Griffiths Electrodynamics Chapter 1 Summary: Vector Analysis

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Contents

Vector Algebra

Vector addition is **commutative**, and **associative**. Each vector also has an **additive inverse**.

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$$

$$(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$$

$$\boldsymbol{A} + (-\boldsymbol{A}) = \boldsymbol{0}$$

Scalar multiplication is **distributive**.

$$a(\boldsymbol{A} + \boldsymbol{B}) = a\boldsymbol{A} + a\boldsymbol{B}$$

The **dot product** of two vectors is

$$\boldsymbol{A} \cdot \boldsymbol{B} = AB\cos(\theta)$$

with θ being the angle between each vector. The dot product is **commutative**, and **distributive**.

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$$

$$\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$$

The **cross product** of two vectors is

$$\boldsymbol{A} \times \boldsymbol{B} = AB\sin(\theta)\hat{\boldsymbol{n}}$$

with $\hat{\boldsymbol{n}}$ pointing normal to both vectors with the sign following the right-hand rule. The cross product is **distributive**, and **anti-commutative**

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) + (\boldsymbol{A} \times \boldsymbol{C})$$

$$(\boldsymbol{A} \times \boldsymbol{B}) = -(\boldsymbol{B} \times \boldsymbol{A})$$

Vectors can be broken up into their components. Using Cartesian coordinates, a vector $\boldsymbol{A}$ will be

$$\boldsymbol{A} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}$$

Adding vectors is as simple as adding components. For a dot product, you multiply identical components.

$$\alpha\boldsymbol{A} + \beta\boldsymbol{B} = (\alpha A_x + \beta B_x)\hat{\boldsymbol{x}} + (\alpha A_y + \beta B_y)\hat{\boldsymbol{y}} + (\alpha A_z + \beta B_z)\hat{\boldsymbol{z}}$$

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$$

The cross product is a little more complicated, but can be found taking a determinant of a special matrix.

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - B_y A_z)\hat{\boldsymbol{x}} + (A_z B_x - A_x B_z)\hat{\boldsymbol{y}} + (A_x B_y - A_y B_x)\hat{\boldsymbol{z}}$$

The **scalar triple product** is $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$. The dot and cross product can be interchanged.

$$\boldsymbol{A}\dot{(}\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C}$$

The **vector triple product** is $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$ and is simplified with the **BAC-CAB rule**.

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$$

It is important to recognize that cross-products are not associative.

The **position vector $\boldsymbol{r}$** in Cartesian coordinates is

$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

with a magnitude

$$r = \sqrt{x^2 + y^2 + z^2}$$

We can then define a **displacement vector** $d\boldsymbol{l}$ as

$$d\boldsymbol{l} = dx\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}} + dz\hat{\boldsymbol{z}}$$

We want to define one more vector, the **separation vector $\boldsymbol{\mathcal{R}}$** which is

$$\boldsymbol{\mathcal{R}} = \boldsymbol{r} - \boldsymbol{r}'$$

with $\boldsymbol{r}'$ being the source point, where the electric charge is located, and $\boldsymbol{r}$ being the field point, where you are calculating the electric or magnetic field.

Example: Dot Product

Let $\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$. The dot product of $\boldsymbol{C}$ with itself is

$$\boldsymbol{C} \cdot \boldsymbol{C} = (\boldsymbol{A} - \boldsymbol{B}) \cdot (\boldsymbol{A} - \boldsymbol{B}) = \boldsymbol{A} \cdot \boldsymbol{A} - \boldsymbol{A} \cdot \boldsymbol{B} - \boldsymbol{B} \cdot \boldsymbol{A} + \boldsymbol{B} \cdot \boldsymbol{B}$$

$$C^2 = A^2 + B^2 - 2AB\cos(\theta)$$

Example: Cube Angles

We want to find the angle between the face diagonals of a cube. Using the unit cube will make it easiest, so our two angles are

$$\boldsymbol{A} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{z}}$$

$$\boldsymbol{B} = \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}$$

We can then take the dot product to find what we need.

$$\boldsymbol{A} \cdot \boldsymbol{B} = 1 = 2\cos(\theta)$$

So

$$\cos(\theta) = \frac{1}{2}$$

or

$$\theta = \frac{\pi}{3}$$

Differential Calculus

The differential of a scalar function T can be represented as

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz$$

Using dot products and gradients, this can be written as

$$dT = (\boldsymbol{\nabla} T) \cdot d\boldsymbol{l}$$

There are three different types of differential operators for vector calculus: the **gradient**, the **divergence**, and the **curl**. These can be represented by the following representations in Cartesian, spherical, and cylindrical coordinates, respectively.

The Gradient:

$$\boldsymbol{\nabla} = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}$$

$$= \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin(\theta)}\frac{\partial}{\partial \phi}$$

$$= \hat{\boldsymbol{s}}\frac{\partial}{\partial s} + \frac{\hat{\boldsymbol{\phi}}}{s}\frac{\partial}{\partial \phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}$$

The Divergence:

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}(v_\theta \sin(\theta)) + \frac{1}{r\sin(\theta)}\frac{\partial v_\phi}{\partial \phi}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

The Curl:

$$\boldsymbol{\nabla} \times \boldsymbol{v} = \Big(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\Big)\hat{\boldsymbol{x}} + \Big(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\Big)\hat{\boldsymbol{y}} + \Big(\frac{v_y}{\partial x} - \frac{\partial v_x}{\partial y}\Big)\hat{\boldsymbol{z}}$$

$$= \frac{1}{r\sin(\theta)}\Big(\frac{\partial}{\partial \theta}(v_\phi \sin(\theta)) - \frac{\partial v_\theta}{\partial \phi}\Big)\hat{\boldsymbol{r}} + \frac{1}{r}\Big(\frac{1}{\sin(\theta)}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\Big)\hat{\boldsymbol{\theta}} + \frac{1}{r}\Big(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\Big)\hat{\boldsymbol{\phi}}$$

$$= \Big(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\Big)\hat{\boldsymbol{s}} + \Big(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\Big)\hat{\boldsymbol{\phi}} + \frac{1}{s}\Big(\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\Big)\hat{\boldsymbol{z}}$$

Another important differential operator is the **Laplacian** ∇^2.

The Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial 62}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\Big(\sin(\theta)\frac{\partial}{\partial \theta}\Big) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial \phi^2}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}\Big(s\frac{\partial}{\partial s}\Big) + \frac{1}{s^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

The product rules are:

$$\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$$

$$\boldsymbol{\nabla}(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) + (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A}$$

$$\boldsymbol{\nabla}\cdot(f\boldsymbol{A}) = f(\boldsymbol{\nabla}\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot(\boldsymbol{\nabla} f)$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})$$

$$\boldsymbol{\nabla}\times(f\boldsymbol{A}) = f(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\times(\boldsymbol{\nabla} f)$$

$$\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{A}(\boldsymbol{\nabla}\cdot\boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{A})$$

Integral Calculus

Line Integrals

A line integral can be expressed as

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{v} \cdot d\boldsymbol{l}$$

where $\boldsymbol{v}$ is some vector function and $d\boldsymbol{l}$ is an infinitesimal displacement vector that points in the direction of the path. There is a special notation for a line integral over a closed path which is

$$\oint \boldsymbol{v} \cdot d\boldsymbol{l}$$

Example: Line Integral Calculation

We want to calculate the line integral with the vector function $\boldsymbol{v} = y^2\hat{\boldsymbol{x}} + 2x(y+1)\hat{\boldsymbol{y}}$ from $\boldsymbol{a} = (1,1,0)$ to the point $\boldsymbol{b} = (2,2,0)$ along two separate paths.

For path (1), the horizontal portion will have a displacement vector of $d\boldsymbol{l} = dx\hat{\boldsymbol{x}}$. The y-value is also unchanged at $y = 1$. The vertical portion will have a displacement vector of $d\boldsymbol{l} = dy\hat{\boldsymbol{y}}$. The x-value will be unchanged at $x = 2$. We can solve the line integral by adding up the line integrals for each segment.

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{v} \cdot d\boldsymbol{l} = \int_1^2 y^2 dx + \int_1^2 2x(y+1)dy = \int_1^2 dx + \int_1^2 4(y+1)dy = 11$$

For path (2), $x = y$, $dx = dy$, so the displacement vector is $d\boldsymbol{l} = dx\hat{\boldsymbol{x}} + dx\hat{\boldsymbol{y}}$. This will give us the integrand $\boldsymbol{v} \cdot d\boldsymbol{l} = (3x^2 + 2x)dx$. Integrating this gives us

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{v} \cdot d\boldsymbol{l} = \int_1^2 (3x^2 + 2x)dx = 10$$

Since these values are not the same, the integral is not path independent and $\boldsymbol{v}$ is not conservative.

Surface Integrals

A surface integral is of the form

$$\int_{\mathcal{S}} \boldsymbol{v} \cdot d\boldsymbol{a}$$

and has the closed form

$$\oint_{\mathcal{S}} \boldsymbol{v} \cdot d\boldsymbol{a}$$

The $\boldsymbol{v}$ is some vector function, and the integral is over some specified surface $\mathcal{S}$.

Example: Surface Integral Calculation

We are going to calculate the surface integral of the vector function $\boldsymbol{v} = 2xz\hat{\boldsymbol{x}} + (x+2)\hat{\boldsymbol{y}} + y(z^2-3)\hat{\boldsymbol{z}}$ over 5 sides of a cube with side length 2.

For side (i) we have $x = 2$, $d\boldsymbol{a} = dydz\hat{\boldsymbol{x}}$, $\boldsymbol{v} \cdot d\boldsymbol{a} = 2xzdydz = 4zdydz$.

$$\int \boldsymbol{v} \cdot d\boldsymbol{a} = 4\int_0^2 \int_0^2 zdydz = 16$$

For side (ii) we have $x = 0$, and $d\boldsymbol{a} = -dydz\hat{\boldsymbol{x}}$ so this integral will be zero.

For side (iii) we have $y = 2$, $d\boldsymbol{a} = dxdz\hat{\boldsymbol{y}}$, $\boldsymbol{v} \cdot d\boldsymbol{a} = (x+2)dxdz$ so

$$\int \boldsymbol{v} \cdot d\boldsymbol{a} = \int_0^2 \int_0^2 (x+2)dxdz = 12$$

For side (iv) we have $y = 0$, $-dxdz\hat{\boldsymbol{y}}$, $\boldsymbol{v} \cdot d\boldsymbol{a} = -(x+2)dxdz$ so

$$\int \boldsymbol{v} \cdot d\boldsymbol{a} = -\int_0^2 \int_0^2 (x+2)dxdz = -12$$

For side (v) we have $z = 2$, $d\boldsymbol{a} = dxdy\hat{\boldsymbol{z}}$, $\boldsymbol{v} \cdot d\boldsymbol{a} = y(z^2-3)dxdy = ydxdy$ so

$$\int \boldsymbol{v} \cdot d\boldsymbol{a} = \int_0^2 \int_0^2 ydxdy = 4$$

Adding these all together gives us

$$\int_{\mathcal{S}} \boldsymbol{v} \cdot d\boldsymbol{a} = 20$$

Volume Integrals

Volume integrals will be of the form

$$\int_{\mathcal{V}} Td\tau$$

or

$$\int_{\mathcal{V}} \boldsymbol{v}d\tau$$

with T being some scalar function.

Example: Volume Integral Calculation

We are going to evaluate the integral of scalar function $T = xyz^2$ over the prism.

$$\int Td\tau = \int_0^3 \int_0^1 \int_0^{1-y} xyz^2dxdydz = \frac{3}{8}$$

The Fundamental Theorem of Calculus

In one dimension, this theorem looks like

$$\int_a^b \left(\frac{df}{dx}\right)dx = f(b) - f(a)$$

This can also be applied in multiple dimensions for gradients.

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} (\boldsymbol{\nabla} T) \cdot d\boldsymbol{l} = T(\boldsymbol{b}) - T(\boldsymbol{a})$$

This tells us that gradients have line integrals that are path independent.

Example: The Fundamental Theorem for Gradients Calculation

We will apply the fundamental theorem of calculus for gradients to a scalar field $T = xy^2$ from $\boldsymbol{a} = \boldsymbol{0}$ to $\boldsymbol{b} = (2, 1, 0)$ for two separate paths.

For path (1), we can split it up into two parts. For part (i), we have $y = 0$, $d\boldsymbol{l} = dx\hat{\boldsymbol{x}}$, $\boldsymbol{\nabla} \cdot d\boldsymbol{l} = y^2 dx = 0$ so

$$\int_{(i)} \boldsymbol{\nabla} T \cdot d\boldsymbol{l} = 0$$

For part (ii), we have $x = 2$, $d\boldsymbol{l} = dy\hat{\boldsymbol{y}}$, $\boldsymbol{\nabla} \cdot d\boldsymbol{l} = 2xydy = 4ydy$ so

$$\int_{(ii)} \boldsymbol{\nabla} T \cdot d\boldsymbol{l} = \int_0^1 4ydy = 2$$

Adding these two parts together gives us

$$\int_{(1)} \boldsymbol{\nabla} T \cdot d\boldsymbol{l} = 2$$

For path 2, we have $y = \frac{1}{2}x$, $dy = \frac{1}{2}dx$, $\boldsymbol{\nabla} T \cdot d\boldsymbol{l} = y^2 dx + 2xydy = \frac{3}{4}x^2 dx$ so

$$\int_{(2)} \boldsymbol{\nabla} T \cdot d\boldsymbol{l} = \int_0^2 \frac{3}{4}x^2 dx = 2$$

The Fundamental Theorem for Divergences

$$\int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \boldsymbol{v}) d\tau = \oint_{\mathcal{S}} \boldsymbol{v} \cdot d\boldsymbol{a}$$

This can also be called Gauss's Theorem, Green's theorem, or the divergence theorem.

Divergence Theorem Calculation

We will check the divergence theorem using $\boldsymbol{v} = y^2\hat{\boldsymbol{x}} + (2xy + z^2)\hat{\boldsymbol{y}} + (2yz)\hat{\boldsymbol{z}}$ on a unit cube. We can start off with the divergence integral. The divergence of our vector function is

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 2(x + y)$$

Evaluating our integral gives us

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{v} d\tau = \int_{\mathcal{V}} 2(x+y) d\tau = 2\int_0^1 \int_0^1 \int_0^1 (x+y) dxdydz = 2$$

We now need to look at the surface integral. For side (i) we have

$$\int \boldsymbol{v} \cdot d\boldsymbol{a} = \int_0^1 \int_0^1 y^2 dydz = \frac{1}{3}$$

Side (ii):

$$\int \boldsymbol{v} \cdot d\boldsymbol{a} = -\int_0^1 \int_0^1 y^2 dydz = -\frac{1}{3}$$

Side (iii):

$$\int \boldsymbol{v} \cdot d\boldsymbol{a} = \int_0^1 \int_0^1 (2x + z^2) dxdz = \frac{4}{3}$$

Side (iv):

$$\int \boldsymbol{v} \cdot d\boldsymbol{a} = -\int_0^1 \int_0^1 z^2 dxdz = -\frac{1}{3}$$

Side (v):

$$\int \boldsymbol{v} \cdot d\boldsymbol{a} = \int_0^1 \int_0^1 2y dxdy = 1$$

Side (vi):

$$\int \boldsymbol{v} \cdot d\boldsymbol{a} = -\int_0^1 \int_0^1 0 dxdy = 0$$

So

$$\oint_{\mathcal{S}} \boldsymbol{v} \cdot d\boldsymbol{a} = 2$$

The Fundamental Theorem for Curls

The fundamental theorem for curls is often called Stokes' Theorem.

$$\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot d\boldsymbol{a} = \oint_{\mathcal{P}} \boldsymbol{v} \cdot d\boldsymbol{l}$$

Example: Stokes' Theorem Calculation

We are going to check Stokes' Theorem with the vector function $\boldsymbol{v} = (2xz + 3y^2)\hat{\boldsymbol{y}} + (4yz^2)\hat{\boldsymbol{z}}$ on a unit square. We have

$$\boldsymbol{\nabla} \times \boldsymbol{v} = (4z^2 - 2x)\hat{\boldsymbol{x}} + 2z\hat{\boldsymbol{z}}$$

and

$$d\boldsymbol{a} = dydz\hat{\boldsymbol{x}}$$

The curl integral will then be

$$\int (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot d\boldsymbol{a} = \int_0^1 \int_0^1 4z^2 dydz = \frac{4}{3}$$

We can break the line integral into four segments and will find for side (i):

$$x, z = (0,0),\ \boldsymbol{v} \cdot d\boldsymbol{l} = 3y^2 dy$$

$$\int \boldsymbol{v} \cdot d\boldsymbol{l} = \int_0^1 3y^2 dy = 1$$

side (ii):

$$x, y = (0,1),\ \boldsymbol{v} \cdot d\boldsymbol{l} = 4z^2 dz$$

$$\int \boldsymbol{v} \cdot d\boldsymbol{l} = \int_0^1 4z^2 dz = \frac{4}{3}$$

side (iii):

$$x, z = (0,1),\ \boldsymbol{v} \cdot d\boldsymbol{l} = 3y^2 dy$$

$$\int \boldsymbol{v} \cdot d\boldsymbol{l} = \int_1^0 3y^2 dy = -1$$

side (iv):

$$x, y = (0,0),\ \boldsymbol{v} \cdot d\boldsymbol{l} = 0$$

$$\int \boldsymbol{v} \cdot d\boldsymbol{l} = 0$$

The sum of all of these integrals will then be

$$\oint \boldsymbol{v} \cdot d\boldsymbol{l} = \frac{4}{3}$$

Integration by Parts

This is one of the most powerful tools in a physicists arsenal. It is basically the product rule but backwards.The product rule is

$$\frac{d}{dx}(fg) = f\Big(\frac{dg}{dx}\Big) + g\Big(\frac{df}{dx}\Big)$$

We can integrate both sides, and use the fundamental theorem to get

$$\int_a^b \frac{d}{dx}(fg)dx = fg\Big|_a^b = \int_a^b f\Big(\frac{dg}{dx}\Big)dx + \int_a^b g\Big(\frac{df}{dx}\Big)dx$$

$$\int_a^b f\Big(\frac{dg}{dx}\Big)dx = -\int_a^b g\Big(\frac{df}{dx}\Big)dx + fg\Big|_a^b$$

We can also exploit the product rule in more than one dimension with the vector calculus product rules. One of those rules is

$$\boldsymbol{\nabla} \cdot (f\boldsymbol{A}) = f(\boldsymbol{\nabla} \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot (\boldsymbol{\nabla} f)$$

We can integrate this over a volume and use divergence theorem to get

$$\int_{\mathcal{V}} f(\boldsymbol{\nabla} \cdot \boldsymbol{A})d\tau = -\int_{\mathcal{V}} \boldsymbol{A} \cdot (\boldsymbol{\nabla} f)d\tau + \oint_{\mathcal{S}} d\boldsymbol{A} \cdot d\boldsymbol{a}$$

Example: Integration by Parts

We are going to evaluate

$$\int_0^\infty xe^{-x}dx$$

We can express the exponent as a derivative

$$e^{-x} = \frac{d}{dx}(-e^{-x})$$

which gives us

$$\int_0^\infty xe^{-x}dx = \int_0^\infty e^{-x}dx - xe^{-x}\Big|_0^\infty = 1$$

Spherical Coordinates

$$x = \sin(\theta)\cos(\phi)$$
$$y = \sin(\theta)\sin(\phi)$$
$$z = \cos(\theta)$$
$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$
$$\phi = \arctan\left(\frac{y}{x}\right)$$
$$\hat{\boldsymbol{x}} = \sin(\theta)\cos(\phi)\hat{\boldsymbol{r}} + \cos(\theta)\cos(\phi)\hat{\boldsymbol{\theta}} - \sin(\phi)\hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{y}} = \sin(\theta)\sin(\phi)\hat{\boldsymbol{r}} + \cos(\theta)\sin(\phi)\hat{\boldsymbol{\theta}} + \cos(\phi)\hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{z}} = \cos(\theta)\hat{\boldsymbol{r}} - \sin(\theta)\hat{\boldsymbol{\theta}}$$
$$\hat{\boldsymbol{r}} = \sin(\theta)\cos(\phi)\hat{\boldsymbol{x}} + \sin(\theta)\sin(\phi)\hat{\boldsymbol{y}} + \cos(\theta)\hat{\boldsymbol{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos(\theta)\cos(\phi)\hat{\boldsymbol{x}} + \cos(\theta)\sin(\phi)\hat{\boldsymbol{y}} - \sin(\theta)\hat{\boldsymbol{z}}$$
$$\hat{\boldsymbol{\phi}} = -\sin(\phi)\hat{\boldsymbol{x}} + \cos(\phi)\hat{\boldsymbol{y}}$$
$$d\boldsymbol{l} = dr\hat{\boldsymbol{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin(\theta)d\phi\hat{\boldsymbol{\phi}}$$
$$d\tau = r^2\sin(\theta)drd\theta d\phi$$

Example: Spherical Volume

Let's find the volume of a generic sphere with radius R.

$$V = \int d\tau = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2\sin(\theta)drd\theta d\phi = \frac{4}{3}\pi R^3$$

Cylindrical Coordinates

$$x = s\cos(\phi)$$
$$y = s\sin(\phi)$$
$$z = z$$
$$s = \sqrt{x^2 + y^2}$$
$$\phi = \arctan\left(\frac{y}{x}\right)$$
$$z = z$$
$$\hat{\boldsymbol{x}} = \cos(\phi)\hat{\boldsymbol{s}} - \sin(\phi)\hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{y}} = \sin(\phi)\hat{\boldsymbol{s}} + \cos(\phi)\hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$$
$$\hat{\boldsymbol{s}} = \cos(\phi)\hat{\boldsymbol{x}} + \sin(\phi)\hat{\boldsymbol{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin(\phi)\hat{\boldsymbol{x}} + \cos(\phi)\hat{\boldsymbol{y}}$$
$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$$

The Dirac Delta Function

The dirac delta function is a distribution with the following properties.

$$\delta(x) = \left\{ \begin{array}{ll} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{array} \right\}$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

If $f(x)$ is some ordinary function, if it is paired with a dirac delta function, the integral becomes easy to evaluate.

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

Another property of the dirac delta function is

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

A similar function, the Heaviside step function $\theta(x)$ can be defined as

$$\theta(x) = \left\{ \begin{array}{ll} 1, & \text{if } x\text{g}0 \\ 0, & \text{if } x \leq 0 \end{array} \right\}$$

The dirac delta function is the derivative of the heaviside step function.

$$\frac{d\theta(x)}{dx} = \delta(x)$$

The dirac celta function can be naturally generalized to more than one dimension.

$$\delta^3(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z)$$

$$\int_{allspace} \delta^3(\boldsymbol{r})d\tau = 1$$

Using this, we can find the result of a particularly tricky divergence problem.

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = 4\pi\delta^3(\boldsymbol{r})$$

or even better

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathcal{R}})$$

Since

$$\boldsymbol{\nabla}\left(\frac{1}{\mathcal{R}}\right) = -\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2}$$

it follows that

$$\nabla^2 \frac{1}{\mathcal{R}} = -4\pi\delta^3(\boldsymbol{\mathcal{R}})$$

The Theory of Vector Potentials

According to Helmholtz theorem, if the divergence $D(\boldsymbol{r})$ and the curl $\boldsymbol{C}(\boldsymbol{r})$ of a vector function $\boldsymbol{F}(\boldsymbol{r})$ are specified, and if they both go to zero faster than $1/r^2$ as $r \to \infty$, and if $\boldsymbol{F}(\boldsymbol{r})$ goes to zero as $r \to \infty$, then $\boldsymbol{F}$ is given uniquely by

$$\boldsymbol{F} = -\boldsymbol{\nabla}U + \boldsymbol{\nabla} \times \boldsymbol{W}$$

with

$$U(\boldsymbol{r}) = \frac{1}{4\pi}\int \frac{D(\boldsymbol{r}')}{\mathcal{R}}d\tau'$$

and

$$\boldsymbol{W}(\boldsymbol{r}) = \frac{1}{4\pi}\int \frac{\boldsymbol{C}(\boldsymbol{r}')}{\mathcal{R}}d\tau'$$

If the curl of a vector field vanishes everywhere, then it can be written as the gradient of a scalar potential.

$$\boldsymbol{\nabla} \times \boldsymbol{F} \leftrightarrow \boldsymbol{F} = -\boldsymbol{\nabla}\Phi$$

If the divergence of a field vanishes everywhere, then it can be written as the curl of a vector potential.

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0 \leftrightarrow \boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

We can, therefore, describe any field by a sum of a gradient of a scalar potential and the curl of a vector potential.

$$\boldsymbol{F} = -\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla} \times \boldsymbol{A}$$

Irrotational Fields

An irrotational field will have the following properties:

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$$

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{F} \cdot d\boldsymbol{l} \text{ is path independent}$$

$$\oint \boldsymbol{F} \cdot d\boldsymbol{l} = 0$$

$$\boldsymbol{F} = -\boldsymbol{\nabla}\Phi$$

Solenoidal Fields

A solenoidal field will have the following properties:

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0$$

$$\int \boldsymbol{F} \cdot d\boldsymbol{a} \text{ is surface independent}$$

$$\oint \boldsymbol{F} \cdot d\boldsymbol{a} = 0$$

$$\boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{A}$$