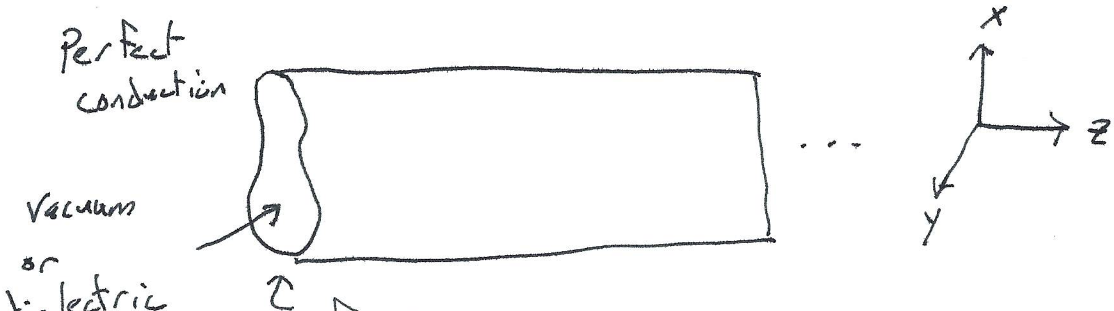
Phys 110B 05 Oct 20

Hollow Generalized-Cylindrical waveguides
with conducting walls



Fixed cross-section shape

time translation invariant $\Rightarrow e^{-i\omega t}$

No transverse electric and magnetic fields if wave propagating in z, then $\nexists$ modes for wave $E_z = 0$, $B_z = 0$

no TEM modes

(Transverse Electric Magnetic fields cf to propagation axis)

$$\underset{0}{\underset{\shortparallel}{\oint \vec{B}\cdot d\vec{r}}} = \mu_0 \underset{0}{\underset{\shortparallel}{\iint \vec{J}_r \cdot d\vec{a}}} + \mu_0 \epsilon_0 \underset{0}{\underset{\shortparallel}{\iint \frac{\partial}{\partial t}\vec{E}\cdot d\vec{a}}}$$

0

Can decompos modes into TE and TM

transverse electric   Transverse magnetic

TE mode: $E_z = 0$ $B_z \neq 0$

TM mode: $B_z = 0$ $E_z \neq 0$

TE Modes

$\vec{\nabla}\cdot\vec{E} = 0$

$(\nabla^2 + K^2)\vec{E} = \vec{0}$ Vector Helmholtz equation

$K^2 = \frac{\omega^2}{c^2}$

Assume $\vec{E} \propto e^{\pm ikz}$ $k = k_z$ = longitudinal wavenumber

"Pilot vector technique"

Scalar Helmholtz equation $(\nabla^2 + k^2)\psi = 0$

then $(\nabla^2 + k^2)\hat{z}\psi = \vec{0}$

$(\nabla^2 + k^2)\vec{\nabla}\psi = \vec{0}$

$(\nabla^2 + k^2)\vec{\nabla}\times(\hat{z}\psi) = \vec{0}$

$\vec{\nabla}\cdot(\vec{\nabla}\times(\hat{z}\psi)) = \vec{0}$

$\underset{\substack{\uparrow \\ \text{generating} \\ \text{Function}}}{\psi(x,y,z)} = e^{ikz}\,\phi(x,y)$

$\vec{E} = \vec{\nabla}\times(\underset{\substack{\uparrow \\ \text{pilot vector}}}{\hat{z}}\psi) = \psi(\overset{\nearrow 0}{\vec{\nabla}\times\hat{z}}) + \vec{\nabla}\psi\times\hat{z} = -\hat{z}\times\vec{\nabla}\psi$

$\vec{\nabla}\psi = \hat{z}ik\psi + \vec{\nabla}_\perp\psi$

$\vec{E} = \vec{\nabla}_\perp\psi\times\hat{z} = e^{ikz}\,\vec{\nabla}_\perp\phi\times\hat{z}$

$\vec{\nabla}\cdot\vec{E} = 0$ TE mode $\Rightarrow E_z = 0 \Rightarrow \hat{z}\cdot\vec{E} = 0$

$(\nabla^2 + K^2)\vec{E} = 0$ $\hat{z}\cdot\vec{E} = \hat{z}\cdot(\vec{\nabla}\psi\times\hat{z}) = 0$

$$i\omega \vec{B} = \vec{\nabla}\times\vec{E} = \vec{\nabla}\times(\vec{\nabla}\times(\hat{z}\psi)) = \vec{\nabla}(\vec{\nabla}\cdot(\hat{z}\psi)) - \nabla^2\hat{z}\psi$$

$$(\nabla^2+k^2)\vec{B} = \vec{0}$$

$$i\omega\vec{B} = \vec{\nabla}(ik\psi) + K^2\hat{z}\psi$$

$$i\omega\vec{B} = ik\vec{\nabla}_\perp\psi + (K^2-k^2)\psi\hat{z}$$

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times(\vec{\nabla}\times(\hat{z}\psi))) = 0$$

$\Rightarrow \vec{E}\cdot\vec{B} = 0$ locally

B.C.: $0 = \hat{n}\times\vec{E}\big|_{\partial R} = \hat{n}\times(-\hat{z}\times\vec{\nabla}_\perp\psi)$

$$= (-\hat{n}\cdot\vec{\nabla}_\perp\psi)\hat{z} + \underbrace{(\hat{n}\cdot\hat{z})}_{\to 0}\vec{\nabla}_\perp\psi$$

$\hat{n}$ normal to ∂R

$\hat{n}\cdot\vec{\nabla}_\perp\psi\big|_{\partial R} = 0$ Neuman Boundary Condition

$(\nabla^2+K^2)\psi = 0$ inside R

$\vec{B}$ B.C. also satisfied

TE modes

Solve $(\nabla^2+k^2)\psi = 0$ inside the cylinder

$\hat{n}\cdot\vec{\nabla}_\perp\psi\big|_{\partial R} = 0$

$\vec{E} = \vec{\nabla}\times(\hat{z}\psi)$ $\quad \vec{B} = \frac{1}{i\omega}\vec{\nabla}\times\vec{E}$

$\psi = e^{ikz}\phi(x,y)$

TM - transverse Magnetic $B_z = 0$ $E_z = 0$

$$(\nabla^2 + K^2)\psi = 0 \qquad \psi = e^{ikz}\phi(x,y)$$

$$\vec{B} = \vec{\nabla} \times (\psi \hat{z}) \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \hat{z}\cdot \vec{B} = 0$$

$$\mu_0 \epsilon_0 (-i\omega)\vec{E} = \vec{\nabla}\times\vec{B} = \dots (K^2 - k^2)\hat{z}\psi + ik\vec{\nabla}_\perp \psi$$

$$\hat{z}\cdot\vec{E}\big|_{\partial R} = 0 \Rightarrow \psi\big|_{\partial R} = 0$$

Dirichlet Boundary Condition

$$\Rightarrow \cdots \Rightarrow \hat{n}\times\vec{E}\big|_{\partial R} = \vec{0}$$

$$(\nabla^2 + K^2)\psi = 0 \Rightarrow (\nabla^2 + K^2)(e^{ikz}\phi) = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial z^2} + \nabla_\perp^2 + K^2\right)(e^{ikz}\phi) = 0$$

$$\left((ik)^2 + \nabla_\perp^2 + K^2\right)e^{ikz}\phi = 0$$

$$\left(\nabla_\perp^2 + (K^2 - k^2)\right)\phi = 0$$

2D scalar Helmholtz equation

Basically eigenvalue equation $\qquad K = \frac{\omega}{c} \qquad k = ?$

$$-\nabla_\perp^2 \phi = \underbrace{(K^2 - k^2)}_{= k_\perp^2}\phi$$

$k_\perp^2$ ← constant for any 1 mode

$$\frac{\omega^2}{c^2} - k^2 = k_\perp^2$$

$$\boxed{\omega^2 = c^2 k_\perp^2 + c^2 k^2}$$ dispersion relation

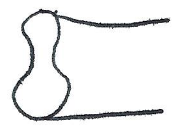
eigenvalue^2 $\qquad$ longitudinal wavenumber

cutoff

$-\nabla_\perp^2 \phi = k_\perp^2 \phi$ 2d eigenvalue problem

Boundary conditions $\Rightarrow$ discrete eigenvalues $k_\perp^2$

Close to 2D Schrödinger equation for a particle in a 2d "box" of shape R



$k_{\perp 1}, k_{\perp 2}, \ldots,$

$\omega^2 = c^2 k_{\perp i}^2 + c^2 k^2$ $\quad i = 1, 2, \ldots,$ modes