Phys 110B 21 Sep 20

$$\partial_\nu G^{\mu\nu} = 0 \qquad \partial_\nu F^{\mu\nu} = \mu_0 J^\mu \qquad G^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad \partial_\mu J^\mu = 0 \qquad \Box A^\mu = -\mu_0 J^\mu \qquad \Box = \partial_\mu \partial^\mu$$

$$F^{\mu\nu}F_{\mu\nu} \qquad F^{\mu\nu}G_{\mu\nu} \qquad G^{\mu\nu}G_{\mu\nu}$$

Field invariants: $\vec{E}\cdot\vec{B}$ $\quad \vec{E}(\vec{x},t)\cdot\vec{B}(\vec{x},t)$

$$|\vec{E}(\vec{x},t)|^2 - c^2|\vec{B}(\vec{x},t)|^2$$

constrains how $\vec{E}$ and $\vec{B}$ can transform

Purcell ~~Purcell~~ Electricity + relativity = Magnetism (Not entirely true)

only have $\vec{E}$ fields $\quad \vec{E}\cdot\vec{B} = 0$

$$|\vec{E}|^2 - c|\vec{B}|^2 \geq 0$$

if $\{ \; \vec{E}\cdot\vec{B} \neq 0 \quad |\vec{E}|^2 - c^2|\vec{B}|^2 < 0 \}$ are true

$$J^\mu J_\mu = J^{\mu\prime} J^\prime_\mu = -(c\rho')(c\rho') + \underbrace{(\rho'\vec{v}')\cdot(\rho'\vec{v}')}_{\to 0} \propto (\rho')^2$$

rest frame for charge

$\frac{1}{c}\sqrt{-J^\mu J_\mu}$ is proper charge density

Conservation of charge $\partial_\mu J^\mu = 0$

Invariance of electric charge: $q$ is a Lorentz scalar

$\rho\, d^3\vec{x} = dq$ in frame $S$

$d^3x = \frac{d^3\vec{x}'}{\gamma}$ in Frame $S'$ rest frame

$$\rho = \gamma\rho' \qquad \rho\, d^3\vec{x} = (\gamma\rho')\left(\frac{1}{\gamma}d^3\vec{x}'\right) = \rho'\, d^3\vec{x}' = \rho'\, d^3\vec{x}'$$

Lorentz - Force Law

$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ consistent with relativity but not manifestly so.

$\vec{F} \cdot \vec{v} = q \vec{E} \cdot \vec{v}$

$$\frac{d}{d\tau} p^\mu = m \frac{d}{d\tau} u^\mu = \boxed{K^\mu = q F^{\mu\nu} u_\nu}$$

covariant Formulation of Lorentz Force

$J^\mu \quad A^\mu$

can → canonical

$$\begin{bmatrix} \mathcal{E}/c \\ \vec{p} \end{bmatrix} + \begin{bmatrix} q\Phi/c \\ q\vec{A} \end{bmatrix} = \begin{bmatrix} \mathcal{E}_{can}/c \\ \vec{p}_{can} \end{bmatrix} = \begin{bmatrix} H/c \\ \vec{P}_{can} \end{bmatrix}$$

$\mathcal{E} = \gamma m c^2$ = rest energy + relativistic kinetic energy but not potential

$\vec{A} \qquad \vec{A}'$ 4-vector

$\vec{E}, \vec{B} \longrightarrow \vec{E}', \vec{B}'$

$F^{\mu\nu} \qquad F'^{\mu\nu}$ rank-2 tensor

$$\Rightarrow \vec{E}'/c = \gamma(\vec{E}/c + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{E}/c)$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}/c) - \frac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{B})$$

$\underset{\sim}{A}(\underset{\sim}{X})$ $\quad A^\mu$ is a 4-vector

means

$\underset{\sim}{A}'(?) = \Lambda \underset{\sim}{A}(?)$

$\underset{\sim}{A}'(\underset{\sim}{X}') = \Lambda \underset{\sim}{A}(\underset{\sim}{X})$ but how $\underset{\sim}{X}$ related to $\underset{\sim}{X}'$?

$= \Lambda \underset{\sim}{A}(\Lambda^{-1} \underset{\sim}{X}')$

# generalizing Ohm's Law Relativistically

in rest frame of a conductor $\vec{J}(\vec{x},t) = \sigma \vec{E}(\vec{x},t)$

starting point

some number/object
could be number/matrix

Find covariant generalization of that = ?

Linearity higher-order terms $\vec{J}, \vec{E}, \vec{B}$ are ruled out

<u>Only</u> 4-vectors $\cancel{J^\mu}$ $J^\alpha$, $u^\alpha$, $F^{\alpha\beta}u_\beta$, $\cancel{F^{\alpha\beta}}$ $F^{\alpha\beta}J_\alpha$

of medium

Must be linear combination of these 4-vectors with scalar coefficients

$F^{\alpha\beta}J_{\alpha} \to \vec{E}\rho$ in rest frame $\times$ doesn't occur

$$J^\alpha = K F^{\alpha\beta} u_\beta + M u^\alpha$$

unknown scalars

$1, c, \sigma$ $\quad u_\beta J^\beta$ $\quad c^2 J_\alpha u^\alpha$, $J_\alpha F^{\alpha\beta} u_\beta$

Lead to terms that don't exist

dimensional analysis $\Rightarrow K = \frac{\sigma}{c} K'$ $\quad M = \frac{1}{c^2} u_\beta J^\beta M'$

dimensionless $\quad$ dimensionless

agreement in rest frame $\to K' = 1$, $M' = 1$

$$\boxed{J^\alpha = \frac{\sigma}{c} F^{\alpha\beta} u_\beta + \frac{1}{c^2} u_\beta J^\beta u^\alpha}$$

avoids overcounting

bulk motion of conductor

in a frame moving at $\vec{v} = \vec{\beta} c$

$$\vec{J} = \gamma\sigma\left(\vec{E} + \vec{v}\times\vec{B} - (\vec{\beta}\cdot\vec{E})\vec{\beta}\right) + \rho\vec{v}$$

replace field with lorentz force