Phys 110B 23Sep20

EM Waves (Chapter 9 Griffiths

$$\left(\nabla^2 - \frac{1}{v^2}\partial_t^2\right)\vec{\psi} = 0 \quad \text{" wave equation"}$$

wavevector $\vec{k}$

$$\vec{\psi} = \frac{1}{2}\left[\hat{\epsilon} A e^{i\vec{k}\cdot\vec{x} - i\omega t} + c.c.\right] = \mathrm{Re}\left(\hat{\epsilon} A e^{i\vec{k}\cdot\vec{x} - i\omega t}\right)$$

polarization amplitude frequency wavenumber $|\vec{k}|$

1D wave equation: $\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0 \quad \psi(x,t) = f(x-vt) + g(x+vt)$

$f(x), g(x)$ arbitrary functions

linearity $\qquad \psi_1, \psi_2$ are solution $\to \alpha\psi_1 + \beta\psi_2$ is also a solution

translation invariance → time / space / both

continuity

linear time translation invariant system

irreducible solutions $\longleftrightarrow$ behave as simply as possible under symmetry

$\psi(t)$

$\psi(t+\tau) = \alpha(\tau)\,\psi(t)$

assume $\psi(0) = 1$ (can re-scale later)

$\psi(t+\tau) = \alpha(\tau)\psi(t)$

$\psi(\tau) = \alpha(\tau)\psi(0) = \alpha(\tau) \Rightarrow \alpha(t) = \psi(t)$

$f(t) = \log\psi(t)$

$f(t+\tau) = f(t) + f(\tau) \qquad f(0) = 0$

$f(t+t) = f(t) + f(t) = 2f(t)$

$f(t+t+\cdots+t) =$

$f(nt) = n\,f(t)$

let $t = \frac{m}{n} t'$

$f(\frac{m}{n} t') = \frac{1}{n} f(n \frac{m}{n} t') = \frac{1}{n}(f\, m t') = \frac{m}{n} f(t')$

$f(\frac{m}{n} t') = \frac{m}{n} f(t')$

by continuity $f(\lambda t) = \lambda f(t)$ $\cancel{\forall \lambda, \forall t}$

$\forall \lambda \in \mathbb{R}$, or $\mathbb{C}$

$\forall t \in \mathbb{R}$, or $\mathbb{C}$

let $t = 1$

$f(\xi) = \xi f(1) \Rightarrow f$ is a linear function

$f(t) \propto t = -i\omega$

$\psi(t) \propto e^{-i\omega t}$

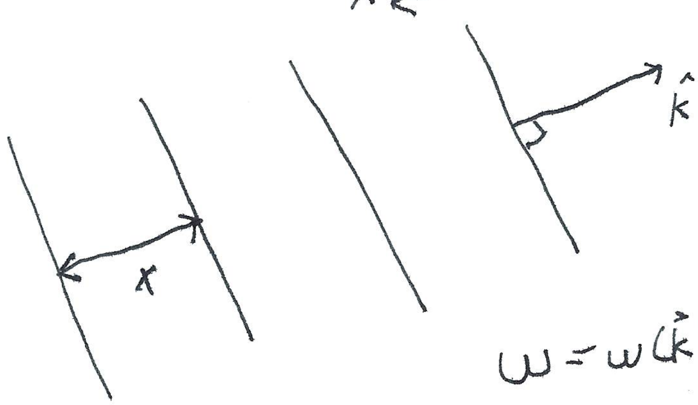
$e^{-i\omega t}$ irreducible solutions

$e^{i\vec{k}\cdot\vec{x} - i\omega t}$

ω = angular frequency = $\frac{2\pi}{T}$ ← period   wiggles in time

$\vec{k}$ = wave vector $k = |\vec{k}| = \frac{2\pi}{\lambda}$ ← wavelength   wiggles in space

$e^{i\vec{k}\cdot\vec{x} - i\omega t}$



$\omega = \omega(\hat{k})$ dispersion relation

Phase velocity of wave

apparent velocity of the motion of the phase fronts

$\theta = \vec{k} \cdot \vec{x} - \omega t$

phase speed $\qquad v_\phi = \dfrac{\omega}{|\vec{k}|}$

phase velocity $\qquad \vec{v}_\phi = \dfrac{\omega}{|\vec{k}|} \hat{k}$ $\qquad$ No limit on phase velocity

Group velocity $\quad$ wave packet velocity

Group velocity $\vec{v}_g = \dfrac{d\omega}{d\vec{k}}$ $\qquad \omega = \omega(\vec{k})$

$\omega = c|\vec{k}|$ $\quad v_\phi = c$

$\qquad\qquad V_g = c$

$\begin{bmatrix} \omega/c \\ \vec{k} \end{bmatrix}$ 4-vector $\qquad \omega = \omega(\vec{k})$

may be frame dependent

sound wave

e.g. $\omega = v|\vec{k}|$ $\quad$ speed of sound (in mass at rest) phase velocity $\vec{v}_\phi = \frac{\omega}{|\vec{k}|^2}\vec{k}$ no reason for it to be a 4-vector

group velocity

$$\vec{v}_g = \frac{d\omega}{d\vec{k}} \quad \text{Looks backwards} \quad \vec{v} = \frac{\delta \vec{x}}{\delta t}$$

Claim: $\underset{\sim}{G} = \gamma \begin{bmatrix} c \\ \vec{v}_g \end{bmatrix}$ is a 4-vector

$\underset{\sim}{K} = \begin{bmatrix} \omega/c \\ \vec{k} \end{bmatrix}$ is a 4-vector $\quad \delta \underset{\sim}{K} = \begin{bmatrix} \delta\omega/c \\ \delta\vec{k} \end{bmatrix}$ is a 4-vector by linearity

$$\begin{bmatrix} \delta\omega/c \\ \delta\vec{k} \end{bmatrix} = \begin{bmatrix} \frac{1}{c}\frac{\partial\omega}{\partial\vec{k}} \cdot \delta\vec{k} \\ \delta\vec{k} \end{bmatrix} = \begin{bmatrix} \vec{v}_g/c \cdot \delta\vec{k} \\ \delta\vec{k} \end{bmatrix}$$

wavepacket

$$\underset{\sim}{G} \cdot \delta\underset{\sim}{K} = \gamma \begin{bmatrix} c \\ \vec{v}_g \end{bmatrix} \cdot \begin{bmatrix} \vec{v}_g/c \cdot \delta\vec{k} \\ \delta\vec{k} \end{bmatrix} = \gamma\left[-\vec{v}_g \cdot \delta\vec{k} + \vec{v}_g \cdot \delta\vec{k}\right] = 0$$

$\delta\vec{k}$ are arbitrary

the only way for this to be true is for $\underset{\sim}{G}$ to be a 4-vector

$\vec{v}_g$ does transform as a 3-velocity under Lorentz transformation,

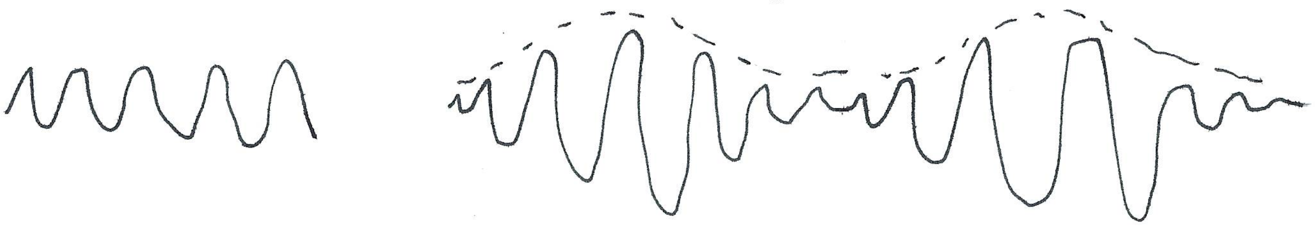
$\underset{\sim}{G}$ transforms as a 4-vector

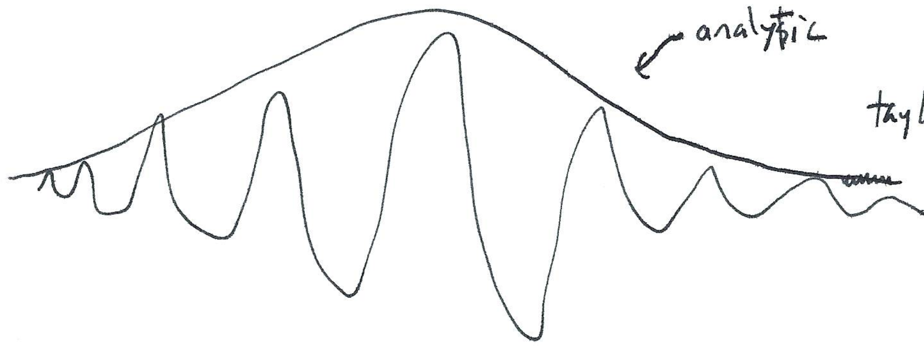
in quantum mechanics $\quad$ group velocity $\longleftrightarrow$ particle velocity

usually $|\vec{v}_g| < c$

usually $\vec{v}_g$ represents the velocity of (i) information transport
(ii) energy transport

In some "active" media (like lasers) $\vec{v}_g$ can be faster than c





taylor expand
can deduce
the
rest

discontinuity travels $\leq c$

information/energy transport can't exceed c

e.g.: $\boxed{v_g v_\phi = c^2}$

free-space Maxwell's equations

look for irreducible solutions

look at insulating media