Phys 110B 18 Sep 20

4-velocity $\underset{\sim}{U} = \begin{bmatrix} c\gamma \\ \vec{v}\gamma \end{bmatrix}$

$\underset{\sim}{U} \cdot \underset{\sim}{U} = U^{\mu} U_{\mu} = -c^2$

Some books define $\underset{\sim}{U}$ to be normalized 4-velocity $\begin{bmatrix} \gamma \\ \gamma\vec{\beta} \end{bmatrix}$

$\underset{\sim}{U} \cdot \underset{\sim}{U} = -1$

massless particles? How do we handle them?

Newtonian: $\frac{1}{2}mv^2 \to 0$ $\quad m\vec{v} \to \vec{0}$

$\mathcal{E} = \overset{\infty}{\overset{\uparrow}{\gamma}} \cancel{m}c^2$ $\quad \gamma m \to$ finite

$\vec{p} = \gamma\vec{\beta} mc$

$\mathcal{E}^2 = c^2|\vec{p}|^2 + m^2c^4$

$\vec{\beta} = \frac{c\vec{p}}{E}$

$m \to 0$

$E^2 = c^2|\vec{p}|^2 \Rightarrow E = c|\vec{p}|$

$|\vec{\beta}| = \frac{|c\vec{p}|}{E} = 1$

No relationship between $|\vec{\beta}|$ and $|\vec{v}|$ since $|\vec{v}| = c$

$\vec{p} = \hbar\vec{k}$

$\mathcal{E} = \hbar\omega$

$\frac{\mathcal{E}}{|\vec{p}|} = \frac{\omega}{|\vec{k}|} = c$

Electromagnetic Potentials $\vec{A}$ Φ

$$\left.\begin{aligned} \vec{B} &= \vec{\nabla}\times\vec{A} \\ \vec{E} &= -\vec{\nabla}\Phi - \frac{\partial}{\partial t}\vec{A} \end{aligned}\right\} \Rightarrow \begin{aligned} \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial}{\partial t}\vec{B} \end{aligned}$$

$$\vec{\nabla}\cdot\left(-\vec{\nabla}\Phi - \frac{\partial}{\partial t}\vec{A}\right) = \frac{1}{\epsilon_0}\rho$$ Gauss Law

$$\boxed{\nabla^2\Phi + \partial_t \vec{\nabla}\cdot\vec{A} = -\frac{1}{\epsilon_0}\rho}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}\Phi - \frac{\partial}{\partial t}\vec{A}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$ and rearrange $c = \sqrt{\epsilon_0\mu_0}$

$$\boxed{\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\vec{A} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\partial_t\Phi\right) = -\mu_0\vec{J}}$$

redundancy of description → gauge invariance
gauge freedom

$$\vec{A} + \vec{\nabla}\chi = \vec{A}'$$

$$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{\nabla}\chi = \vec{\nabla}\times\vec{A}$$

$$-\vec{\nabla}\Phi' - \partial_t(\vec{A} + \vec{\nabla}\chi) = -\vec{\nabla}\Phi - \partial_t\vec{A}$$

$$\vec{\nabla}(\Phi' - \Phi) = -\vec{\nabla}\partial_t\chi$$

$$\Phi' = \Phi - \partial_t\chi$$

$$\left.\begin{aligned}\vec{A} &\to \vec{A} + \vec{\nabla}\chi \\ \Phi &\to \Phi - \partial_t\chi\end{aligned}\right\}$$ $\vec{E}, \vec{B}$ are unchanged

χ the generator of the gauge transformation

"fixing the gauge" $\chi = \chi(\vec{x}, t)$

Could choose $\Phi(\vec{x}, t) = 0$ temporal gauge / Hamiltonian Gauge / Radiation gauge / Weyl gauge

$$A(\vec{x}, t) = 0$$

Coulomb Gauge: $\vec{\nabla}\cdot\vec{A} = 0$ $\vec{A} = \vec{A}_\perp$ functional transversality ie divergence-free

Lorenz - Lorentz gauge: $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\partial_t \dot{\Phi} = 0$

Multipolar $\vec{x}\cdot\vec{A} = 0$

Foch - Schwinger

Dirac gauge

Feynman gauge

In L-L gauge:

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\vec{A} = -\mu_0\vec{J} = \Box\vec{A} \qquad c = \sqrt{\epsilon_0\mu_0}$$

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\Phi = -\frac{1}{\epsilon_0}\rho = \Box\Phi$$

$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right) =$ "wave operator" = d'Alembertian

$$= -\left(\frac{\partial}{\partial ct}\right)^2 + \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 = \partial_\mu\partial^\mu = \partial^\mu\partial_\mu$$

invariant scalar operator

$$\partial^\mu\partial_\mu = \partial'^\mu\partial'_\mu$$

$$\nabla^2 - \frac{1}{c^2}\partial_t^2 = \nabla'^2 - \frac{1}{c^2}\partial_t'^2$$

$\partial_\mu\partial^\mu = \Box$ "box" operator

griffiths instead uses $\Box^2$

$$\Box A^\mu = -\mu_0 J^\mu \qquad \underset{\sim}{J} = \begin{bmatrix} c\rho \\ \vec{J} \end{bmatrix} \quad \underset{\sim}{A} = \begin{bmatrix} \Phi/c \\ \vec{A} \end{bmatrix}$$

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\partial_t\vec{A} = 0$$

$\boxed{\partial_\mu A^\mu = 0}$ Lorenz-Lorentz gauge

↑ does not uniquely fix a gauge if $\partial_\mu A^\mu = 0$ then $A'^\mu = A^\mu + \partial^\mu\chi$

then $\partial_\mu A'^\mu = 0$ provided $\Box\chi = \partial_\mu\partial^\mu\chi = 0$

Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$

Lorenz-Lorentz gauge: $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\partial_t \Phi = 0$

Velocity gauges: $\vec{\nabla}\cdot\vec{A} + \alpha \frac{1}{c}\partial_t \Phi = 0$

Field Strength Tensor / EM tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & E^x/c & E^y/c & E^z/c \\ -E^x/c & 0 & B^z & -B^y \\ -E^y/c & -B^z & 0 & B^x \\ -E^z/c & B^y & -B^x & 0 \end{bmatrix}$$

Griffiths Convention $(-+++)$ signature

rank 2 tensor

"dual" Field strength tensor $G^{\mu\nu}$, $\mathcal{F}^{\mu\nu}$

$\vec{E}/c \to \vec{B}$ $\quad \vec{B} \to -\vec{E}/c$

$$G^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$$

completely antisymmetric tensor (Levi-Civita symbol)

$\epsilon = 1$ if $\alpha\beta\gamma\delta$ is an even permutation

$\epsilon = -1$ if odd permutation

$\epsilon = 0$ otherwise

$$G^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{bmatrix}$$

Maxwell's Equations:

$$\partial_\nu G^{\mu\nu} = 0$$

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$$

$$J^\mu J_\mu \quad F^{\mu\nu}F_{\mu\nu} \quad G^{\mu\nu}G_{\mu\nu} \quad F^{\mu\nu}G_{\mu\nu}, \ldots$$