Phys 110B 16 Oct 20

Constitutive Relations

$\vec{D}$ in terms of $\vec{E}$ and $\vec{B}$

$\vec{H}$

$\epsilon(\omega)$ $\quad \mu(\omega)$ $\quad \sigma(\omega)$

$\epsilon(\omega, x)$ $\quad \mu(\omega, x)$ $\quad \sigma(\omega, x)$

$\overleftrightarrow{\epsilon}$ $\quad \overleftrightarrow{\mu}$ $\quad \overleftrightarrow{\sigma}$

Magneto-optical effects $\vec{D}(\vec{E}, \vec{B})$

$\vec{H}(\vec{E}, \vec{B})$

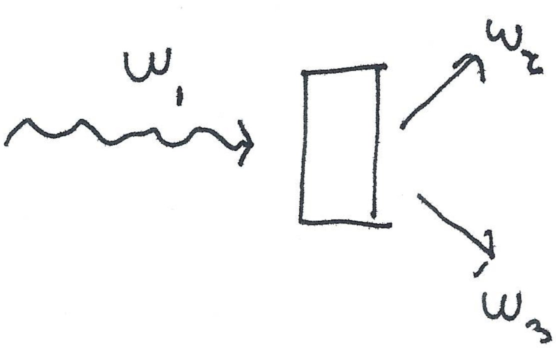
Non-linear optics

$\vec{D} = \epsilon \vec{E}$ in general $\vec{D}(\vec{E}, \vec{B})$

nonlinear

$$\vec{D}(\vec{E}) = \epsilon \vec{E} + \beta |\vec{E}| \vec{E} + \dots$$

2nd order, 3rd order nonlinear optical effects



$\omega_1 = \omega_2 + \omega_3$

Quantum optics and nonlinear optics

Potential Formulation

$$\vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{E} = -\vec{\nabla}\phi - \partial_t\vec{A}$$

Satisfy homogeneous Maxwell Equations

$$\nabla^2\phi - \partial_t(\vec{\nabla}\cdot\vec{A}) = -\frac{1}{\epsilon_0}\rho$$

$$\nabla^2\vec{A} - \mu_0\epsilon_0\partial_t^2\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\partial_t\phi) = -\mu_0\vec{J}$$

redundancy → gauge "freedom" gauge "invariance"

$$\phi' = \phi - \partial_t\chi$$
$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$

Gauge fixing constraint Gauge Condition

eg. Temporal / Hamilton gauge $\phi = 0$

Coulomb / transverse gauge $\vec{\nabla}\cdot\vec{A} = 0$

Lorentz-Lorenz gauge $\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\partial_t\phi = 0$

$$\partial_\mu A^\mu = 0$$

L-L gauge

$$\Box A^\mu = -\mu_0 J^\mu$$

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\vec{A} = -\mu_0\vec{J}$$

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\phi = -\frac{1}{\epsilon_0}\rho$$

Inhomongeneous or sourced wave equations

Coulomb Gauge

$$\vec{\nabla}\cdot\vec{A} = 0$$

Helmholtz decomposition of $\vec{A}$

gauge dependent

$$\vec{A}(\vec{x},t) = \vec{A}_\perp(\vec{x},t) + \vec{A}_\parallel(\vec{x},t)$$ gauge independent

functionally transverse and longitudinal

$$\vec{\nabla}\cdot\vec{A}_\perp = 0 \qquad \vec{\nabla}\times\vec{A}_\parallel = \vec{0}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\vec{A}_\perp \qquad \vec{A}_\parallel = \vec{0}$$

$$\vec{A} = \vec{A}_\perp$$

$$-\nabla^2\phi = \frac{1}{\epsilon_0}\rho \quad \leftarrow$$ looks like electrostatics

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\vec{A}_\perp = -\mu_0\vec{J} + \frac{1}{c^2}\vec{\nabla}\partial_t\phi$$

in Coulomb gauge ϕ is given by Coulomb's Law with respect to the instantaneous charge density ρ

$$-\nabla^2\phi(x,t) = \frac{1}{\epsilon_0}\rho(\vec{x},t)$$

potential changes instantaneously

$dt\ \pi^+$ $+$

ϕ and $\vec{A}$ are not physically observable
They are book-keeping devices

physical fields $\vec{E},\vec{B}$ don't have superluminal dependence

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\vec{A}_\perp = -\mu_0 \vec{J} + \frac{1}{c^2}\vec{\nabla}\partial_t\phi = -\mu_0 \vec{J}_\perp$$

$\vec{\nabla}\cdot\vec{A}_\perp = 0$

transverse current density

Divergence of LHS = 0 divergence of RHS must be 0

$$\vec{J} = \vec{J}_\perp + \vec{J}_\parallel$$

$$\vec{J}_\parallel = \epsilon_0 \vec{\nabla}\partial_t\phi \qquad \left(\frac{1}{c^2}\vec{\nabla}\partial_t\phi\right) = \mu_0 \vec{J}_\parallel$$

$$\vec{\nabla}\times\vec{J}_\parallel = \vec{0}$$

$$\vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot(\vec{J}_\perp + \vec{J}_\parallel) = \vec{\nabla}\cdot\vec{J}_\parallel$$

$$\vec{\nabla}\cdot\vec{J}_\parallel = \vec{\nabla}\cdot(\epsilon_0\vec{\nabla}\partial_t\phi) = \epsilon_0\partial_t\vec{\nabla}\cdot\vec{\nabla}\phi$$

$$= \epsilon_0\partial_t\nabla^2\phi = \epsilon_0\partial_t\left(-\frac{1}{\epsilon_0}\rho\right) = -\partial_t\rho$$

$$= -(-\vec{\nabla}\cdot\vec{J}) = \vec{\nabla}\cdot\vec{J}$$

$\vec{E} = -\vec{\nabla}\phi - \partial_t\vec{A}_\perp$ has no "action at a distance"

$\vec{B} = \vec{\nabla}\times\vec{A}$

Helmholtz Theorem

$$\vec{J}_\perp = -\frac{1}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}\cdot\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'$$

$$\vec{J}_\parallel = \frac{1}{4\pi}\vec{\nabla}\times\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'$$

see Griffiths Appendix

Inhomogeneous wave equation (with source)

Green functions / impulse response functions

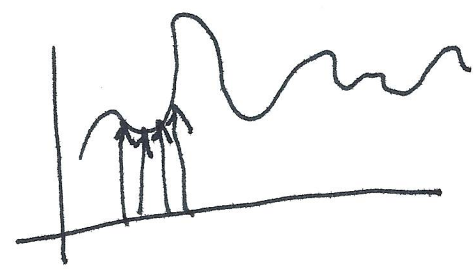
e.g. Harmonic Oscillator

$$m\ddot{x} = -m\omega^2 x - \gamma \dot{x} + \cancel{F(t)} + f(t)$$

$$m\ddot{x} + m\omega^2 x + \gamma \dot{x} = \cancel{f(t)}\ F(t) = \delta(t) = 0$$ except at $t=0$

impulsive forcing

matching homogeneous solutions at the impulse



Green Function itself = solution to the impulsive forcing

$$F(t) = \int dt'\, f(t')\,\delta(t-t')$$

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right) G(\vec{x}, t; \vec{x}', t') = \text{constant}\cdot \delta(\vec{x}-\vec{x}')\,\delta(t-t')$$

observation coordinates

location of impulsive force

ξ

impulse response G

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\phi = -\frac{1}{\epsilon_0}\rho$$

$$\phi(\vec{x}, t) = \int d^3\vec{x}'\,dt'\, \frac{G(\vec{x}, t; \vec{x}', t')}{\xi}\left(-\frac{1}{\epsilon_0}\rho(\vec{x}', t')\right)$$

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)$$