Phys 110B 19 Oct 20

Green Function / Impulse Response Functions

forced/source linear equation

e.g. Electrostatics

source

$$\underbrace{\phi(\vec{r})}_{\text{observation}} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'$$

Green function

Coulomb's Law

$$\cancel{\nabla^2\phi} \quad \nabla^2 G(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,')$$

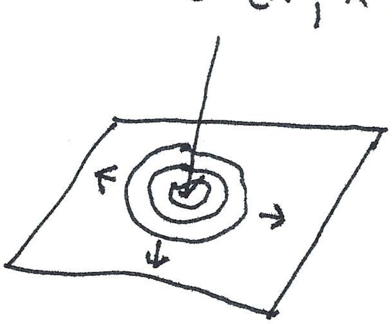
$$\underbrace{\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)}_{\text{wave operator}} G(\underbrace{\vec{x},t}_{\text{observation}};\underbrace{\vec{x}\,',t'}_{\text{source}}) = -4\pi\,\delta(\vec{x}-\vec{x}\,')\,\delta(t-t')$$

space-time domain

frequency domain

$$(\nabla^2 + k^2)\,G_k(\vec{x};\vec{x}\,') = -4\pi\,\delta(\vec{x}-\vec{x}\,'), \quad k = \frac{\omega}{c}$$

Claim: $G^{\pm}(\vec{x};\vec{x}\,') = \dfrac{e^{\pm ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|}$ spherical waves

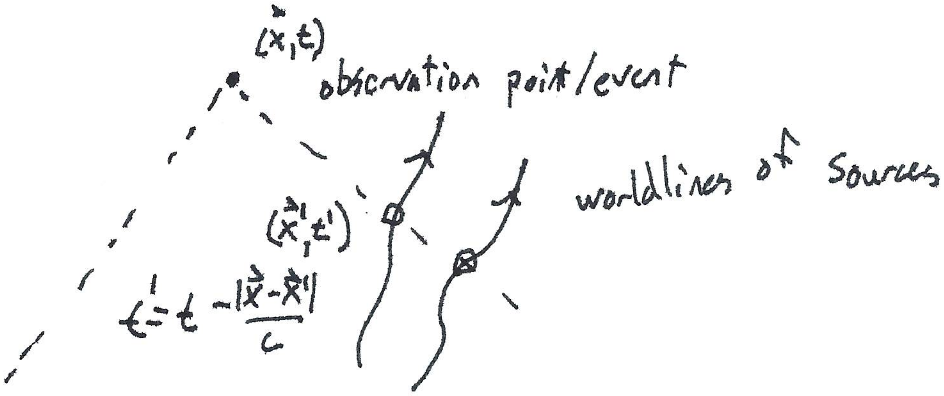


+ expanding spherical wave

− converging spherical wave

$$G^{\pm}(\vec{x},t;\vec{x}',t') = \frac{\delta\left(t' - \left(t \mp \frac{|\vec{x}-\vec{x}'|}{c}\right)\right)}{|\vec{x}-\vec{x}'|}$$

nonzero when $t' = t \mp \dfrac{|\vec{x}-\vec{x}'|}{c}$



t = current time, present, observer time

$t' = t_r$ = retarded time (backwards along past light cone of observation point)

= emitter time

any worldline of a charged particle intersects the past light cone at most one point

G^+ = retarded Green function

causal Green function

G^- = advanced Green function

Acausal Green function

$$t' = t_a = t + \frac{|\vec{x}-\vec{x}'|}{c}$$

Green functions about information, not causation

One and the same physical solution using either green function.

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\psi(\vec{x},t) = -4\pi f(\vec{x},t)$$

$$\psi(x,t) = \underbrace{\psi_{in}(\vec{x},t)}_{\text{homogeneous solution}} + \underbrace{\iint G^+(\vec{x},t;\vec{x}',t')f(\vec{x}',t')\,d^3\vec{x}'\,dt'}_{\text{inhomogeneous solution}}$$

everything is present in ψ before f turns on

extrapolated over all time according to source-free wave equation

"Incident" fields

$$\psi(\vec{x},t) = \psi_{out}(\vec{x},t) + \iint G^{-}(\vec{x},t;\vec{x}',t')f(\vec{x}',t')d^3\vec{x}'dt'$$

"stuff" present after f turns off

extrapolated over all t

Usually $\psi(\vec{x},t) = \iint G^{+}(\vec{x},t;\vec{x}',t')d^3\vec{x}'dt'$

Lorenz-

~~Lorentz~~ gauge

$$\phi_L(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}'\frac{(\rho(\vec{r}',t'))_{ret}}{|\vec{r}-\vec{r}'|} \qquad t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$$

$$\vec{A}_L(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3\vec{r}'\frac{(\vec{J}(\vec{r}',t'))_{ret}}{|\vec{r}-\vec{r}'|} \qquad t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$$

Coulomb gauge

$$\phi_c(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}'\frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$ ← introduces observer time

$$\vec{A}_\perp(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3\vec{r}'\frac{(\vec{J}_\perp(\vec{r}',t'))_{ret}}{|\vec{r}-\vec{r}'|}$$

Covariant formulation

$$\Box G = \delta^4(X-X')$$

$$R = |\vec{x}-\vec{x}'|$$

$$G_r = \Theta(x^0-x^{0\prime})\frac{\delta(x^0-x^{0\prime}-R)}{4\pi R} = \frac{1}{2\pi}\Theta(x^0-x^{0\prime})\delta[(\underset{\sim}{X}-\underset{\sim}{X}')^2]$$

$$G_a = \Theta(x^{0\prime}-x^0)\frac{\delta(x^0-x^{0\prime}+R)}{4\pi R} = \frac{1}{2\pi}\Theta(x_0'-x_0)\delta[(\underset{\sim}{X}-\underset{\sim}{X}')^2]$$