac{\hat{R}}{R^2}(\rho(\vec{x}\,',t'))_{ret} + \frac{\hat{R}}{cR}\left(\frac{\partial\rho(\vec{x},t')}{\partial t'}\right)_{ret} + \frac{-1}{c^2R}\left(\frac{\partial\vec{J}(\vec{x},t')}{\partial t'}\right)_{ret} \right.$$

$$\vec{R} = \vec{x} - \vec{x}\,' \qquad R = |\vec{R}| \qquad \hat{R} = \frac{\vec{R}}{R}$$

$$\vec{B}(\vec{x},t) = \frac{\mu}{4\pi}\int d^3\vec{x}\,' \left\{ (\vec{J}(\vec{x},t'))_{ret} \times \frac{\hat{R}}{R^2} + \left(\frac{\partial\vec{J}(\vec{x}\,',t')}{\partial t'}\right)_{ret} \times \frac{\hat{R}}{cR} \right.$$

Extra terms are precisely what accounts for radiation.