Phys 110B 28Sep20

EM waves in LHI medium

$$\vec{D} = \epsilon \vec{E} \qquad \vec{H} = \frac{1}{\mu}\vec{B} \qquad \vec{J}_f = \sigma \vec{E}$$

$$\epsilon \vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} + \mu\epsilon\partial_t \vec{E}$$

Think about midterm
2 hour?
24 hour?

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\partial_t \vec{\nabla}\times\vec{B} = -\mu\sigma\partial_t \vec{E} - \mu\epsilon\partial_t^2 \vec{E}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}$$

$$\boxed{\begin{aligned} \nabla^2\vec{E} - \mu\sigma\partial_t\vec{E} - \mu\epsilon\partial_t^2\vec{E} &= \vec{0} \\ \nabla^2\vec{B} - \mu\sigma\partial_t\vec{B} - \mu\epsilon\partial_t^2\vec{B} &= \vec{0} \end{aligned}}$$

1st order derivatives

Special case: insulating medium $\sigma = 0$

$$(\nabla^2 - \mu\epsilon\partial_t^2)\vec{E} = \vec{0}$$

phase velocity $v_\phi = \frac{1}{\sqrt{\mu\epsilon}}$

irreducible solutions $\sim e^{-i\omega t}$

$$(\nabla^2 + k^2)\vec{E} = \vec{0} \qquad \text{Helmholtz Equation}$$

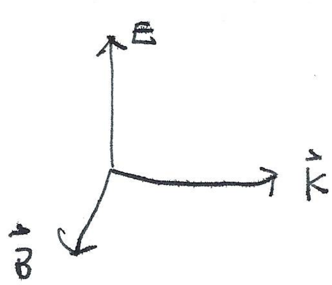
$$\vec{E} = \hat{e} E_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

Substitute into Maxwell Equations

$$\omega^2 = v_\phi^2 |\vec{k}|^2 = v\, \vec{k}\cdot\vec{k}$$

$$\vec{k}\cdot\vec{E} = 0 \qquad \vec{k}\cdot\vec{B} = 0$$

$$\vec{k}\times\vec{E} = \frac{\omega}{|\vec{k}|}\vec{B} \qquad \hat{k}\times\vec{B} = \frac{-\omega}{|\vec{k}|v^2}\vec{E}$$



$$\frac{1}{\sqrt{\mu\epsilon}} = v_\phi = \frac{\omega}{k} = \frac{c}{n}$$ index of refraction

$$n = \frac{ck}{\omega} = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$$

$\sigma \neq 0$

$$\nabla^2\vec{E} - \mu\sigma\partial_t\vec{E} - \mu\epsilon\,\partial_t^2\vec{E} = \vec{0}$$

$$\vec{E} = \hat{e}E_0\,e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\vec{k}\cdot\vec{k} = k^2 = \omega^2\mu\epsilon + i\omega\mu\sigma$$

time-harmonic fields $\rightarrow \omega > 0 \rightarrow k$ is complex

$k = k_r + ik_i \qquad k_i, k_r \in \mathbb{R}$ $\qquad k_r$ is real component

k_i is imaginary component

$k_r^2 - k_i^2 = \mu\epsilon\omega^2$ $\qquad$ assuming μ, ϵ, σ are real

$2k_rk_i = \mu\sigma\omega$

$$k_r = \omega\sqrt{\frac{\mu\epsilon}{2}}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right)^{1/2}$$

$$k_i = \omega\sqrt{\frac{\mu\epsilon}{2}}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right)^{1/2}$$

$\frac{\omega\epsilon}{\sigma} = Q$ = "quality factor" $\qquad$ how fast decays on per wiggle basis

$\propto \frac{\epsilon/\sigma}{2\pi/\omega}$ $\leftarrow$ decay time-scale due to conductivity

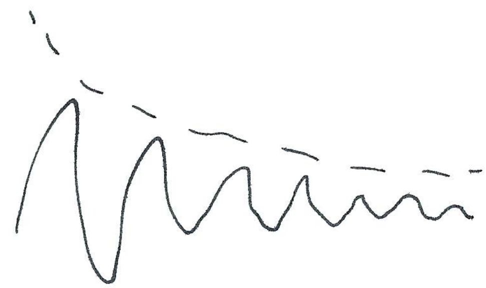
$\leftarrow$ oscillation time scale

$\propto \frac{\left|\frac{\partial\vec{D}}{\partial t}\right|}{|\vec{J}_f|}$ $\leftarrow$ displacement current

$\leftarrow$ free current

$K = |k| e^{i\Phi}$ $\quad k_r = |k|\cos\Phi$ $\quad k_i = |k|\sin\Phi$

$$|k| = \omega\sqrt{\mu\epsilon}\left(1+\frac{1}{Q^2}\right)^{1/4}$$



$$\tan\Phi = \frac{k_i}{k_r} = \sqrt{1+Q^2} - Q$$

$$\vec{E} = \hat{e} E_0 e^{-k_i z} e^{i(k_r z - \omega t)}$$

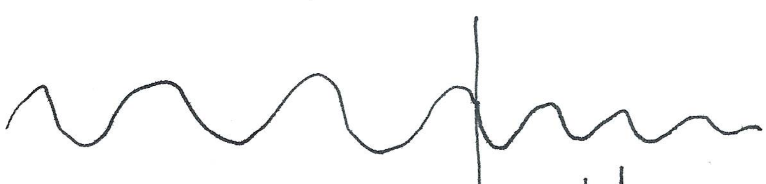
oscillatory part

damping part

"phase velocity" $v_\phi = \frac{\omega}{k_r} = \frac{1}{\sqrt{\mu\epsilon}}\left(\frac{1}{\left(1+\frac{1}{Q^2}\right)^{1/2}+1}\right)^{1/2}$

Not the apparent velocity of the peaks

Attenuation distance "skin depth" $\quad \delta = \frac{1}{k_i} = \frac{4\pi}{\mu\sigma\omega\lambda}$



Complex wave velocity $V = \frac{\omega}{K} = \frac{\omega}{k_r + ik_i} = \frac{\omega}{|k|} e^{-i\Phi}$

complex refractive index $\quad N = \frac{c}{V} = \frac{c(k_r+ik_i)}{\omega} = \underbrace{\frac{c}{v}}_{= n} + i\frac{ck_i}{\omega}$

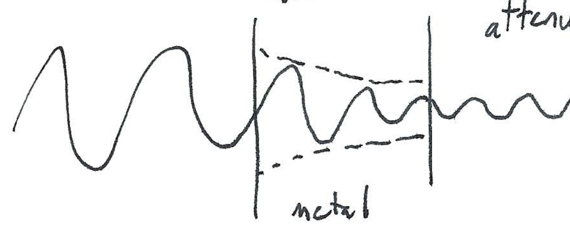
Wave moving in $\hat{z}$ direction

$$\vec{E} = \hat{e}\,\epsilon_0\, e^{-k_r z} e^{i(k_r - \omega t) + i\theta} \qquad \hat{e}\cdot\hat{z} = 0$$

$$\vec{B} = \frac{k}{\omega}\,\hat{z}\times\vec{E} \Rightarrow \vec{B} = \frac{\hat{z}\times\vec{E}}{V}$$

$$\vec{B} = (\hat{z}\times\hat{e})\,\frac{|\vec{k}|}{\omega}\,\epsilon_0\, e^{-k_i z} e^{i(k_r z - \omega t + \theta + \Phi)}$$

$\vec{B}$ and $\vec{E}$ are no longer <u>in phase</u>



$$\frac{|B_r|}{|E_r|} = \frac{|k|}{\omega} = \sqrt{\mu\epsilon}\left(1 + \frac{1}{Q^2}\right)^{1/4}$$

monochromatic $\rightarrow$ fixed, real ω

Limiting cases

near insulator $Q \gg 1$ use $\frac{1}{Q}$ as small parameter for expansion

$$k_r \cong \omega\sqrt{\mu\epsilon}\left(1 + \frac{1}{8Q^2} + \dots\right)$$

$$\frac{1}{\delta} = k_i \cong \frac{\omega\sqrt{\mu\epsilon}}{Q} = \frac{\sigma}{2}\left(\frac{\mu}{\epsilon}\right)^{1/2} \qquad = \frac{\sigma}{2}\left(\frac{\mu}{\epsilon}\right)^{1/2}$$

$$|k| = \omega\sqrt{\mu\epsilon}\left(1 + \frac{1}{4Q^2} + \dots\right) \qquad \tan\Phi = \frac{1}{2Q} = \frac{\sigma}{2\omega\epsilon}$$

$$v = \frac{1}{\sqrt{\mu\epsilon}}\left(1 - \frac{1}{8Q^2} + \dots\right) \qquad \lambda = \frac{2\pi}{\omega\sqrt{\mu\epsilon}}\left(1 - \frac{1}{8Q^2} + \dots\right)$$

$$\frac{|\vec{B}|}{|\vec{E}|} = \sqrt{\mu\epsilon}\left(1 + \frac{1}{4Q^2} + \dots\right)$$

Good conductor $Q \ll 1$ $\quad Q$ is a small parameter for expansion

$$k_r = \left(\tfrac{1}{2}\mu\sigma\omega\right)^{1/2}\left(1 + \frac{Q}{2} + \cdots\right)$$

$$\frac{1}{\delta} = k_i = \left(\tfrac{1}{2}\mu\sigma\omega\right)^{1/2}\left(1 - \tfrac{1}{2}Q + \cdots\right)$$

$$|k| = (\mu\sigma\omega)^{1/2} \qquad \tan\Phi = 1 - Q$$

$$v = \frac{2\omega}{\mu\sigma}\left(1 - \frac{Q}{2}\right) \qquad \lambda = 2\pi\left(\frac{2}{\mu\sigma\omega}\right)^{1/2}\left(1 - \tfrac{1}{2}Q\right)$$

$$\frac{|\vec{B}|}{|\vec{E}|} = \left(\frac{\mu\sigma}{\omega}\right)^{1/2} = \left(\frac{\mu\epsilon}{Q}\right)^{1/2}$$

~~No~~ $|\vec{B}|$ is large compared to $|\vec{E}|$

phase difference is $\sim \frac{\pi}{4}$

$$\delta \sim \lambda_{\text{in conductor}} \ll \lambda_{\text{vacuum}}$$

$$\cancel{\vec{H} = \frac{k}{\mu\omega}\, E}$$

$$\vec{H} = \frac{k}{\mu\omega}\,\hat{k}\times\vec{E} = \frac{\hat{k}\times\vec{E}}{\mu v} = \frac{\hat{k}\times\vec{E}}{Z}$$

capital Z = "wave impedance"

eg for $\sigma = 0$ $\quad \sqrt{\frac{\mu}{\epsilon}}$

$\sigma \neq 0$ $\quad \sqrt{\frac{i\omega\mu}{\sigma + i\omega\epsilon}}$

So far we assumed $\epsilon\ \mu\ \sigma$ constants

more generally they can be dispersive $\epsilon(\omega)$, $\mu(\omega)$, $\sigma(\omega)$

eg simple model for conducting medium "plasma"
tenuous ionized gas

look at 1 electron (can do the same for ions)

$$\vec{F} = m\vec{a} = -e\vec{E} - \zeta\vec{v}$$

Assum $\vec{E} \propto \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$\vec{v} \propto \vec{v}_0 e^{-i\omega t}$ steady state

$$\vec{v} = \frac{-e\vec{E}}{\zeta - im\omega}$$

forced damped oscillator
(no internal restoring force)

$$J_F = n_e(-e)\vec{v} = \frac{n_e e^2 \vec{E}}{\zeta - im\omega}$$

$\zeta \sim \frac{1}{\tau} \leftarrow$ collision time

$$\sigma(\omega) = \frac{n_e e^2}{\zeta - im\omega} = \frac{\frac{n_e e^2}{\zeta}}{1 - \frac{im\omega}{\zeta}}$$

using Ohm's Law

$$\sigma_0 = \frac{n_e e^2}{\zeta}$$

$$\sigma(\omega) = \sigma_r(\omega) + i\sigma_i(\omega)$$

$$= \frac{\sigma_0}{1 + \left(\frac{\sigma_0 m\omega}{n_e e^2}\right)^2}\left(1 + i\frac{\sigma_0 m\omega}{n_e e^2}\right)$$

σ is ω-dependent and complex

Complexify $\epsilon \to \epsilon(\omega) = \epsilon_r(\omega) - \frac{i\sigma_i(\omega)}{\omega}$