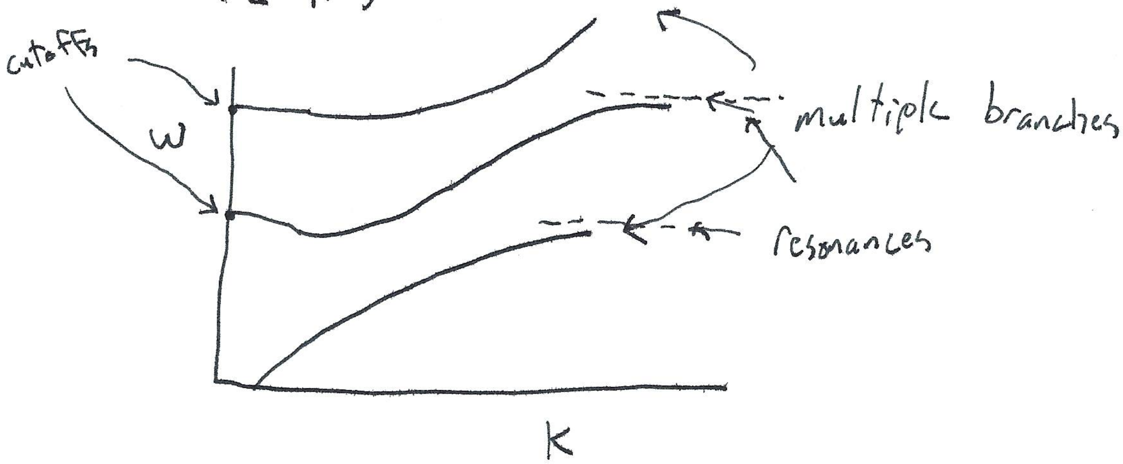
Phys 110B 25Sep20

Dispersion Relations $\omega = \omega(\vec{k})$

Not a function because it doesn't often come in this form.

$D(\omega, \vec{k}) = 0$



Cutoffs or resonances

↓ Finite non-zero ω at zero $|\vec{k}|$

↓ Finite ω as $|\vec{k}| \to \infty$ horizontal asymptotes

EM type waves

index of refraction $|\vec{v}_\phi| = v_\phi = \frac{c}{n}$

speed of light in vacuum

index of refraction

$n = \frac{c}{v_\phi} = \frac{c}{\frac{\omega}{|\vec{k}|}} = \frac{ck}{\omega}$, n as a function of ω or k

cutoff $\longleftrightarrow$ zero ~~of n~~ of n

resonance $\longleftrightarrow$ pole/divergence of n

Microscopic Maxwell's Equations

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \qquad \vec{\nabla}\times\vec{E} = -\partial_t\vec{B}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\partial_t\vec{E}$$

Functional decomposition into $\perp$, $\parallel$ components

$\perp$ = functionally transverse
divergence - free
solenoidal

$\parallel$ = functionally longitudinal
curl - free
irrotational

$$\vec{E}(\vec{x},t) = \vec{E}_\perp(\vec{x},t) + \vec{E}_\parallel(\vec{x},t)$$

where $\vec{\nabla}\cdot\vec{E}_\perp(\vec{x},t) = 0 \qquad \vec{\nabla}\times\vec{E}_\parallel(\vec{x},t) = 0$

Helmholtz-Hodge Decompositions

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{E}_\parallel = \frac{1}{\epsilon_0}\rho$$

$$\vec{B}_\parallel = \vec{0} \qquad \vec{0} = \mu_0\vec{J}_\parallel + \epsilon_0\mu_0\partial_t\vec{E}_\parallel$$

~~$0 = \mu_0 \Delta$~~

$$0 = \mu_0 \underbrace{\vec{\nabla}\cdot\vec{J}_\parallel}_{= \vec{\nabla}\cdot\vec{J} = -\partial_t\rho} + \epsilon_0\mu_0\partial_t \underbrace{\vec{\nabla}\cdot\vec{E}_\parallel}_{= \frac{1}{\epsilon_0}\rho} = \mu_0\epsilon_0(-\partial_t\rho + \partial_t\rho) = 0$$

$\vec{\nabla}\cdot\vec{E}_\perp = 0 \qquad \nabla\times\vec{E}_\perp = -\partial_t \vec{B}_\perp$

$\vec{\nabla}\cdot\vec{B}_\perp = 0 \qquad \vec{\nabla}\times\vec{B}_\perp = \mu_0 \vec{J}_\perp + \epsilon_0 \mu_0 \partial_t \vec{E}_\perp$

$\vec{E} = \vec{E}_\perp + \vec{E}_\parallel \qquad \vec{B} = \vec{B}_\perp + \vec{B}_\parallel \qquad \vec{J} = \vec{J}_\perp + \vec{J}_\parallel$

Fourier Decomposition

$\omega \ \vec{k}$

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int dk\, e^{ikx}\, \Psi(k)$$

$$\Psi(k) = \frac{1}{\sqrt{2\pi}} \int dx\, e^{-ikx}\, \psi(x)$$

$$\int |\psi(x)|^2 dx = \int dk\, |\Psi(k)|^2$$

$$\phi(t) = \frac{1}{\sqrt{2\pi}} \int d\omega\, e^{-i\omega t}\, \Phi(\omega)$$

$\frac{1}{\sqrt{2\pi}}$ is a normalization factor

$$\Phi(\omega) = \frac{1}{\sqrt{2\pi}} \int dt\, e^{i\omega t}\, \phi(t)$$

$$\vec{E}(\vec{x},t) \longleftrightarrow \vec{E}(\vec{k},\omega)$$

Fourier-space Maxwell Equations

$i\vec{k}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \qquad i\vec{k}\times\vec{E} = i\omega\vec{B}$

$i\vec{k}\cdot\vec{B} = 0 \qquad i\vec{k}\times\vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0(-i\omega)\vec{E}$

$\vec{\nabla}\cdot\vec{E}_\perp = 0 \longleftrightarrow i\vec{k}\cdot\vec{E}_\perp = 0$

Functionally transverse
not equivalent to
geometric transverse
in real space

$\vec{E}_\perp(\vec{k},\omega)$ is $\perp$ to $\vec{k}$
geometric transversality
in $\vec{k}$-space

$\vec{\nabla}\times\vec{E}_\parallel = \vec{0} \longleftrightarrow i\vec{k}\times\vec{E}_\parallel = \vec{0}$

Functionally longitudinal

$\vec{E}_\parallel(\vec{k},\omega)$ is $\parallel$ to $\vec{k}$

Free space (vacuum) Maxwell Equations

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \epsilon_0 \mu_0 \partial_t \vec{E}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\partial_t \vec{\nabla}\times\vec{B} = -\partial_t(\epsilon_0\mu_0\partial_t\vec{E})$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}$$

$$-\nabla^2\vec{E} = -\epsilon_0\mu_0\partial_t^2\vec{E}$$

$$(\nabla^2 - \epsilon_0\mu_0\partial_t^2)\vec{E}(\vec{x},t) = \vec{0}$$ "wave equation"

wave speed $c = \dfrac{1}{\sqrt{\epsilon_0\mu_0}}$

$$(\nabla^2 - \epsilon_0\mu_0\partial_t^2)\vec{B}(\vec{x},t) = \vec{0}$$

We need to make sure they both give solutions to the original equations

irreducible solutions: monochromatic plane waves

$$\vec{E} = \hat{e} E_0 e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad \vec{B} = \hat{b} B_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$k^2 = \vec{k}\cdot\vec{k} = \frac{\omega^2}{c^2}$$ vacuum dispersion relation

$$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow i\vec{k}\cdot(\hat{e}E_0) = 0 \Rightarrow \hat{e}\cdot\hat{k} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \hat{b}\cdot\hat{k} = 0$$

$$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B} \qquad \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\partial_t\vec{E}$$

$$i\vec{k}\times\vec{E} = i\omega\vec{B} \qquad i\vec{k}\times\vec{B} = \mu_0\epsilon_0(-i\omega)\vec{E}$$

$$B_0 = \frac{E_0}{c}$$

$$\hat{b} = \hat{k}\times\hat{e}$$

Covariantly $\Box A^\mu = 0$ Lorentz-Lorenz gauge

$\partial_\mu A^\mu = 0$

$$A^\mu(\underset{\sim}{X}) = C^\mu e^{ik^\alpha X_\alpha}$$

$\Box A^\mu = 0 \Rightarrow k^\alpha k_\alpha = 0$ dispersion relation

$\underset{\sim}{k}$ is lightlike

$\partial_\mu A^\mu = 0 \Rightarrow k^\mu C_\mu = 0$

transversality constraint

$$F^{\mu\nu} = i(C^\mu k^\nu - C^\nu k^\mu)\, e^{ik^\alpha X_\alpha}$$

covariant plane wave

Linear Homogeneous Isotropic Medium (LHI)

conductivity

$\vec{D} = \epsilon \vec{E}$ $\quad \vec{H} = \frac{1}{\mu}\vec{B}$ $\quad \vec{J}_f = \sigma \vec{E}$

constant

$\epsilon \vec{\nabla}\cdot\vec{E} = 0$ no free charge $\qquad \vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$

$\vec{\nabla}\cdot\vec{B} = 0$ $\qquad \vec{\nabla}\times\vec{B} = +\mu_0\sigma\,\partial_t\vec{E} + \mu_0\epsilon_0\partial_t^2\vec{E}$

conduction current $\qquad$ displacement current