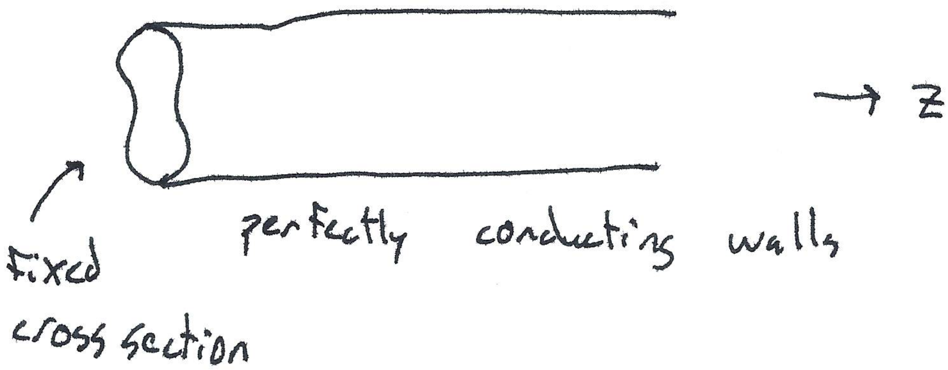
Phys 110B 07Oct20

Metallic Waveguides



"Pilot vector approach"

$$\psi(\vec{x}) = e^{ikz}\phi(\vec{x}_\perp)$$

TE "transverse electric"

$$(\nabla^2 + k_\perp^2)\phi = 0$$

$\hat{n}\cdot\vec{\nabla}_\perp\phi = 0$ on boundary

$E_z = 0 \quad B_z \propto \phi(\vec{x}_\perp) \qquad \vec{\nabla}\times(\hat{z}\psi)$

TM (transverse magnetic)

$$(\nabla^2 + k_\perp^2)\phi = 0$$

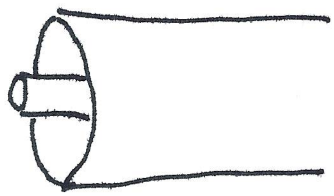
$\phi = 0$ on boundary

$B_z = 0 \quad E_z \propto \phi \qquad \vec{\nabla}\times(\hat{z}\psi)$

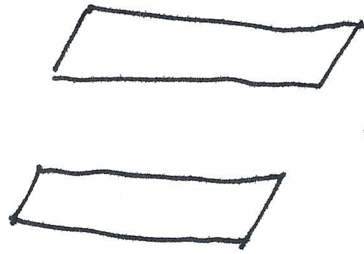
TEM modes "transverse electromagnetic

$E_z = B_z = 0$ often unsupportable

eg if walls consist of a single conductor then $\nexists$ TEM modes



coaxial cable



stripline

$$\vec{\nabla}_\perp \cdot \vec{E} = \vec{\nabla} \cdot \vec{E} = 0$$

$$i\omega \vec{B} = \vec{\nabla} \times \vec{E} = ik\hat{z} \times \vec{E} + \cancel{\vec{\nabla}_\perp \times \vec{E}} + \overbrace{\vec{\nabla}_\perp \times \vec{E}}^{\propto \hat{z}} \Rightarrow \vec{\nabla}_\perp \times \vec{E} = \vec{0}$$

$\perp$ to $\hat{z}$ $\qquad$ $\perp$ to $\hat{z}$

$$\vec{\nabla}_\perp \cdot \vec{E} = 0$$
$$\vec{\nabla}_\perp \times \vec{E} = \vec{0}$$
$$\hat{n} \times \vec{E} = \vec{0} \text{ on boundary}$$

2D electrostatics problem

in one cross section

$$\vec{E} = -\vec{\nabla}_\perp \phi$$
$$-\nabla_\perp^2 \phi = 0$$

ϕ to be constant on each connected component of the boundary

Head on example

$\phi =$ constant everywhere

$\vec{E} = \vec{0}$ $\quad$ $\vec{B} = \vec{0}$

eg

$\phi = \text{constant}_1$

$\phi = \text{constant}_2$

$$\omega^2 = c^2 k_\perp^2 + c^2 k^2$$
$$\omega^2 = c^2 k^2$$

do not take on constant profile in a transverse plane

$$\vec{B} = \frac{1}{c} \hat{z} \times \psi \qquad \psi = \phi e^{ikz}$$

$$(\nabla^2 + k_\perp^2)\phi = 0 \qquad k^2 = K^2$$

but with $k_\perp^2 = 0$

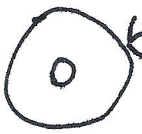
$\vec{E}, \vec{B}$ are in phase

$\vec{E} \perp \vec{B}$

$\vec{E} \perp \hat{z}$

$\vec{B} \perp \hat{z}$

shields as well

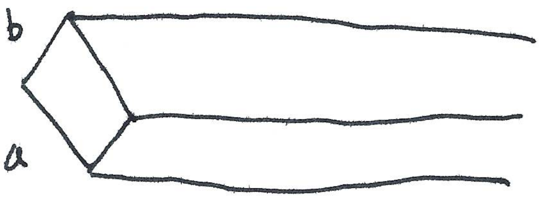


shielding

gives a TEM mode that a rectangular waveguide does not

dispersion-free propagation $\omega = \cancel{c(k)}\ c|k|$

eg. rectangular waveguide



TEM modes? None

TE modes? $(\partial_x^2 + \partial_y^2 + k_\perp^2)\phi = 0$ $\quad B_z \propto \phi$

$x=0, x=a \quad \frac{\partial \phi}{\partial x} = 0$

$y=0, y=b \quad \frac{\partial \phi}{\partial y} = 0$

$$\phi(x,y) = \phi_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$m,n \in \{0,1,2,3,\ldots\}$, $mn \neq 0$

$$k_{\perp mn}^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

Lowest TE mode

$m=1 \quad n=0$

$$\omega^2 = c^2 k_{\perp mn}^2 + c^2 k^2$$

TM modes $E_z \propto \phi$

$$(\partial_x^2 + \partial_y^2 + k_\perp^2)\phi = 0$$

at $x=0$, $x=a$ ~~$\phi=0$~~ $\phi =$ constant

$y=0$, $y=b$

$$\phi \propto \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

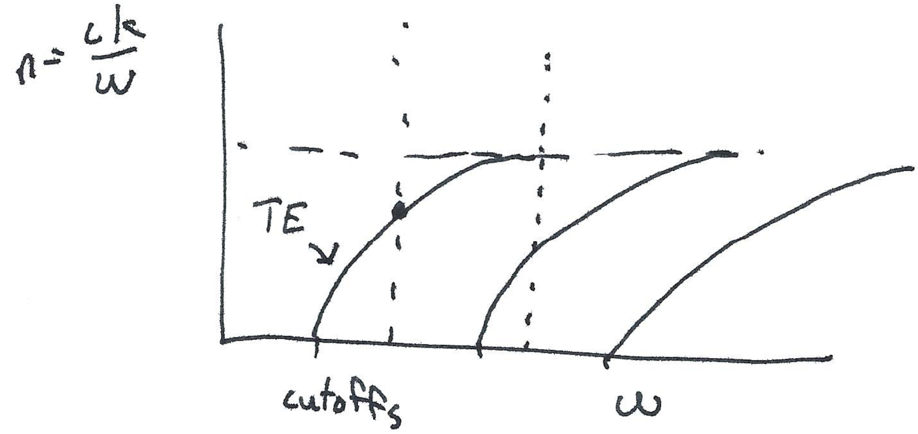
$m, n \in \{\cancel{0}, 1, 2, 3, \ldots\}$

neither can be zero

Lowest TM mode $m=1$, $n=1$

infinite # of TE modes infinite # of TM modes

No TEM modes



only waves that can transmit information are real waves

decompose total $\vec{E}, \vec{B}$ fields into modes for each ω

decompose further into TE, TM, TEM modes

Orthogonality

More complicated than QM because we have more types of boundary conditions

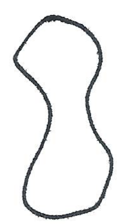
Green's Identities in the transverse plane

$$\int_R \psi_1 \nabla^2 \psi_2^* = \int_{\partial R} dl\, \psi_1 \hat{n} \cdot \vec{\nabla} \psi_2^* - \int_R da\, \vec{\nabla}\psi_1 \cdot \vec{\nabla}\psi_2^*$$

$$k_\perp^2 \geq 0$$

$\int_R \phi_j \phi_i^* \, da = \delta_{ji}$ orthogonality of the generating functions

$$\int_R \hat{z} \cdot (\vec{E}_j \times \vec{B}_i^*) da \propto \int \vec{\nabla}\psi_j \cdot \vec{\nabla}\psi_i^* da \propto \int \psi_j \psi_i^* da$$



decompose into modes using functional inner products

given fields in 1 cross section

propagate each mode "forward"

No power is carried in cross terms

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu} \propto \sum_{\text{modes } j} \Big(\underbrace{a_j \cancel{q} \hat{z} |\vec{\nabla}_\perp \psi_j|^2}_{\text{energy flux}} + \overbrace{i b_j \underbrace{\psi_j \vec{\nabla}_\perp \psi_j^*}_{\text{real}}}^{\text{reactive power flow}} \Big) da$$

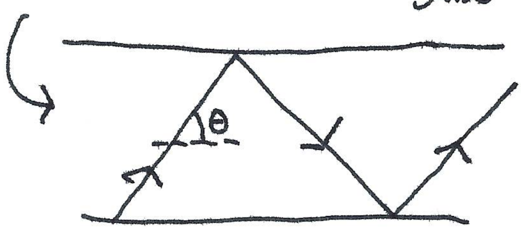
so $\frac{1}{2} \mathrm{Re}(\vec{S}) \propto \hat{z}$

Energy is transported at $v_g = \frac{ck}{\omega}$

different
for each mode

Vacuum

cross-section/slice
of waveguide



$\rightarrow z$

$k_z = k_{tot} \cos\theta$

$v_g = c \cos\theta$

$v_\phi = \frac{c}{\cos\theta}$

eg rectangular waveguide

$\lambda_x = \frac{2a}{m} \qquad \lambda_y = \frac{2b}{n}$