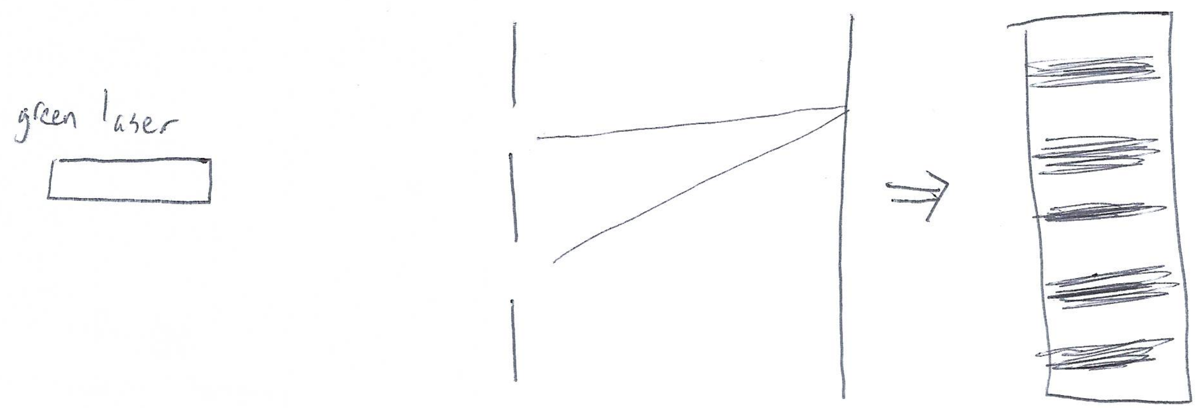
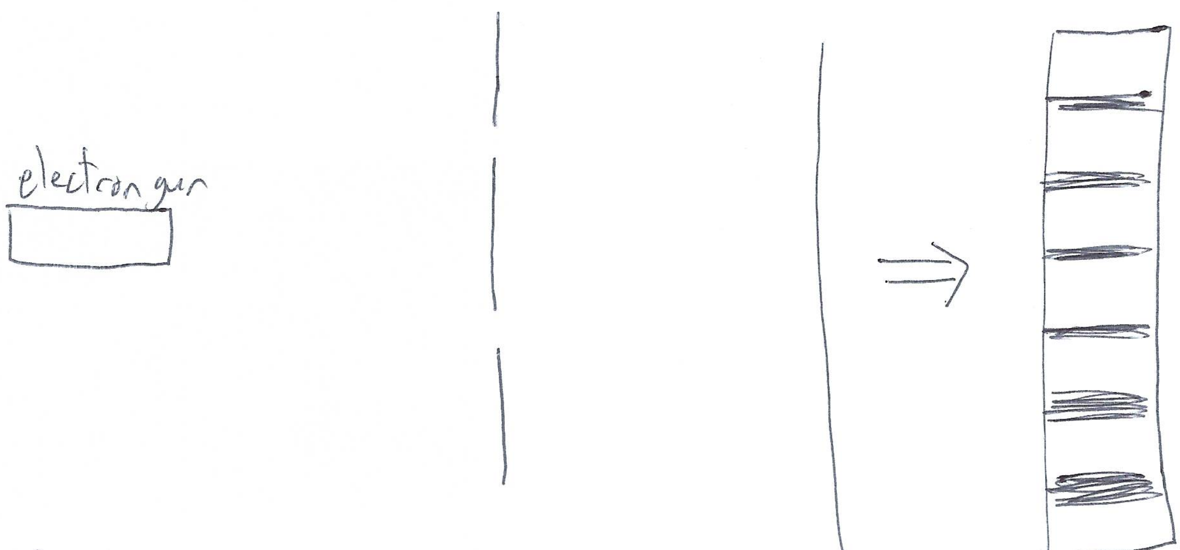
Phys 137A 24Jun19

Austin Hedeman
aphysicist28@berkeley.edu
office: 443 Birge
office hours: M 12-1, Th 11-12, F 1-2 in 429 Birge

Midterm 26Jul19
Final 16Aug19
1 page notes allowed one-sided





Interference is caused by the square of something obeying a wave equation
intensity for matter waves is a probability density
There is a linear theory (QM) with some "wavy object/amplitude"
that when absolute-value-squared gives an intensity which tells us probabilities

The "wavy object" $\rightarrow$ the wave function or probability amplitude

Postulate I (states)

The state of a quantum system at a given time t is given by a complex, normalized wave function $\Psi(x,t)$.

Phys 137A 25 Jun 19

Postulate I (states)

The state of a quantum system at time t_0 is given by a complex normalized wave function $\Psi(x, t_0)$

→ start with a single particle (mass m) moving in one dimension

$$\Psi(x,t_0) \mapsto e^{i\phi}\Psi(x,t_0)$$ nothing changes

only matters with more than one phase added together

$$(|\Psi\rangle = |\psi_1\rangle + |\psi_2\rangle)$$

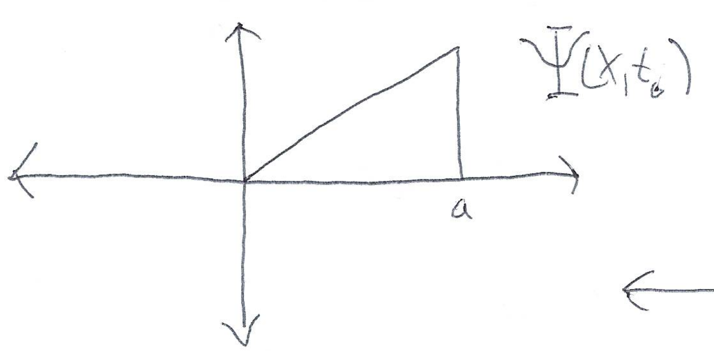
$$\frac{dP}{dx} = |\Psi(x,t_0)|^2 = \Psi^*(x,t_0)\,\Psi(x,t_0)$$

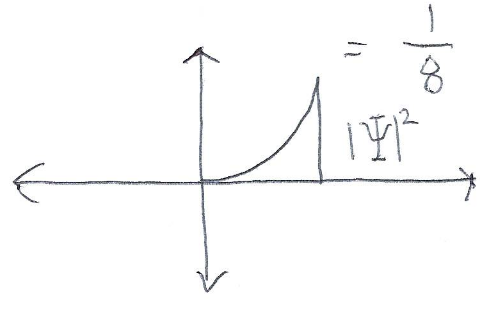
$$dP = |\Psi(x,t_0)|^2 dx$$

$$P(a \le x \le b) = \int_a^b |\Psi(x,t_0)|^2 dx$$

$$\Psi(x,t_0) = \begin{cases} \sqrt{\frac{3}{a^3}}\,x & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

$$P(0 \le x \le \tfrac{a}{2}) = \int_0^{a/2} \frac{3}{a^3} x^2 dx = \frac{1}{a^3} x^3 \Big|_0^{a/2} = \frac{1}{8}$$





What does probability mean?

Frequentist: Many identical systems and measure all of them "ensemble" systems

Bayesian: "Plausibility", How expected is it that we get a result.

Ψ must be normalized or probability makes no sense

$$\boxed{\int_{-\infty}^{\infty} |\Psi|^2 dx = 1}$$

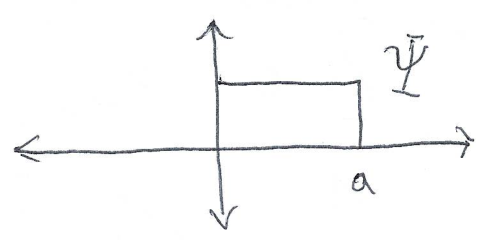
$\Psi(x,t_0) = e^x$ can't be normalized

Normalizable wave functions represent physical situations and ~~a~~ non-normalizable wave functions don't

$$\left.\int_{-\infty}^{\infty} |f(x)|^2 dx = \text{finite} \right\} \text{"Square-integrable"}$$

Example: Normalizing a top hat wave function $L^2(R)$

$$\Psi(x,t) = \begin{cases} N & 0 \leq x \leq a \\ 0 & \text{else} \end{cases}$$



$$\int_0^a |\Psi|^2 dx = N^2 \int_0^a dx = N^2 a = 1 \Rightarrow |N| = \frac{1}{\sqrt{a}}$$

$$\Psi(x,t) = \begin{cases} \frac{1}{\sqrt{a}} & 0 \leq x \leq a \\ 0 & \text{else} \end{cases} \qquad N = \frac{1}{\sqrt{a}} e^{i\phi}$$

Phase symmetry leads to conserved total probability

<u>Noether's (3) Theorem</u>

Number of siblings	0	1	2	3	4
$N(x)$	2	11	1	0	2
$P(x)$	1/8	11/16	1/16	0	1/8

$N_{tot} = 16$

$$\sum_x P(x) = 1$$

$$\langle x \rangle = \frac{0(2) + 11(1) + 2(1) + 3(0) + 4(2)}{16} = \frac{21}{16} \cong 1.3125$$

$$\langle x \rangle = \sum_x x P(x)$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\langle x^2 \rangle = \cancel{2 \cdot 0 + 11} \sum_x x^2 P(x) = 0^2 \tfrac{1}{8} + 1^2\left(\tfrac{11}{16}\right) + 2^2\left(\tfrac{1}{16}\right) + 3^2(0) + 4^2\left(\tfrac{1}{8}\right)$$

$$= \frac{11}{16} + \frac{1}{4} + 2 = \frac{47}{16} \cong 2.9375$$

$$\sigma_x = \sqrt{2.9375 - 1.3125^2} \cong 1.10$$

<u>Operator</u>

Something that eats a function and spits out another function

<u>Position operator</u>: $\hat{x}\left[\Psi(x,t_0)\right] = x\,\Psi(x,t_0)$

<u>Momentum operator</u>: $\hat{p}\left[\Psi(x,t_0)\right] = \frac{\hbar}{i}\frac{\partial \Psi(x,t_0)}{\partial x}$

<u>Linear operator</u>: $\hat{A}\left[a\Psi_1(x,t_0) + b\Psi_2(x,t_0)\right] = a\hat{A}\left[\Psi_1(x,t_0)\right] + b\hat{A}\left[\Psi_2(x,t_0)\right]$

<u>Postulate II (Observables)</u>

Observable/Measurable quantities are represented by linear operators acting on wave functions.

$KE = \frac{p^2}{2m} = T \quad V(x) \quad H(x,p) = \frac{p^2}{2m} + V(x)$

$\hat{T} = \frac{\hat{p}^2}{2m} \qquad \hat{T}[\Psi] = \frac{1}{2m}\hat{p}\left[\hat{p}[\Psi]\right] = \frac{1}{2m}\hat{p}\left[\frac{\hbar}{i}\frac{\partial\Psi}{\partial x}\right] = \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$

$\hat{H} = \hat{T} + \hat{V} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

$\cancel{\hat{H}\Psi = \frac{-\hbar^2}{2m}} \qquad \hat{H}\Psi = \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi$

Observable A

Operator $\hat{A}$

State $\Psi(x,t_0)$

Expectation value $\langle A\rangle = \int_{-\infty}^{\infty}\Psi^*\hat{A}\Psi\,dx \qquad \langle p\rangle = \frac{\hbar}{i}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,dx$

$\langle x\rangle = \int_{-\infty}^{\infty}\Psi^* x\,\Psi\,dx \qquad \langle x^2\rangle = \int_{-\infty}^{\infty}x^2|\Psi|^2\,dx \qquad \sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

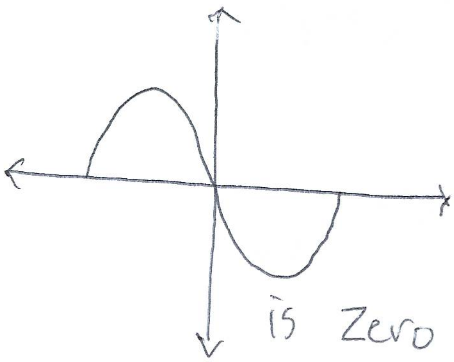
Phys 137A 26 Jun 19

Integration Rules

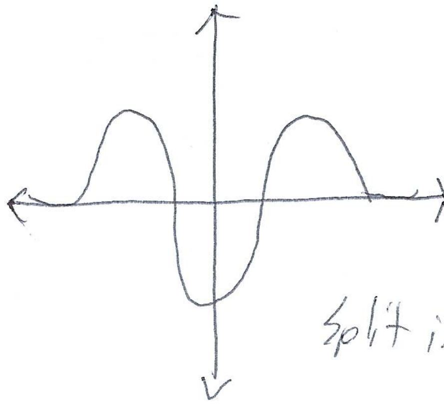
1) Think before you integrate

Know what your integral means

Exploit Symmetry



is Zero



Split in half

2) Tricks

u-substitution

Integration by parts

Simplify

add 0

multiply by 1

Use identity operator

Trig is awful, change to exponentials

Dirty-Tricks: $\int_{-\infty}^{\infty} e^{-bx^2}dx = \sqrt{\frac{\pi}{b}}$ $\quad \int_{-\infty}^{\infty} x^2 e^{-bx^2}dx = -\int_{-\infty}^{\infty} \frac{d}{db} e^{-bx^2}dx$

$$= \frac{-d}{db}\int_{-\infty}^{\infty} e^{-bx^2}dx = \frac{-d}{db}\sqrt{\frac{\pi}{b}} = \frac{1}{2b}\sqrt{\frac{\pi}{b}}$$

3) Look it up

Gaussian Wave Packet

$$\Psi(x,t_0) = Ne^{-x^2/4\sigma^2}$$

$$\int_{-\infty}^{\infty} N^2 e^{-x^2/2\sigma^2} dx = N^2\sqrt{\pi 2\sigma^2} = 1$$

$$N^2 = \frac{1}{\sqrt{2\pi\sigma^2}} \Rightarrow N = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4}$$

$$\Psi(x,t_0) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} e^{-x^2/4\sigma^2}$$

$$\langle x \rangle = \langle \Psi | \hat{x} | \Psi \rangle = 0 = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} e^{-x^2/2\sigma^2} x \, dx$$

$$\langle x^2 \rangle = \sigma^2$$

$$\langle p^2 \rangle = \langle \Psi | \hat{p}^2 | \Psi \rangle = \frac{\hbar^2}{4\sigma^2}$$

$$\langle p \rangle = \langle \Psi | \hat{p} | \Psi \rangle = 0$$

$$\sigma_x \sigma_p = \frac{\hbar}{2}$$

Uncertainty

Observable Q
Operator $\hat{Q}$
State $\Psi(x,t_0)$

$$\langle Q \rangle = \langle \Psi | \hat{Q} | \Psi \rangle$$

$$\sigma_Q = \sqrt{\langle Q^2 \rangle - \langle Q \rangle^2}$$

Uncertainty principle

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

Caused by position and momentum not commuting

As $\sigma \to 0$ in the Gaussian Wave Packet the wave function becomes a delta function

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1 \qquad \int_{-\infty}^{\infty} \delta(x) f(x)\,dx = f(0)$$

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\,dx = f(a)$$

Each $\hat{Q}$ has special values q and functions $\psi_q(x)$

eigen values q and eigenfunctions $\psi_q(x)$

<u>Eigen value Equation</u>

$$\hat{Q}|\psi_q\rangle = q|\psi_q\rangle$$

Claim: If $\Psi(x,t_0) = \psi_q(x)$

Then $\langle Q \rangle = q$, $\sigma_Q = 0$

$$\langle Q \rangle = \int_{-\infty}^{\infty} \psi_q^*(x)\, \hat{Q}\, \psi_q\, dx = \int_{-\infty}^{\infty} \psi_q^*(x)\, q\, \psi_q(x)\, dx = q$$

$$\hat{Q}^2\psi = \hat{Q}\hat{Q}\psi = \hat{Q}(q\psi) = q\hat{Q}\psi = q^2\psi$$

If $\hat{Q}\psi_q = q\psi_q$

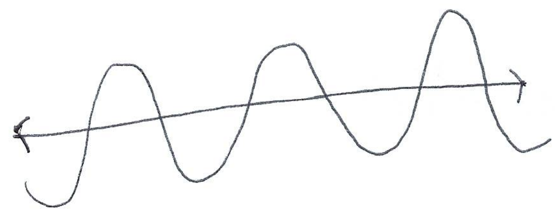
Then $f(\hat{Q})\psi_q = f(q)\psi_q$

Momentum

$$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$\hat{p}|\psi_{p_0}\rangle = p_0|\psi_{p_0}\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi_{p_0} \Rightarrow \frac{\partial \psi_{p_0}}{\partial x} = \left(\frac{ip_0}{\hbar}\right)\psi_{p_0}$$

$$\psi_{p_0} = N e^{\frac{ip_0 x}{\hbar}}$$



Postulate III (Measurement Possibilities)

A precise measurement of observable $\hat{Q}$ can only yield one of the eigenvalues in the spectrum of $\hat{Q}$.

For now we will assume that our eigenvalues each belong to a unique normalized eigenfunction up to a phase.

$$\hat{Q}\psi_{q_i} = q_i \psi_i$$

Postulate IV (Measurement Probabilities)

at time t_0 state $\Psi(x,t_0)$ operator $\hat{Q}$ with discrete nondegenerate spectrum. At t_0 we measure $\hat{Q}$.

$$P(q_i) = \left|\int_{-\infty}^{\infty} \psi_i^*(x)\Psi(x,t_0)dx\right|^2 \quad \langle q_i|\Psi\rangle = \int_{-\infty}^{\infty}\psi_i^*(x)\Psi(x,t_0)dx$$

$$P(q_i) = |\langle q_i|\Psi\rangle|^2$$

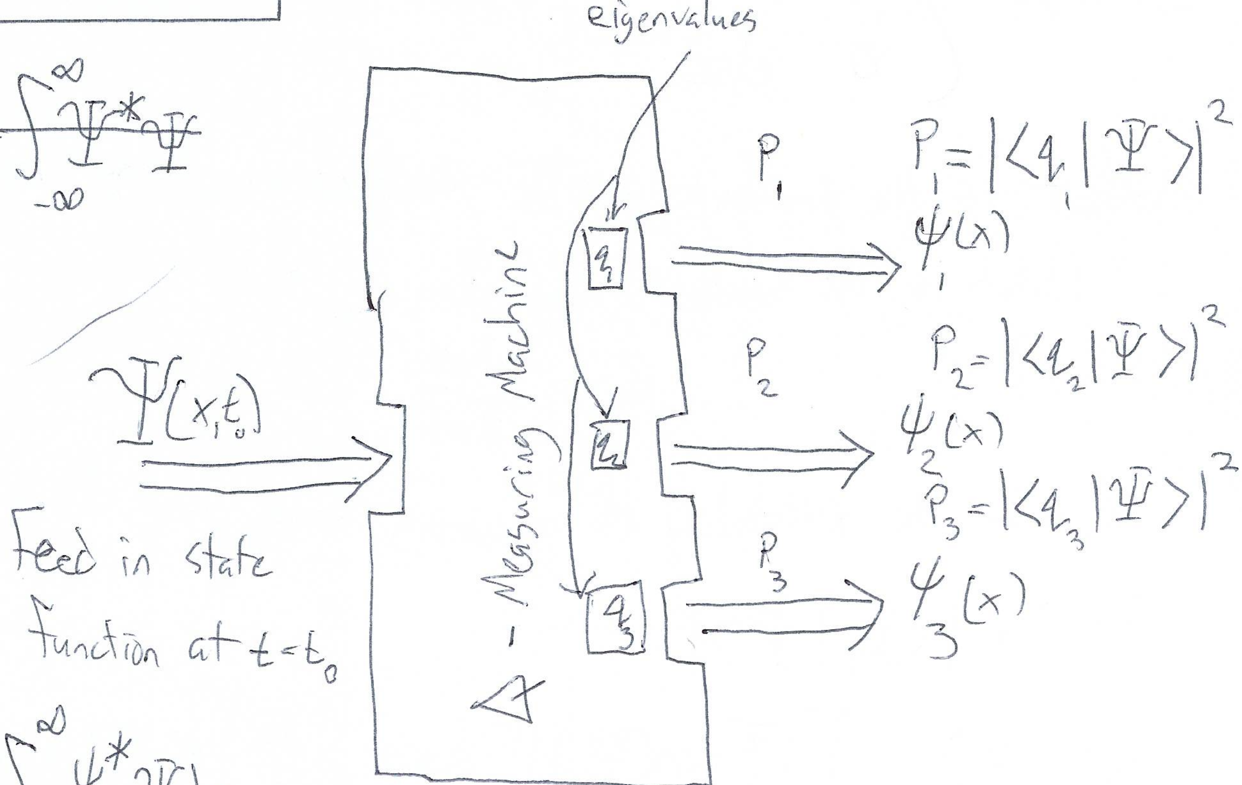
Postulate V (Collapse)

If a measurement of Q at time t_0 yields eigenvalue q_i, then immediately after at $t = t_0 + \delta t$ the wave function collapses into the normalized eigenfunction

$$\Psi(x, t_0 + \delta t) = \psi_{q_i}(x)$$

Phys 137A 27Jun19

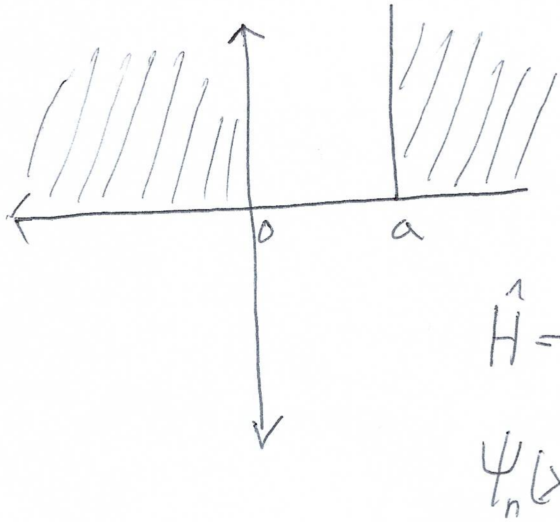
$$\cancel{\langle q_i | \Psi \rangle = \int_{-\infty}^{\infty} \Psi^* \Psi}$$



$$\langle q_i | \Psi \rangle = \int_{-\infty}^{\infty} \psi_i^* \Psi \, dx$$

Infinite Square Well

$$V(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & \text{else} \end{cases}$$



Energy

$$H = T + V = \frac{p^2}{2m} \text{ inside the box}$$

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & 0 \le x \le a \\ 0 & \text{else} \end{cases} \qquad n \in \mathbb{N}$$

$$\hat{H}\psi_n = \frac{-\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} = \frac{-\hbar^2}{2m}\sqrt{\frac{2}{a}}\left(\frac{n\pi}{a}\right)^2\left(-\sin\left(\frac{n\pi}{a}\right)\right) = \frac{n^2\hbar^2\pi^2}{2ma^2}\psi_n(x)$$

$$E_n = \frac{n^2\hbar^2\pi^2}{2ma^2}$$

$$\boxed{\hat{H}\psi_n = E_n \psi_n}$$

$$\Psi(x,0) = \begin{cases} \sqrt{\frac{1}{a}} & 0 \le x \le a \\ 0 & \text{else} \end{cases}$$ Inside Infinite Square Well

What is the smallest possible result of an energy measurement?
(What is the smallest eigenvalue of $\hat{H}$?)

$$E_1 = \frac{\hbar^2 \pi^2}{2ma^2}$$

What is the probability of measuring Energy and finding E_1?

$$P(E_1) = |\langle E_1 | \Psi \rangle|^2$$

$$\langle E_1 | \Psi \rangle = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \sqrt{\frac{1}{a}}\, dx = \frac{\sqrt{2}}{a} \int_0^a \sin\left(\frac{\pi x}{a}\right) dx$$

$$= -\frac{a\sqrt{2}}{\pi a} \cos\frac{\pi x}{a} \Big|_0^a = \cancel{\frac{a}{\pi}}\ -\frac{\sqrt{2}}{\pi}(-1-1) = \frac{2\sqrt{2}}{\pi} \cong$$

$$P(E_1) = \frac{8}{\pi^2} \cong 0.81$$

$$\langle q_1 | q_2 \rangle = \int_{-\infty}^{\infty} \psi_1^* \psi_2\, dx = \begin{cases} 1 & q_1 = q_2 \\ 0 & q_1 \neq q_2 \end{cases}$$

Postulate $\underline{\text{VI}}$ Time Evolution

The time-evolution of a quantum system is governed by the time-dependent Schrödinger Equation (TDSE)

$$\boxed{i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \hat{H}\, \Psi(x,t)}$$

Infinite Square Well

$$\Psi(x,0) = \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

Assume $\Psi(x,t) = \psi_n(x)e^{if(t)}$

$$i\hbar\, i f'(t)\psi_n(x)e^{if(t)} = \cancel{\frac{\hbar^2}{}}\hat{H}\psi_n(x)e^{if(t)} = E_n\psi_n(x)e^{if(t)}$$

$$-\hbar f'(t) = E_n$$

$$f'(t) = \frac{-E_n}{\hbar} \Rightarrow f(t) = \frac{-E_n t}{\hbar} + c$$

$$\Psi_n(x,t) = \psi_n(\cancel{x})e^{-\frac{iE_n t}{\hbar}} = \psi_n(x)e^{-\frac{iE_n t}{\hbar}} \qquad \frac{E_n}{\hbar} = \omega_n = f$$

If $\Psi_{Clark}(x,t)$ solves TDSE

and $\Psi_{Bex}(x,t)$ solves TDSE

then $a\Psi_{Clark} + b\Psi_{Bex}$ solves TDSE

~~The~~ TDSE $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$

Consider separable solutions

$$\Psi(x,t) = \psi(x)\phi(t) = \psi(x)T(t)$$

xt

$x^2t^2 \qquad axt + bx^2t^2$

Stationary States

All measurements are independent of time

A general solution may always be expressed as

$$\Psi(x,t) = \sum_n c_n \psi_n \varphi_n$$

Separation of Variables

1) SoV ansatz eg. $f(x,y,z) = X(x)Y(y)Z(z)$

$$\Psi_n(x,t) = \psi_n(x)\,\phi_n(t)$$

2) Plug in ansatz and simplify

$$i\hbar \frac{\partial}{\partial t}\left[\psi(x)\phi(t)\right] = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(\psi\phi) + V(x)\psi\phi$$

3) $\cancel{i\hbar\phi\partial}$ $i\hbar\psi\frac{d\phi}{dt} = \frac{-\hbar^2\phi}{2m}\frac{d^2\psi}{dx^2} + V\psi\phi$

$$\frac{i\hbar}{\phi}\frac{d\phi}{dt} = \frac{-\hbar^2}{2m\psi}\frac{d^2\psi}{dx^2} + V$$

4) Group terms

$$f(t) = g(x) = C = k^2 = E$$

$$\frac{i\hbar\phi_n'}{\phi_n} = k^2 = E \qquad \frac{-\hbar^2\psi_n''}{2m\psi_n} + V = k^2 = E_n$$

Time $\frac{d\phi}{\phi_n} = \frac{-iE\cancel{\hbar}\,dt}{\hbar}$ $\cancel{\phi = e^{-i\hbar C}}$ $\phi_n = e^{-i\frac{E_n t}{\hbar}}$

Space $\boxed{\hat{H}\psi_n = E_n\psi_n \quad \text{TISE}}$

$$\Psi(x,t) = \psi e^{-i\frac{E_n t}{\hbar}}$$

Phys 137A 01Jul19

TDSE: $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$ TISE: $\hat{H}\psi = E_n\psi$

$\hat{H}\Psi = \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\cdot\Psi$ $\Psi = \psi e^{-iE_nt/\hbar}$

SoV: $\Psi = \psi\varphi$

$\varphi = e^{-iE_nt/\hbar} \longrightarrow$ needs definite energy to use it

separable solutions are stationary states (expectation values of observable are independent of time)

Observable $Q \Rightarrow$ Operator $\hat{Q}$

$\langle Q\rangle$ for a separable solution $\Psi(x,t) = \psi_n(x)e^{-iE_nt/\hbar}$

$$\langle Q\rangle = \int_{-\infty}^{\infty} \Psi_n^* \hat{Q}\Psi_n \, dx = \int_{-\infty}^{\infty} \psi_n^*(x) e^{iE_nt/\hbar} \hat{Q}\left(\psi_n(x) e^{-iE_nt/\hbar}\right) dx$$

$$\cancel{= \int_{-\infty}^{\infty} \psi^*(x)} = \int_{-\infty}^{\infty} \psi_n^* \hat{Q}\psi_n \, dx = \langle Q\rangle$$

$$\frac{d\langle Q\rangle}{dt} = 0$$

$P(Q=q) = \cancel{\int} |\langle q|\Psi\rangle|^2$ if we have a separable state

$$\langle q|\Psi\rangle = \int_{-\infty}^{\infty} \psi_q^*(x)\Psi(x,t)\,dx = \int_{-\infty}^{\infty} \psi_q^*(x)\psi_n(x)e^{-iE_nt/\hbar}\,dx$$

$$= e^{-iE_nt/\hbar}\int_{-\infty}^{\infty} \psi_q^*\psi_n\,dx = e^{-iE_nt/\hbar}\langle q|n\rangle$$

$$P(Q=q) = |e^{-iE_nt/\hbar}\langle q|n\rangle|^2 = |e^{-iE_nt/\hbar}|^2|\langle q|n\rangle|^2 = |\langle q|n\rangle|^2$$

Infinite Square Well

$V(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & \text{else} \end{cases}$

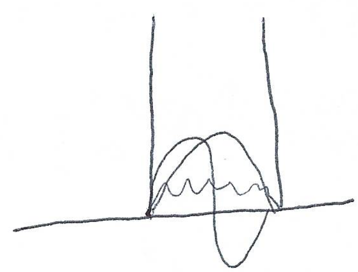
Solve the TISE for ISW

$\Psi(x,t) = 0$ outside the well

Inside $\hat{H}\psi_n = E_n\psi_n = \frac{-\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} + V\psi_n = \frac{-\hbar^2}{2m}\frac{d^2\psi_n}{dx^2}$

$\frac{d^2\psi_n}{dx^2} = \frac{-2mE_n}{\hbar^2}\psi_n \Rightarrow \psi_n = A\sin\sqrt{\frac{2mE_n}{\hbar^2}}x + B\cos\sqrt{\frac{2mE_n}{\hbar^2}}x$

$k = \sqrt{\frac{2mE_n}{\hbar^2}} \Rightarrow \psi_n = A\sin k_n x + B\cos k_n x$



- Continuous at $x=0$ and $x=a$

$0 = \psi_n(0) = B \therefore B = 0$

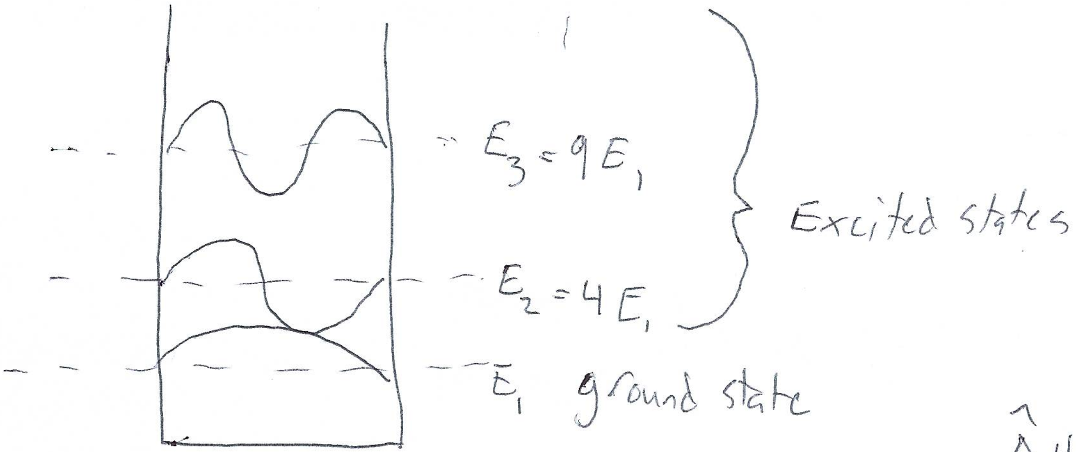
$\psi_n = A\sin k_n x$

$\psi_n(a) = 0 = A\sin k_n a \quad k_n a = \pi n \Rightarrow k_n = \frac{\pi n}{a} \quad n \in \mathbb{N}$

$\boxed{E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}}$ ~~$\psi_n = As($~~

$\psi_n = A\sin\left(\frac{n\pi x}{a}\right) \quad A^2\int_{-\infty}^{\infty}\sin^2\left(\frac{n\pi x}{a}\right)dx = \frac{A^2}{2}\int_{-\infty}^{\infty} 1 - \cos\left(\frac{2\pi n x}{a}\right)dx$

$A = \sqrt{\frac{2}{a}}$ $\boxed{\psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right) \text{ inside}}$ $\Psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right)e^{-iE_nt/\hbar}$



$$\hat{Q}\psi_q(x) = q\,\psi_q(x)$$

Given any physical observable Q with associated operator $\hat{Q}$ the full spectrum of eigenfunctions $\{\psi_q(x)\}$ of $\hat{Q}$ form a complete orthonormal basis of wave functions.

Discrete eigenvalues: Discrete label n, m

Kronecker Delta symbol: $\delta_{n,m} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

$$\boxed{\sum_m c_m^* \delta_{nm} = c_n}$$

Continuous Eigenvalues $\Rightarrow$ Continuous labels α, β

$$c_\alpha \Rightarrow c(\alpha)$$

Dirac Delta Function: $\cancel{\delta = \{}$ $\quad \delta(\alpha - \beta) = \begin{cases} \infty & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$

$$\boxed{\int c(\alpha)\,\delta(\alpha - \beta)\,d\alpha = c(\beta)}$$

$$\int_{-\infty}^{\infty} e^{i\alpha\beta}\,d\alpha = 2\pi\,\delta(\beta) \qquad \int e^{i\alpha\beta/\hbar}\,d\alpha = 2\pi\hbar\,\delta(\beta)$$

Orthonormality

Discrete:

$$\langle n|m\rangle = \int_{-\infty}^{\infty} \psi_n^*(x)\psi_m(x)\,dx = \begin{cases} 1 & n=m \\ 0 & n\neq m \end{cases}$$

Continuous:

$$\cancel{\langle \alpha|m\rangle = \int_{-\infty}^{\infty} \psi_\alpha^*(x)\psi_m(x)\,dx}$$

$$\langle \alpha|\beta\rangle = \int_{-\infty}^{\infty} \psi_\alpha^*(x)\psi_\beta(x)\,dx = \delta(\alpha-\beta)$$

$$\langle p_1|p_2\rangle = \int_{-\infty}^{\infty} A^* e^{-ip_1x/\hbar} A e^{ip_2x/\hbar}\,dx = |A|^2\int_{-\infty}^{\infty} e^{i(p_2-p_1)x/\hbar}\,dx = |A|^2\delta(p_2-p_1)\,2\pi\hbar$$

Postulate IV.b Measurement Probabilities with a continuous spectrum

$$\underbrace{\frac{dP(\alpha)}{d\alpha}}_{\text{Probability density}} = |\langle \alpha|\Psi\rangle|^2$$

$$\langle \alpha|\Psi\rangle \rightarrow \int_{-\infty}^{\infty} \psi_\alpha^*(x)\,\Psi(x,t)\,dx$$

Phys 137A 02Jul19

	Discrete	Continuous
Orthonormality	$\langle n \vert m \rangle = \int_{-\infty}^{\infty} \psi_n^*(x)\, \psi_m(x)\, dx = \delta_{n,m}$	$\langle \alpha \vert \beta \rangle = \int_{-\infty}^{\infty} \psi_\alpha^*(x)\, \psi_\beta(x)\, dx = \delta(\alpha - \beta)$
completeness	$\Psi(x) = \sum_n c_n \psi_n(x)$ $\{c_n\}$ "expansion coefficients"	$\Psi(x) = \int c(\alpha)\, \psi_\alpha(x)\, dx$ $c(\alpha)$ "expansion coefficients"

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \qquad \psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$\Psi(x) = \sum_{n=1}^{\infty} \left(\sqrt{\frac{2}{a}}\, c_n\right) \sin\left(\frac{n\pi x}{a}\right)$$

Q-eigenbasis

Observable Q, Operator $\hat{Q}$, Spectrum $\{q_n\}$, Eigenfunctions $\{\psi_n\}$

How do we find c_n?

Fourier's Trick! $\quad c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi x}{a}\right) \Psi(x)\, dx$

$$c_n = \int_{-\infty}^{\infty} \psi_n^*(x)\, \Psi(x)\, dx = \langle n \vert \Psi \rangle$$

$$c_n = \int_{-\infty}^{\infty} \psi_n^*(x)\,\psi(x)\,dx = \int_{-\infty}^{\infty} \psi_n^*(x) \sum_m c_m \psi_m(x)\,dx$$

$$= \sum_m c_m \int_{-\infty}^{\infty} \psi_n^*(x)\,\psi_m(x)\,dx = \sum_m c_m \,\delta_{n,m} = c_n \qquad \square$$

$$\boxed{c_n = \langle n|\psi\rangle = \int_{-\infty}^{\infty} \psi_n^* \psi \,dx}$$

Energy Eigenbasis

$\{E_n\}$ $\{\psi_n\}$

Solve TISE

$$\Psi(x,0) = \sum_n c_n \psi_n(x)$$

Probability of measuring energy and finding E_n?

$$P(E_n) = \left| \int_{-\infty}^{\infty} \psi_n(x)\cdot\Psi(x,0)\,dx \right|^2 = |\langle n|\Psi\rangle|^2 = \boxed{|c_n|^2 = P(E_n)}$$

$$\psi(x) = \sqrt{\tfrac{1}{4}}\,\psi_1(x) + \frac{i}{\sqrt{2}}\psi_2(x) - \frac{e^{i\pi/3}}{\sqrt{4}}\psi_3(x)$$

$c_1 = \frac{1}{2}$, $c_2 = \frac{i}{\sqrt{2}}$ $c_3 = \frac{-e^{i\pi/3}}{2}$

$P_1 = \frac{1}{4}$ $P_2 = \frac{1}{2}$ $P_3 = \frac{1}{4}$

Normalization: $1 = \sum_n |c_n|^2$

Expectation: $\langle E\rangle = \sum_n E_n |c_n|^2$

This is not special for energy. This works on any discrete observable.

$$\Psi(x,0) = \sum_n c_n \psi_n(x)$$

$$\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_nt/\hbar}$$

$$= \sum_n c_n(t) \psi_n(x) \quad , \; c_n(t) = c_n(0) e^{-iE_nt/\hbar}$$

Free Particle

$$\hat{H} = \frac{\hat{p}^2}{2m} \qquad V(x) = 0$$

TISE: $\frac{-\hbar^2}{2m} \frac{\partial^2 \psi_E}{\partial x^2} = E\psi_E \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$

$$\frac{d^2\psi_E}{dx^2} = -k^2\psi_E \Rightarrow \psi(x) = \boxed{Ae^{ikx}} + \boxed{Be^{-ikx}}$$

$$= \boxed{A\psi_{p_0}(x)} + \boxed{B\psi_{-p_0}(x)}$$

$p_0 = \hbar k = \sqrt{2mE}$

right travelling   left travelling

$$\Psi(x,0) = Ae^{ikx} + Be^{-ikx}$$

$$\Psi(x,t) = (Ae^{ikx} + Be^{-ikx})e^{-iEt/\hbar} \qquad \omega = \frac{E}{\hbar} \Rightarrow E = \frac{p^2}{2m} = \hbar\omega$$

$$= Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)} \qquad \Rightarrow \omega = \frac{\hbar k^2}{2m}$$

$$\hat{p}e^{ipx/\hbar} = pe^{ipx/\hbar} \qquad \psi_p(x) = e^{i(\frac{p}{\hbar})x}$$

Momentum Eigenbasis

$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ eigen value $\{p_0 \in \mathbb{R}\}$ eigenfunction $\{Ne^{ipx/\hbar}\}$

Normalize: $\langle p_1 | p_2 \rangle = \int_{-\infty}^{\infty} \psi_{p_1}^*(x)\psi_{p_2}(x)\,dx = \delta(p_2 - p_1)$

$$|N|^2 \int_{-\infty}^{\infty} e^{-ip_1x/\hbar} e^{ip_2x/\hbar}\,dx = |N|^2 \int_{-\infty}^{\infty} e^{i(p_2-p_1)x/\hbar} = |N|^2 2\pi\hbar\,\delta(p_2 - p_1)$$

$$= \delta(p_2 - p_1) \Rightarrow |N|^2 2\pi\hbar = 1 \Rightarrow N = \frac{1}{\sqrt{2\pi\hbar}}$$

$$\boxed{\text{Normalized Eigenbasis: } \psi_p = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}}$$

$$\boxed{\cancel{\Psi}\Psi(x,0) = \int_{-\infty}^{\infty} c(p)\,\psi_p(x)\,dp}$$

$$\boxed{\Psi(x) = \int_{-\infty}^{\infty} \varphi(p)\,\psi_p(x)\,dp}$$

$\varphi(p)$ coefficients of momentum expansion

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar}\,\varphi(x)\,dp$$

$$\varphi(p) = \langle p | \Psi \rangle = \int_{-\infty}^{\infty} \psi_p^* \Psi$$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\,\Psi(x)\,dx$$

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar}\,\phi(p)\,dp$$

Fourier Transformations

$\phi(p)$ momentum space wave function

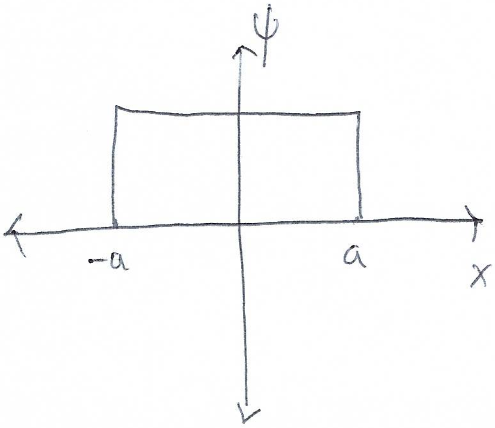
It does everything $\psi(x)$ can

$$\frac{dP(p_0)}{dp} = |\langle p|\psi\rangle|^2 = |\phi(p_0)|^2$$

$$P(p_1 < p < p_2) = \int_{p_1}^{p_2} |\phi|^2 dp \qquad 1 = \int_{-\infty}^{\infty} |\phi(p)|^2 dp$$

$$\langle p\rangle = \int \psi^* \hat{p}\, \psi\, d_p \qquad \hat{p}\,\phi(p) = p\,\phi(p)$$

$$= \int_{-\infty}^{\infty} p\,|\phi(p)|^2 dp$$

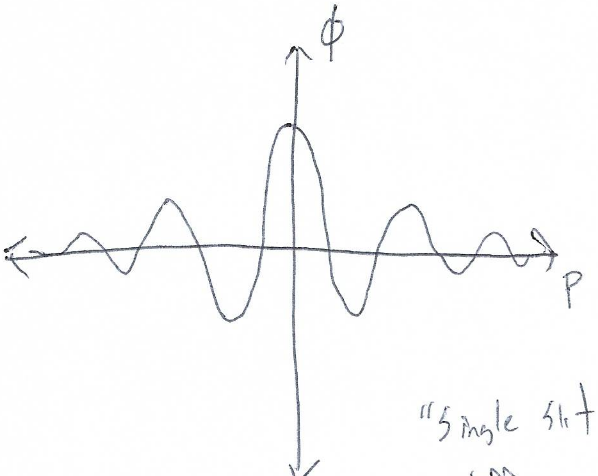


$$\psi(x) = \begin{cases} \frac{1}{\sqrt{2a}} & -a \le x \le a \\ 0 & \text{else} \end{cases}$$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-a}^{a} e^{-ipx/\hbar} \frac{1}{\sqrt{2a}}\, dx$$

$$= \frac{1}{\sqrt{4\pi\hbar a}} \left. \frac{e^{-ipx/\hbar}}{-ip/\hbar} \right|_{-a}^{a}$$

$$= \frac{-\hbar}{\sqrt{\pi a \hbar}} \frac{e^{-ipa/\hbar} - e^{ipa/\hbar}}{2ip} = \frac{\hbar}{p\sqrt{\pi a\hbar}} \sin\left(\frac{pa}{\hbar}\right)$$



"Single slit diffraction"

Phys 137A 03Jul19

Observable	Operator	Eigenvalues	Expansion Coefficients	Probabilities
Position	$\hat{x}$	x	$\langle x \vert \psi \rangle = \psi(x) = \int_{-\infty}^{\infty} \psi_x^*(x)\, \psi(x) dx$	$\frac{dP(x)}{dx} = \vert\psi(x)\vert^2$
Momentum	$\hat{p}$	p	$\langle p \vert \psi \rangle = \phi(p) = \int_{-\infty}^{\infty} \psi_p^*(x) \psi(x) dx$	$\frac{dP(p)}{dp} = \vert\phi(p)\vert^2$
Energy	$\hat{H}$	E_n	$\langle n \vert \psi \rangle = c_n = \int_{-\infty}^{\infty} \psi_n^*(x) \psi(x) dx$	$P(E_n) = \vert c_n\vert^2$

Free Particle

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

$\exists$ Momentum eigenstates are free particle energy eigenstates

$$\hat{p}\,\psi_{p_0} = p_0 \psi_{p_0} \qquad \hat{H}\psi_{p_0} = \frac{p_0^2}{2m}\psi_{p_0} \qquad c_n \Rightarrow \phi(p)$$

$$c_n(t) = c_n(0)\, e^{-iE_n t/\hbar}$$

$$\boxed{\Phi(p,t) = \phi(p)\, e^{-i\left(\frac{p^2}{2m}\right)t/\hbar}}$$

Free Particle Wave Function $\Psi(x,0) = \psi(x)$

Express using momentum eigenbasis

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar}\, \phi(p)\, dp \qquad \phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \psi(x)\, dx$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar}\, \phi(p)\, e^{\frac{-ip^2 t}{2m\hbar}}\, dp$$

Position Eigenbasis

$\hat{X}$

Eigenfunctions $\psi_{x_0}(x) = \delta(x - x_0)$

$$\hat{x}\,\delta(x - x_0) = x\,\delta(x - x_0) = x_0\,\delta(x - x_0)$$

Orthonormality

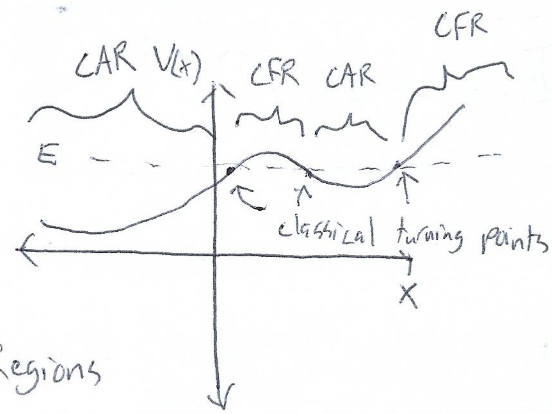
$$\langle x_1 | x_2 \rangle = \delta(x_2 - x_1)$$

$$= \int_{-\infty}^{\infty} \delta(x - x_1)\,\delta(x - x_2)\,dx = \delta(x_2 - x_1)$$

$$\psi(x) = \int_{-\infty}^{\infty} c(x_0)\,\delta(x - x_0)\,dx_0 = c(x)$$

$$\frac{dP(x_0)}{dx_0} = |\langle x_0 | \psi \rangle|^2$$

$$\langle x_0 | \psi \rangle = \int_{-\infty}^{\infty} \delta(x - x_0)\,\psi(x)\,dx = \psi(x_0)$$



TISE $\Rightarrow$ Energy Eigenfunctions

Given an Energy E

Regions where $E \geq V(x) \Rightarrow$ classically Allowed Regions (CARs)

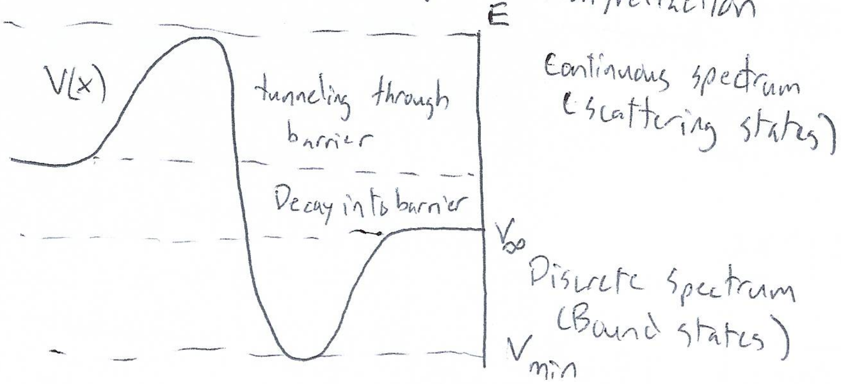
Regions where $E < V(x) \Rightarrow$ classically forbidden Regions (CFRs)

Points ~~Regions~~ where $E = V(x) \Rightarrow$ classical turning points (CTPs)

A quantum bound state is an energy eigenfunction such that no CAR extends to infinity.

Evanescent (?) wave

Frustrated internal internal reflection/refraction



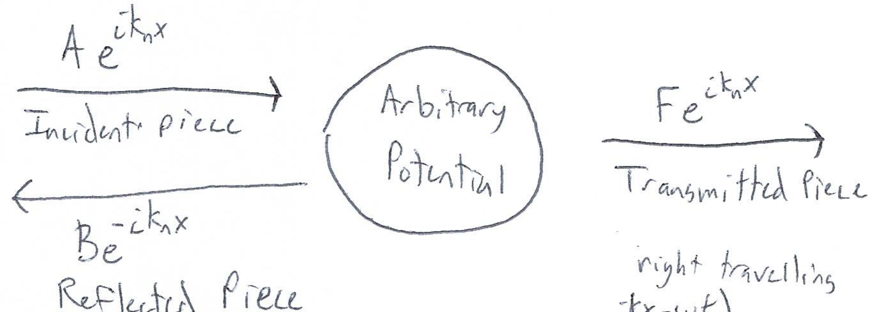
$E \le V_{min}$ no eigenstate

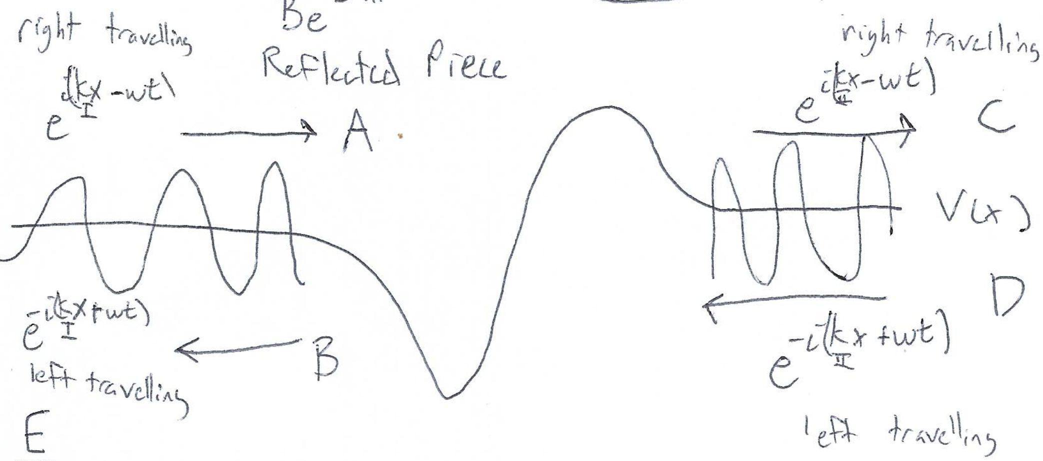
~~$E \le$~~

$V_{min} < E < V_\infty$ Bound state discrete

$E \ge V_\infty$ Scattering state continuous

Scattering states can't be normalized but they can be compared. You get relative probability to each other.





$\omega = \frac{E}{\hbar}$

$p = \pm k_{I,II} \hbar$

	in from $-\infty$	in from $+\infty$
incident	A	D
reflected	B	C
transmitted	C	B
absent	D	A

Probability Current

$$J(x,t) = \frac{i\hbar}{2m}\left(\frac{\partial \Psi^*}{\partial x}\Psi - \Psi^*\frac{\partial \Psi}{\partial x}\right) = -\frac{\hbar}{m}\,\mathrm{Im}\left(\frac{\partial \Psi^*}{\partial x}\Psi\right)$$

speed

incident $Ae^{ik_L x} \Rightarrow J_{incident}(-\infty) = \frac{\hbar k_L}{m}|A|^2$ probability density

reflect $Be^{-ik_L x} \Rightarrow J_{reflect}(-\infty) = -|B|^2\frac{\hbar k_L}{m}$

transmitted $Fe^{ik_R x} \Rightarrow J_{transmit}(+\infty) = |F|^2\frac{\hbar k_R}{m}$

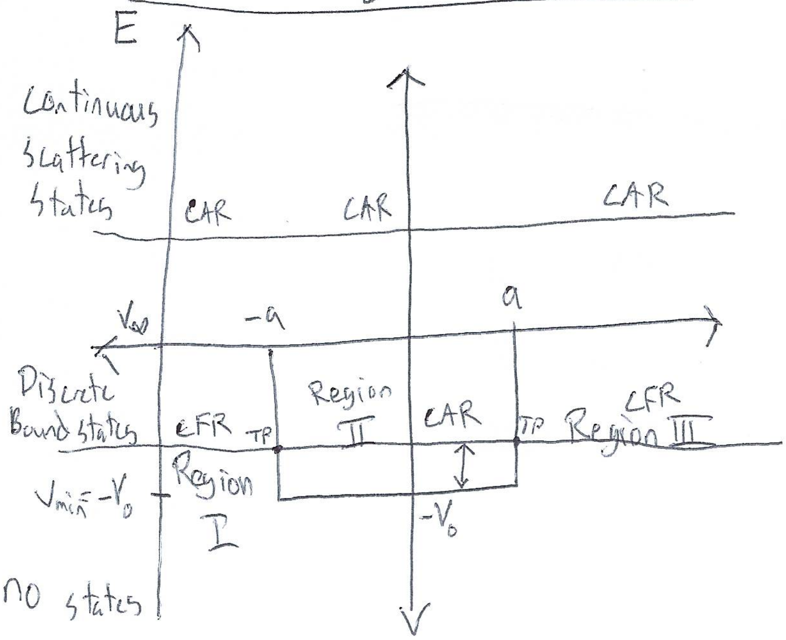
R = reflection coefficient $= \left|\frac{J_{reflect}}{J_{incident}}\right| = \left|\frac{B}{A}\right|^2 = \frac{|B|^2}{|A|^2}$

T = transmitted coefficient $= \left|\frac{J_{trans}}{J_{incident}}\right| = \frac{k_R|F|^2}{k_L|A|^2}$

$$\boxed{1 = R + T}$$

Phys 137A 08Jul19

Finite Square Well



$$V = \begin{cases} 0 & x < -a \quad \text{Region I} \\ -V_0 & -a < x < a \quad \text{Region II} \\ 0 & a < x \quad \text{Region III} \end{cases}$$

Bound States

Solve TISE

$-V_0 < E < 0$

$$\boxed{k \equiv \sqrt{\frac{2m(E+V_0)}{\hbar^2}}}$$

Region II $\quad E > V(x) \quad E - V_0(x) = E + V_0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - V(x))\psi = -k^2\psi \quad \text{for region II}$$

$$\psi_{II} = \cancel{A}^{D}\cos kx + \cancel{B}^{C}\sin kx$$

Region I and III $\quad E < V(x)$

$$\boxed{\sqrt{\frac{2m(V(x)-E)}{\hbar^2}} = \sqrt{\frac{2m|E|}{\hbar^2}} \simeq K}$$

$$\psi_I'' = K^2\psi_I \quad \psi_{III}'' = K^2\psi_{III}$$

$$\psi_I = Ae^{Kx} + Be^{-Kx} \qquad \psi_{III} = Ee^{Kx} + Fe^{-Kx}$$

$\psi_{I} = Ae^{\kappa x} + Be^{-\kappa x}$, $B=0$ due to boundary conditions

$\psi_{I} = Ae^{\kappa x}$

$\psi_{III} = Fe^{\kappa x} + Ge^{-\kappa x}$, $F=0$ due to bondary conditrons

$\psi_{III} = Ge^{-\kappa x}$

$\psi_{II} = C\sin kx + D\cos kx$, ~~$C=0$ due to boundary conditions~~

$\psi_{II} = D\cos kx + C\sin kx$

$$\psi_{even} = \begin{cases} Ae^{\kappa x} & I \\ D\cos kx & II \\ Ae^{-\kappa x} & III \end{cases} \qquad \psi_{odd} = \begin{cases} Ae^{\kappa x} & I \\ C\sin kx & II \\ -Ae^{-\kappa x} & III \end{cases}$$

ψ is continuous

ψ' is continuous

<u>Even solutions</u> continuity at $a=x$ continuity at $-a=x$

$\psi_{II}(a) = \psi_{III}(a)$ continuity of derivative at $a=x$

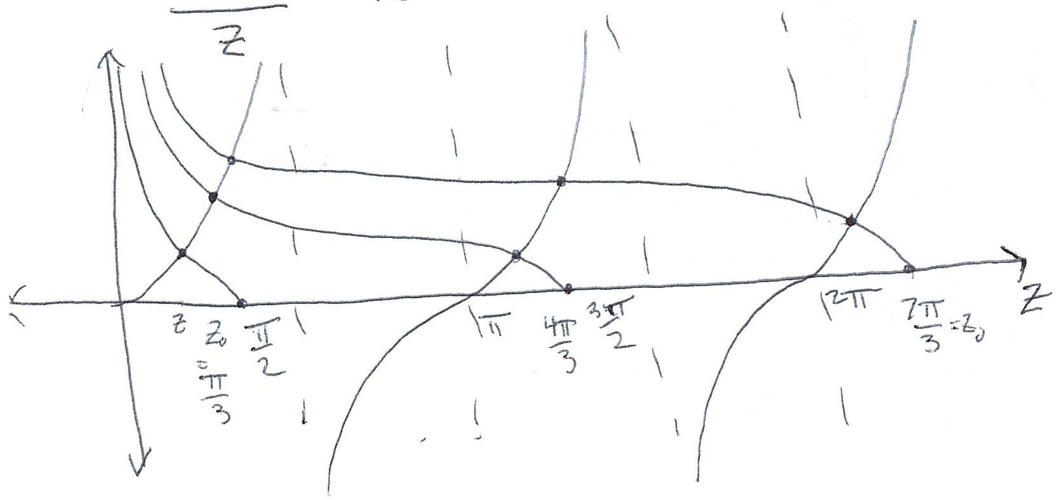
$D\cos ka = Ae^{-\kappa a}$ $\quad -kD\sin ka = A\kappa e^{-\kappa a}$

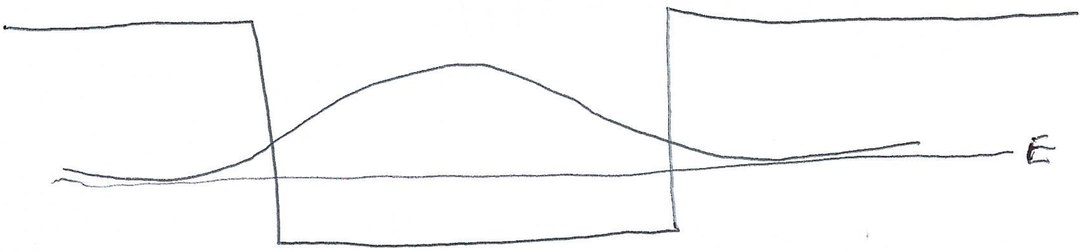
$-k\tan ka = -\kappa \Rightarrow \tan ka = \frac{\kappa}{k}$ $\quad ka\tan ka = \kappa a$

$z\tan z = \sqrt{z_0^2 - z^2}$ $\quad z \equiv ka$

$z_0 \equiv a\sqrt{\frac{2mV_0}{\hbar^2}}$

$\tan z = \frac{\sqrt{z_0^2 - z^2}}{z} = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$

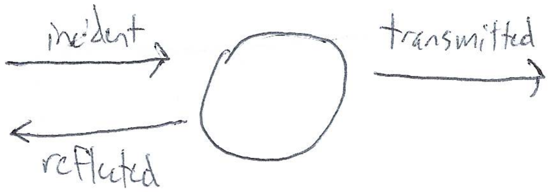




Scattering

$E > 0$

Beam incident from the left $(-\infty)$



Region I and III

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

Region II

$$l = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

$$\psi_I'' = -k^2\psi_I \qquad \psi_{II}'' = -l^2\psi_{II} \qquad \psi_{III}'' = -k^2\psi_{III}$$

$$\psi_I = Ae^{ikx} + Be^{-ikx} = T_R + I_{L\,\text{ref}}$$

$$\psi_{II} = C\sin lx + D\cos lx =$$

$$\psi_{III} = Fe^{ikx} + Ge^{-ikx} = T_R + 0$$

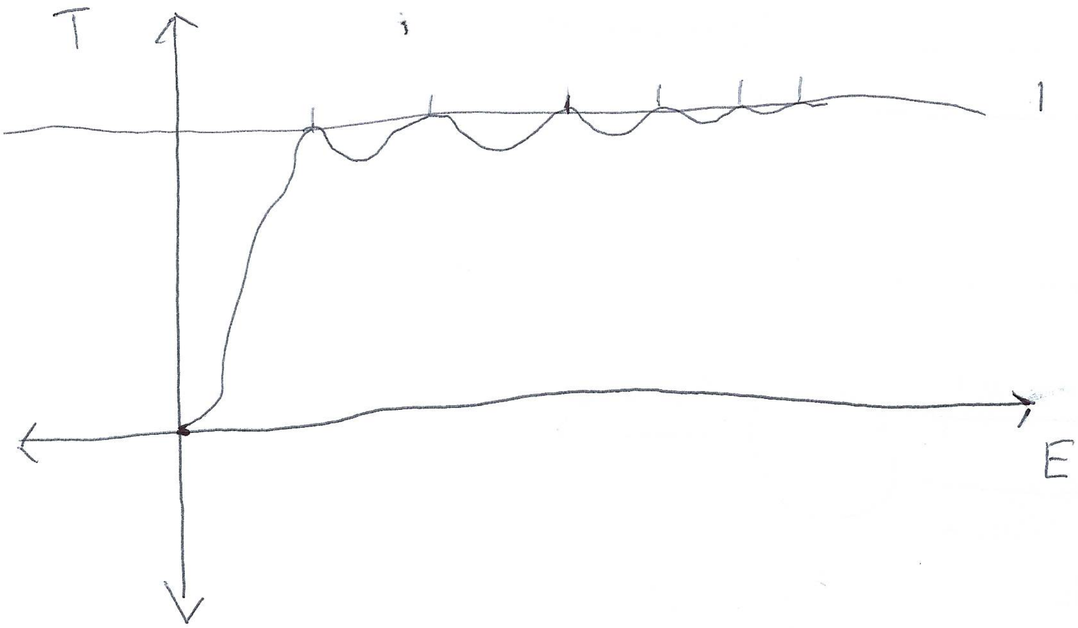
Continuous at $-a$ $\qquad Ae^{-ika} + Be^{ika} = -C\sin la + D\cos la$

der continuous at $-a$ $\qquad ikAe^{-ika} - ikBe^{ika} = lC\cos la + lD\sin la$

continuous at a $\qquad C\sin la + D\cos la = Fe^{ika}$

der continuous at a $\qquad lC\cos la - lD\sin la = ikFe^{ika}$

$$T = \frac{|F|^2}{|A|^2} = \left(1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)\right)^{-1}$$

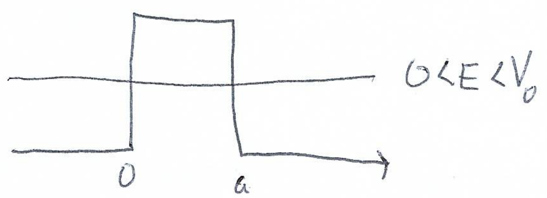


Phys 137A 09Jul19

Tunneling

$$k = \frac{\sqrt{2m(V_0-E)}}{\hbar}$$

$$\Downarrow$$



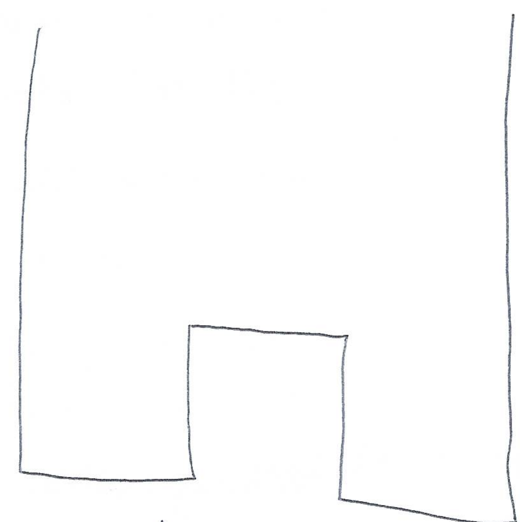
$$T = \frac{4E(V_0-E)}{4E(V_0-E)+V_0^2\sinh^2(ka)}$$

$$\sinh z = \frac{e^{z}-e^{-z}}{2}$$

$$V(x) = \begin{cases} V_0 & 0<x<a \\ 0 & \text{else} \end{cases}$$

$ka \gg 1$ limit

$$T \propto e^{-2ka}$$

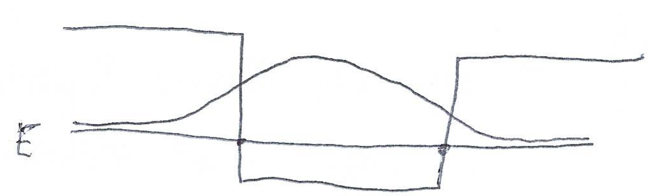


TISE

$$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi$$

$$\psi'' = -\frac{2m(E-V(x))}{\hbar^2}\psi = \frac{2m(V(x)-E)}{\hbar^2}\psi$$

$[E]=J \quad [\psi]=\frac{1}{\sqrt{m}}$



intersections are meaningless for wave function and potential

Intersections of energy and potential matter. They are turning points

Drawing Wave Functions

1. Draw the potential and determine the spectrum
2. Pick relevant energy and sketch it as horizontal line. Determine CARs and CFRs and CTPs
3. General Considerations
   a) Low energy bound states. Determine amount of nodes and locations
      - Symmetric potentials the bound states are either symmetric or anti-symmetric
      - ground state has zero nodes
      - A CFR is bounded on both sides by CARs have at most one node
      - A CFR that extends to infinity can not contain any nodes

   b) High energy bound states
      Sketch " classical probability distribution into the CARs.
      This acts as a rough envelope for the oscillation of the wave function

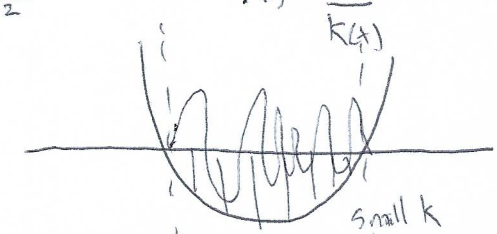
   c) Scattering states. You can use a "guess" of the transmission coefficient T to get an extremely rough comparison of the amplitudes for the incident and transmitted pieces
4. Classically Allowed Regions $k(x) = \sqrt{\frac{2m(E-V(x))}{\hbar^2}}$ $\boxed{\psi'' = -k^2(x)\psi \quad \text{CAR}}$ V(x) constant $\to \psi = \sin + \cos$
   a) in CARs we get oscillations and the concavity bend towards the axis

   Everything bent to zero

   b) define local wave number $k(x) = \sqrt{\frac{2m(E-V)}{\hbar^2}}$

   $\lambda(x) = \frac{2\pi}{k(x)}$

The bigger E-V is
the larger k is and the
smaller λ is and the smaller amplitude



Small k large k Small k

If $V(x)$ constant $\Rightarrow$ amplitude is constant

Only track amplitude changes in "highly" excited states

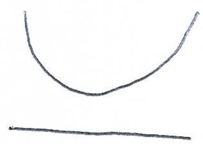
CFRs

$$K(x) = \sqrt{\frac{2m(V-E)}{\hbar^2}} \qquad \psi'' = K^2\psi$$

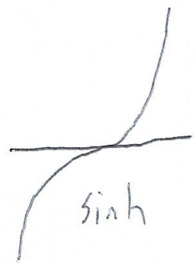
pushes away from axis

$V(x)$ constant

$\psi = e^{Kx}$ and e^{-Kx}



cosh



sinh

wave function exhibit exponential behavior in CFRs

$\frac{1}{K}$ = attenuation length (half-life type deal)

~~(half-life~~

The further E is below V The sharper the attenuation/growth/decay

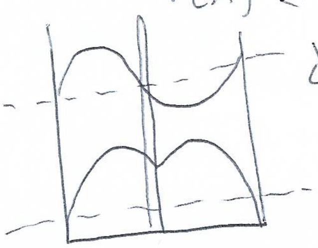
If the CFR extends to infinity we can only have decay to infinity

Boundary conditions at CTPs

Energy Eigenfunctions must be continuous everywhere.

The derivative of an energy eigenfunction must be continuous at all points where our potential is finite.

$V(x) = \alpha\delta(x-x_0)$



discontinuity of ψ', $\Delta\psi'(x_0) = \psi'(x_0^+) - \psi'(x_0^-) = \frac{2m\alpha}{\hbar^2}\psi(x_0)$

V(x) symmetric

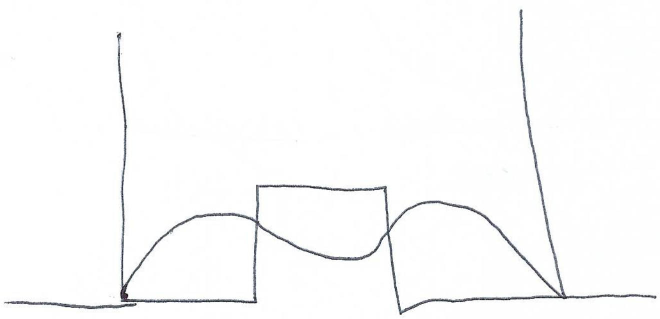
ψ ~~atternates~~ alternates between symmetric and antisymmetric

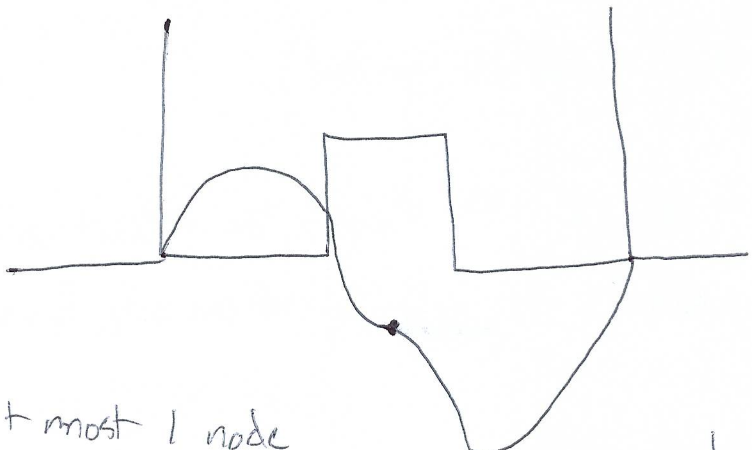
<u>Nodes</u>

Ground states have 0 nodes

1st excited state has 1 node

each successive state gets a node more

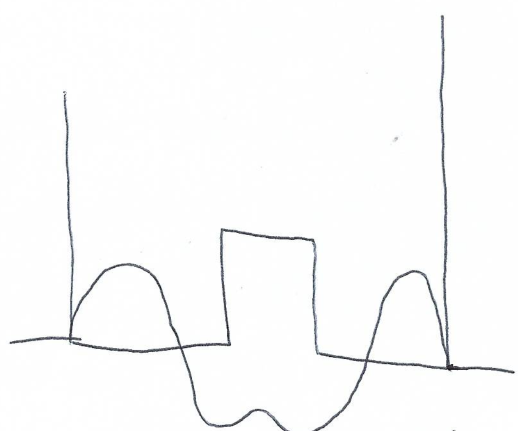




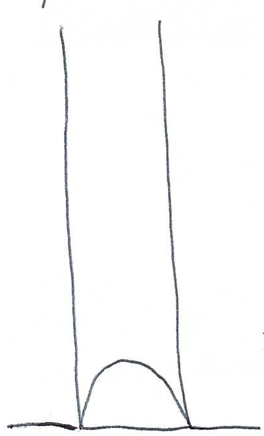
Each CFR contains at most 1 node

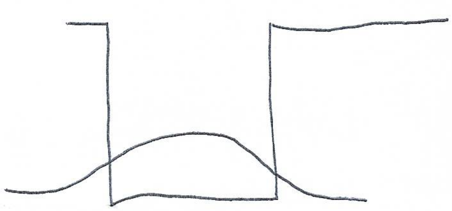
The curviness tells us about our energy

WKB Theory



supposed to be symmetric





Phys 137A 11 Jul 19

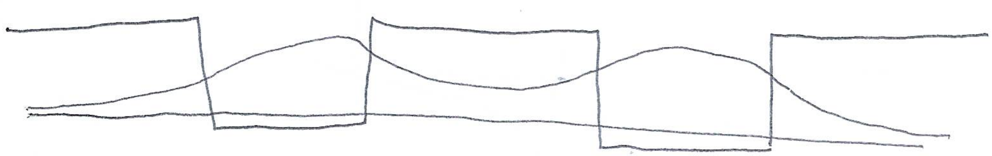
Part 2: Formalism

Two-state system   Two basis states

Spin, neutrinos, Ammonium, Kayons, MRIs, quantum computing

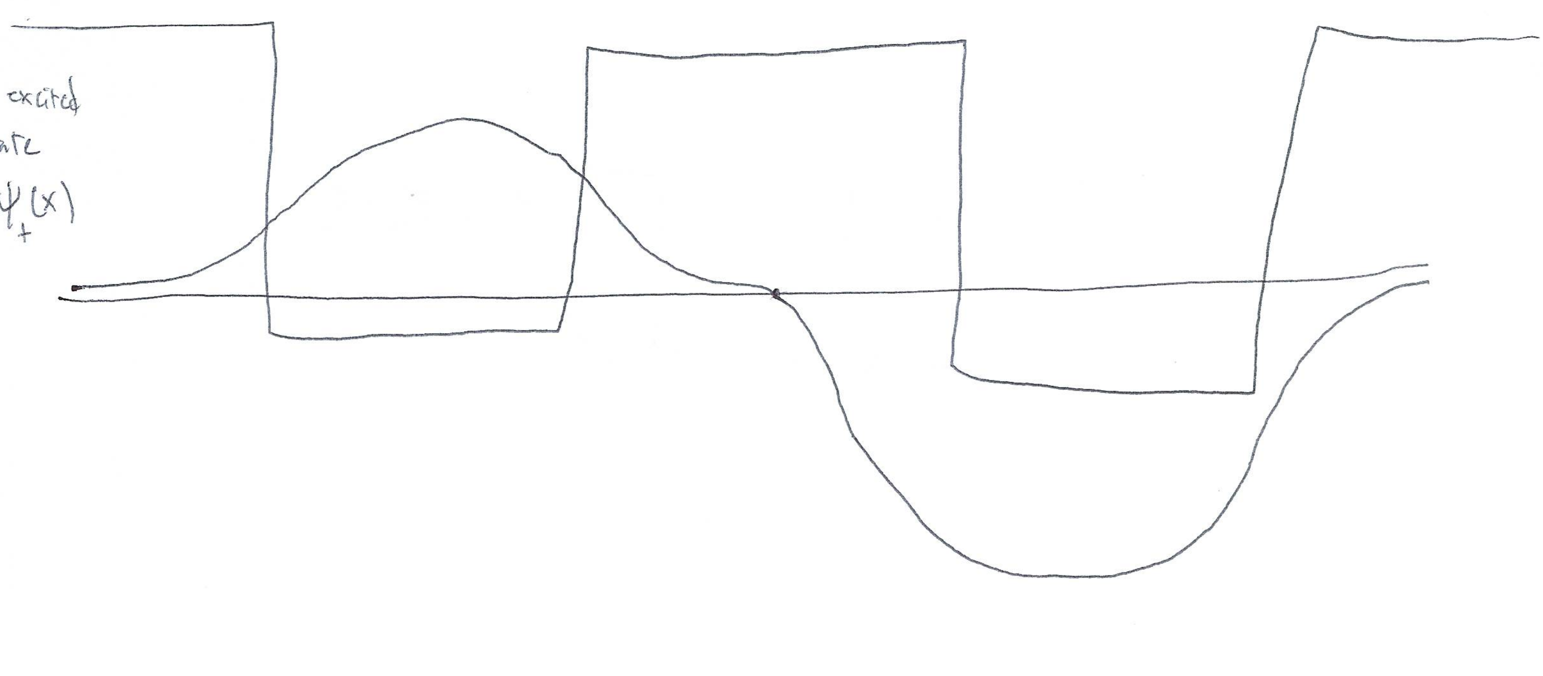
Toy model

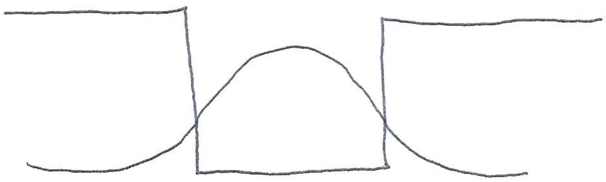
Double Finite Square Well



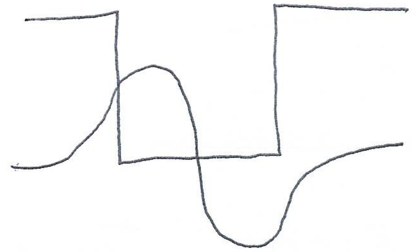
Ground state $\psi_-(x)$

1st excited state $\psi_+(x)$

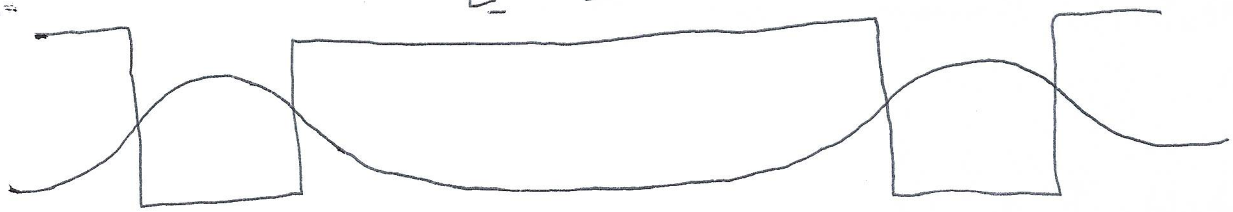




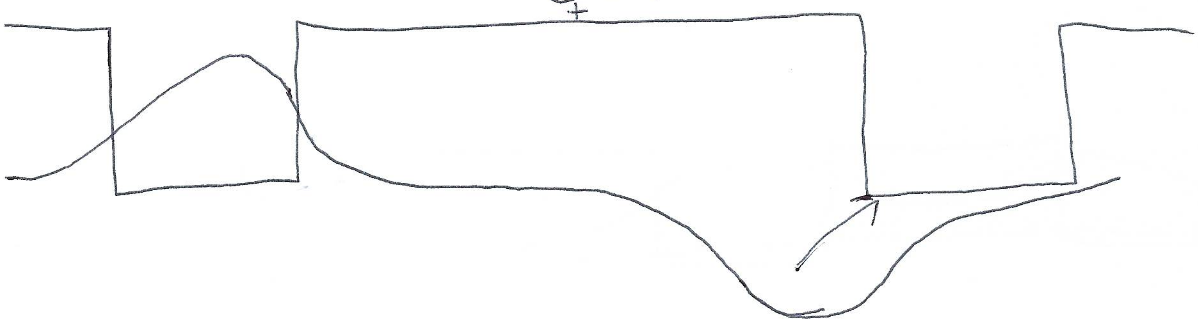
E_0

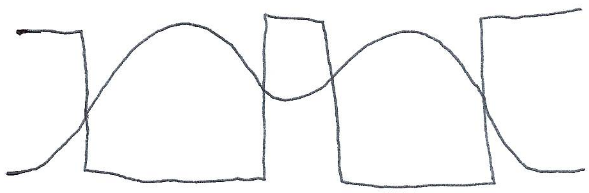


$E_- \approx E_0$

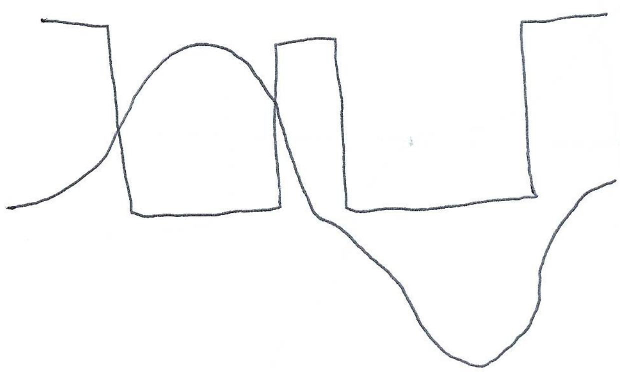


$E_+ \approx E_0$



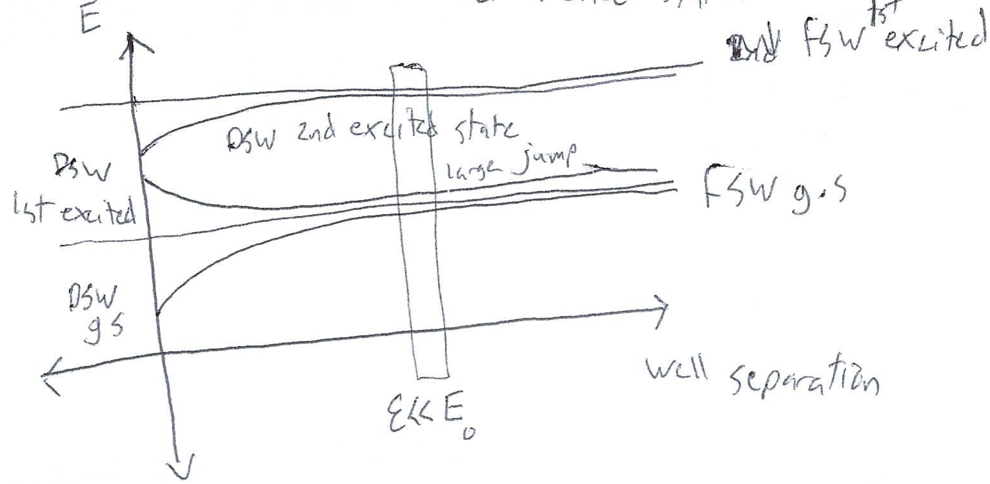


$E_- = E_0 - \varepsilon$



$E_+ = E_0 + \varepsilon$

Second Excited State



Toy Model system

All wave functions in DSW system of the form

$$\psi(x) = c_-\psi_-(x) + c_+\psi_+(x) \quad , \; c_\mp \in \mathbb{C}$$

$$\langle E_-|E_+\rangle = 0 \quad \langle E_+|E_+\rangle = 1 \quad \langle E_-|E_-\rangle = 1$$

Normalization:

$$|c_-|^2 + |c_+|^2 = 1 \qquad c_\mp = \langle E_\mp|\psi\rangle = \int_{-\infty}^{\infty} \psi_\mp^* \psi \, dx$$

Probability$(E_\mp) = |c_\mp|^2$ $\quad \Psi(x,t) = c_-\psi_- e^{-iE_-t/\hbar} + c_+\psi_+ e^{-iE_+t/\hbar}$

$$\Psi(x,t) = e^{-iE_0t/\hbar}\left(c_-\psi_- e^{i\varepsilon t/\hbar} + c_+\psi_+ e^{-i\varepsilon t/\hbar}\right)$$

$$\psi(x) \doteq \begin{bmatrix} c_- \\ c_+ \end{bmatrix} = \begin{bmatrix} \langle E_-|\psi\rangle \\ \langle E_+|\psi\rangle \end{bmatrix}$$

means $\psi(x) = c_-\psi_- + c_+\psi_+$

$$\Psi(x,t) = e^{-iE_0 t/\hbar}\begin{bmatrix} c_- e^{i\varepsilon t/\hbar} \\ c_+ e^{-i\varepsilon t/\hbar}\end{bmatrix}$$

Left well state

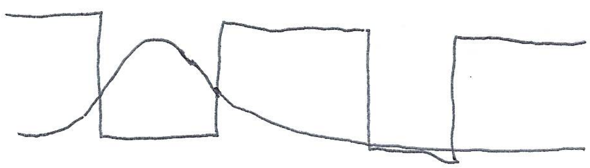
$$\psi_L(x) = c\psi_-(x) + c\psi_+(x) \qquad |c|^2 + |c|^2 = 1 \qquad |c|^2 = \tfrac{1}{2}$$

$$= \frac{1}{\sqrt{2}}\psi_- + \frac{1}{\sqrt{2}}\psi_+ \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \qquad c = \frac{1}{\sqrt{2}}$$

Right well state

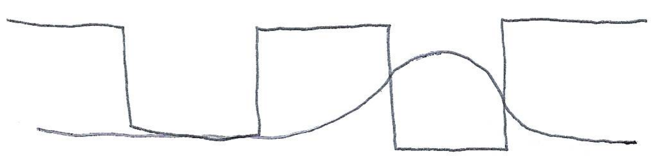
$$\psi_R(x) = \frac{1}{\sqrt{2}}\psi_- - \frac{1}{\sqrt{2}}\psi_+ \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

ψ_L



Not energy eigenfunctions

ψ_R



States are orthogonal

$$\langle E\rangle_L = |c_-|^2(E_0 - \varepsilon) + |c_+|^2(E_0 + \varepsilon) = E_0$$

$$\psi(x) = c_L\psi_L + c_R\psi_R \qquad c_L = \langle L|\psi\rangle \quad c_R = \langle R|\psi\rangle$$

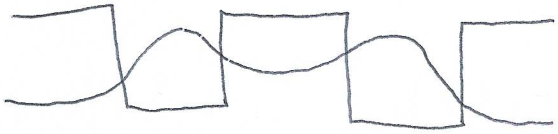
$$\psi(x) \doteq \begin{bmatrix}c_L\\c_R\end{bmatrix} = \begin{bmatrix}\langle L|\psi\rangle\\ \langle R|\psi\rangle\end{bmatrix} \qquad \psi_\mp = \frac{1}{\sqrt{2}}(\psi_L(x) \pm \psi_R(x))$$

$$\psi_+(x) \doteq \begin{bmatrix}0\\1\end{bmatrix}_{[e]} \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}_{[w]}$$

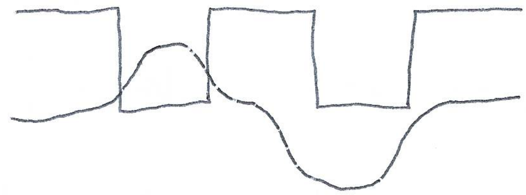
Main Idea The states of a quantum system are represented by vectors and questions that we want to ask about our system are answered with vector manipulations

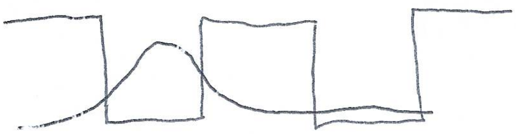
Phys 137A 15 Jul 19

ψ_- # $E_0 - \varepsilon$

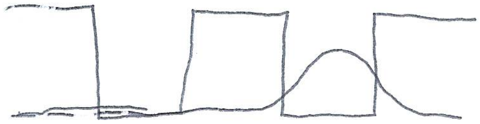


ψ_+ # $E_0 + \varepsilon$





ψ_L



ψ_R

Arrows in the plane $\mathbb{R}^2$ $\vec{V}$

set of objects we can add and scale,

Basis $\hat{y}$ $\hat{x}$ $\quad \vec{V} = V_x \hat{x} + V_y \hat{y} \doteq \begin{bmatrix} V_x \\ V_y \end{bmatrix}_{[x,y]} = \begin{bmatrix} \hat{x}\cdot\vec{V} \\ \hat{y}\cdot\vec{V} \end{bmatrix}$

"is represented by"

3D Kinematics $\mathbb{R}^3$ physical objects

- quantities (e.g. Force)

$\vec{F} \doteq \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$

$\Psi_{arbitrary}(x) = c_- \psi_-(x) + c_+ \psi_+(x) = c_L \psi_L(x) + c_R \psi_R(x)$

$\Psi \doteq \begin{bmatrix} c_L \\ c_R \end{bmatrix}_{well} \doteq \begin{bmatrix} c_- \\ c_+ \end{bmatrix}_{energy}$

1D QM

Space of normalizable functions, $L^2(\mathbb{R})$

$\psi(x)$ like V_x, V_y Components $\psi_{x=1}$ $\psi_{x=0.99}$ $\psi_{x=0.98}$ etc.

$\phi(p)$

$\{c_n\}$

Vectors in QM are called <u>Kets</u>

$|\text{state name}\rangle$ $\quad \psi_- = |-\rangle \quad \psi_+ = |+\rangle \quad \psi_L = |L\rangle \quad \psi_R = |R\rangle$

$$|L\rangle = \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle) \qquad |R\rangle = \frac{1}{\sqrt{2}}(|-\rangle - |+\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle) \qquad |+\rangle = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle)$$

<u>Postulate 1</u>

A given quantum system is described by a Hilbert Space
The state of a quantum system is given by a normalized ket in the Hilbert Space

<u>Basis</u> A set of linearly independent vectors/Kets that span the Hilbert Space

Linear Independence

Basis $\{|e_i\rangle\}$ $\quad$ if $\sum_i c^i |e_i\rangle = 0$ $\quad$ then all $c^i = 0$

Span: Given $|\alpha\rangle$, can always find $\{c^i\}$
so $|\alpha\rangle = \sum_i c^i |e_i\rangle$

$\uparrow \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

does not span
is linear independence

$\uparrow \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \rightarrow \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad \begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

linear independent
Spans

$\uparrow \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \nearrow \doteq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} a \\ b \end{bmatrix} = (a-b)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

linear independent
and Spans

$\begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}2\\8\end{bmatrix}$

not lin ind

does not span

$\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}$

span

not lin ind

Energy basis

$|-\rangle, |+\rangle$

Well basis

$|L\rangle, |R\rangle$

Simple Harmonic Oscillator Basis

n^{th} energy eigenfunction $\psi_n(x) \Rightarrow |n\rangle$ $n = 0, 1, 2, \ldots$

$|0\rangle$ is ground state

$|1\rangle$ is 1st excited state etc

$\cancel{|\alpha\rangle = \sum \langle \alpha | n \rangle |n\rangle}$

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

Matrix Representation

~~Specify~~ Specify a basis

$|\alpha\rangle \doteq \begin{bmatrix}\vdots\\ \vdots\end{bmatrix}$ $\vec{\Psi}_\alpha$ $\vec{\alpha}_{[energy]}$ $\vec{\alpha}_{[well]}$

A Ket is an $(N \times 1)$ matrix or a column vector

$$|\alpha\rangle = \sum_{i=1}^{N} \alpha^i |e_i\rangle \doteq \vec{\Psi}_\alpha = \begin{bmatrix}\alpha^1\\ \vdots \\ \alpha^N\end{bmatrix}$$

In a matrix representation, upper indices represent row and lower indices represent column

$$|L\rangle \doteq \begin{bmatrix}1/\sqrt{2}\\ \frac{1}{\sqrt{2}}\end{bmatrix}_{[e]} \qquad |R\rangle \doteq \begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}_{[e]}$$

A machine that eats kets/vectors and returns a scalar (in a linear way) is a dual vector or bras

In Dirac Notation a bra is written ~~and~~ with a label and decoration

$$\langle bra|$$

$$\langle bra|(|ket\rangle) = \langle bra|ket\rangle \quad \text{a number "bracket"}$$

"Actions" of quantum objects are represented by matrix Multiplication

ket: $(N \times 1)$ matrix

bracket: number (1×1) matrix

bra: $(1 \times N)$ matrix

$$\langle \beta| \doteq [\beta_1 \; \beta_2 \cdots \beta_N] = \vec{\Psi}_\beta^\dagger$$

<u>Dual Basis</u>

$\{|e_i\rangle\}$ basis kets

Define $\{\langle e^i|\}$ basis bras by $\langle e^i|e_j\rangle \equiv \delta^i_j$

$$\langle \beta| = \sum_i \beta_i \langle e^i|$$

Phys 137A 16Jul19

Basis kets $\{|e_i\rangle\}$

Basis Bras $\{\langle e^i|\}$

$\langle e^i|e_i\rangle = \delta^i_j$

States are kets $|\alpha\rangle$

Matrix representation $\Rightarrow$ Column vector

$$|\alpha\rangle = \sum_i \alpha^i |e_i\rangle \doteq \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^N \end{bmatrix}$$

Linear Maps from kets to scalars are bras $\langle\beta|$

Mat Rep $\Rightarrow$ Row vectors

$$\langle\beta| = \sum_i \beta_i \langle e^i| \doteq [\beta_1 \ \beta_2 \cdots \beta_N]$$

Bra acting on ket gives bracket

$$\langle\beta|\alpha\rangle = [\beta_1 \cdots \beta_N] \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^N \end{bmatrix} = \sum_i \beta_i \alpha^i$$

$$\langle\beta|(|\alpha\rangle) = \langle\beta|\alpha\rangle$$

$$\left(\sum_i \beta_i \langle e^i|\right)\left(\sum_j \alpha^j |\alpha_j\rangle\right) = \sum_i \sum_j \beta_i \alpha^j \langle e^i|e_j\rangle = \sum_i \sum_j \beta_i \alpha^j \delta^i_j = \sum_i \beta_i \alpha^i$$

$$\langle e^i|\alpha\rangle = \langle e^i|\left(\sum_j \alpha^j |e_j\rangle\right) = \sum_j \alpha^j \langle e^i|e_j\rangle = \sum_j \alpha^j \delta^i_j = \alpha^i$$

$$|\alpha\rangle \doteq \begin{bmatrix} \langle e^1|\alpha\rangle \\ \vdots \\ \langle e^N|\alpha\rangle \end{bmatrix}$$

$\beta_i = \langle\beta|e_i\rangle$ $\quad \langle\beta| \doteq [\langle\beta|e_1\rangle \cdots \langle\beta|e_N\rangle]$

$$|e_1\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad |e_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\langle e_1| = [1 \ 0 \cdots 0]$

$\langle e_2| = [0 \ 1 \ 0 \cdots 0]$

DSW

energy basis $|-\rangle, |+\rangle$ well basis $|L\rangle, |R\rangle$

$$|-\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle) \qquad |+\rangle = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle)$$

$$|-\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}_{[w]} \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}\langle L|-\rangle\\ \langle R|-\rangle\end{bmatrix}_{[w]} \qquad |+\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}_{[w]} = \frac{1}{\sqrt{2}}\begin{bmatrix}\langle L|+\rangle\\ \langle R|+\rangle\end{bmatrix}_{[w]}$$

$$|-\rangle \doteq \begin{bmatrix}\langle -|-\rangle\\ \langle +|-\rangle\end{bmatrix}_{[e]} = \begin{bmatrix}1\\0\end{bmatrix}_{[e]} \qquad |+\rangle \doteq \begin{bmatrix}\langle -|+\rangle\\ \langle +|+\rangle\end{bmatrix}_{[e]} = \begin{bmatrix}0\\1\end{bmatrix}_{[e]}$$

Inner Product

eats two vectors to get a number

$$\mathbb{I}(|\alpha\rangle, |\beta\rangle)$$

Conjugate Symmetry

$$\mathbb{I}(|\beta\rangle, |\alpha\rangle) = [\mathbb{I}(|\alpha\rangle, |\beta\rangle)]^*$$

Linear in 2nd slot

$$\mathbb{I}(|\alpha\rangle, b|\beta\rangle + c|\gamma\rangle) = b\,\mathbb{I}(|\alpha\rangle, |\beta\rangle) + c\,\mathbb{I}(|\alpha\rangle, |\gamma\rangle)$$

Antilinear in 1st slot

$$\mathbb{I}(a|\alpha\rangle + b|\beta\rangle, |\gamma\rangle) = a^*\mathbb{I}(|\alpha\rangle, |\gamma\rangle) + b^*\mathbb{I}(|\beta\rangle, |\gamma\rangle)$$

sesquilinearity

Positive Definiteness

$$\mathbb{I}(|\alpha\rangle, |\alpha\rangle) \geq 0$$

$\equiv \|\alpha\|^2$ "norm-squared"

w The only ket for which $\mathbb{I}(|\alpha\rangle, |\alpha\rangle) = 0$ is the zero vector

$\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^*$

$\langle \alpha | (b|\beta\rangle + c|\gamma\rangle) = b\langle \alpha|\beta\rangle + c\langle \alpha|\gamma\rangle$

$(a\langle \alpha| + b\langle \beta|) |\gamma\rangle = a\langle \alpha|\gamma\rangle + b\langle \beta|\gamma\rangle$

$\langle \alpha|\alpha\rangle = \|\alpha\|^2 \geq 0$

$\mathbb{I}(|e_i\rangle, |e_j\rangle) = \langle e^i | e_j \rangle = \delta^i_j$

$\mathbb{I}(|\alpha\rangle, |\beta\rangle) = \mathbb{I}(\sum_i \alpha^i |e_i\rangle, \sum_j \beta^j |e_j\rangle)$

$= \sum_i \sum_j (\alpha^i)^* \beta^j \, \mathbb{I}(|e_i\rangle, |e_j\rangle) = \sum_i (\alpha^i)^* \beta^i$

$\|\alpha\|^2 = \sum_i |\alpha^i|^2$ $\qquad \mathbb{I}(|\alpha\rangle, |\beta\rangle) = \sum_i (\alpha^i)^* \beta^i = \int \psi^*_\alpha(x)\, \psi_\beta(x)\, dx = \langle \alpha|\beta\rangle$

Schwartz Inequality

$|\langle \alpha|\beta\rangle|^2 \leq \langle \alpha|\alpha\rangle \langle \beta|\beta\rangle$

Ket $|\alpha\rangle$ $\qquad$ Dual Bra to $|\alpha\rangle$ is $\langle \alpha|$

$\langle \alpha| \equiv \mathbb{I}(|\alpha\rangle, \;)$

$\langle \alpha|\beta\rangle = \mathbb{I}(|\alpha\rangle, |\beta\rangle)$

$\langle \alpha| = |\alpha\rangle^\dagger \qquad |\beta\rangle = \langle \beta|^\dagger$

$|\alpha\rangle^{\dagger\dagger} = |\alpha\rangle \qquad \langle \beta|^{\dagger\dagger} = \langle \beta|$

$(a|\alpha\rangle + b|\beta\rangle)^\dagger = (a|\alpha\rangle)^\dagger + (b|\beta\rangle)^\dagger = a^*\langle \alpha| + b^*\langle \beta|$

$|e_i\rangle^\dagger = \langle e^i|$

$|\alpha\rangle^\dagger = \langle \alpha| = \sum_i (\alpha^i)^* \langle e^i| = \sum_i \alpha_i \langle e^i| \qquad \alpha_i = (\alpha^i)^*$

$$|\alpha\rangle \doteq \begin{bmatrix} a \\ b \end{bmatrix} \qquad \langle\alpha| \doteq \begin{bmatrix} a^* & b^* \end{bmatrix}$$

The matrix representation of a dual bra is the Hermitian conjugate of matrix representation of the ket

$$|\alpha\rangle \doteq \vec{\alpha} \qquad \langle\alpha| \doteq \vec{\alpha}^\dagger = \vec{\alpha}^{T*}$$

Operators

Operators ~~eat~~ are linear functions from ket to other ket

$$\hat{Q} \qquad \hat{Q}|\alpha\rangle = \cancel{|\alpha} \, |\beta\rangle$$

$$\hat{Q}(a|\alpha\rangle + b|\beta\rangle) = a\hat{Q}|\alpha\rangle + b\hat{Q}|\beta\rangle$$

$$\hat{Q}(\langle\beta|) = \langle\beta|\hat{Q} = \langle\gamma|$$

$$(\langle\beta|\hat{Q})|\alpha\rangle = \langle\beta|(\hat{Q}|\alpha\rangle) \equiv \underbrace{\langle\beta|\hat{Q}|\alpha\rangle}_{\text{Matrix Element}}$$

Expectation Value

$$\langle Q\rangle \equiv \langle\alpha|\hat{Q}|\alpha\rangle \qquad |\alpha\rangle \doteq \psi_\alpha(x) \qquad \hat{Q}|\alpha\rangle \doteq \hat{Q}\psi_\alpha(x)$$

$$= \int_{-\infty}^{\infty} \psi_\alpha^* \hat{Q} \psi_\alpha \, dx \qquad \langle\alpha| \doteq \int_{-\infty}^{\infty} \psi_\alpha^* \underline{\quad} \, dx$$

Composition of operators

$$\hat{Q}, \hat{P} \qquad (\hat{Q}\hat{P})|\alpha\rangle = \hat{Q}(\hat{P}|\alpha\rangle)$$

Commutator $[\hat{Q}, \hat{P}] = \hat{Q}\hat{P} - \hat{P}\hat{Q}$

Operator x ket = ket
(NxN) (Nx1) (Nx1)

Operators are represented by square matrices

bra x Operator = bra
(1xN) (NxN) (1xN)

Operator x operator = operator
(NxN) (NxN) (NxN)

Phys 137A 17Jul19

Operators Represented by Square Matrices

$$\hat{A} \doteq \vec{A} = \begin{bmatrix} A^1_1 & \cdots & A^1_N \\ \vdots & \ddots & \vdots \\ A^N_1 & \cdots & A^N_N \end{bmatrix} \qquad A^i_j = \langle e^i|\hat{A}|e_j\rangle$$

$$= \begin{bmatrix} \langle e^1|\hat{A}|e_1\rangle & \cdots & \langle e^1|\hat{A}|e_N\rangle \\ \vdots & \ddots & \vdots \\ \langle e^N|\hat{A}|e_1\rangle & \cdots & \langle e^N|\hat{A}|e_N\rangle \end{bmatrix}$$

$|\mp\rangle$ = ground state $E_0 - \varepsilon$ $\qquad |L\rangle = \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle)$

1st excited state $E_0 + \varepsilon$ $\qquad |R\rangle = \frac{1}{\sqrt{2}}(|-\rangle - |+\rangle)$

$$\hat{H} \doteq \vec{H}_{[energy]} = \begin{bmatrix} \langle -|\hat{H}|-\rangle & \langle -|\hat{H}|+\rangle \\ \langle +|\hat{H}|-\rangle & \langle +|\hat{H}|+\rangle \end{bmatrix}_{[energy]} = \begin{bmatrix} E_0 - \varepsilon & 0 \\ 0 & E_0 + \varepsilon \end{bmatrix}_{[energy]}$$

$$H|-\rangle = (E_0 - \varepsilon)|-\rangle$$

$$H|+\rangle = (E_0 + \varepsilon)|+\rangle$$

$$\vec{H}_{[W]} = \begin{bmatrix} \langle L|\hat{H}|L\rangle & \langle L|\hat{H}|R\rangle \\ \langle R|\hat{H}|L\rangle & \langle R|\hat{H}|R\rangle \end{bmatrix}_{[well]} = \begin{bmatrix} E_0 & -\varepsilon \\ -\varepsilon & E_0 \end{bmatrix}_{[W]}$$

$$\hat{H}|L\rangle = \frac{1}{\sqrt{2}}\left((E_0 - \varepsilon)|-\rangle + (E_0 + \varepsilon)|+\rangle\right) = E_0|L\rangle - \varepsilon|R\rangle$$

$$\hat{H}|R\rangle = \cancel{E_0|L\rangle + \varepsilon|R\rangle} \; E_0|R\rangle - \varepsilon|L\rangle$$

$$\begin{bmatrix}\cdot\\ \cdot\\ \cdot\end{bmatrix}\begin{bmatrix}\cdot & \cdot & \cdot\end{bmatrix} = \begin{bmatrix}\cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\end{bmatrix}$$

ket bra operator

$|\alpha\rangle\langle\beta|$ ketbras

Outer Product

$\hat{A} = |\alpha\rangle\langle\beta| \qquad \hat{A}|\gamma\rangle = |\alpha\rangle\langle\beta|\gamma\rangle = \#\, |\alpha\rangle$

$\langle\gamma|\hat{A} = \langle\gamma|\alpha\rangle\langle\beta| = \#\, \langle\beta|$

$A^i_j = \langle e^i|\hat{A}|e_j\rangle = \langle e^i|\alpha\rangle\langle\beta|e_j\rangle = \alpha^i\beta_j$

$|e_1\rangle \doteq \begin{bmatrix}1\\0\end{bmatrix} \qquad |e_2\rangle \doteq \begin{bmatrix}0\\1\end{bmatrix}$

$\langle e_1| \doteq \begin{bmatrix}1 & 0\end{bmatrix} \qquad \langle e_2| \doteq \begin{bmatrix}0 & 1\end{bmatrix}$

$|e_1\rangle\langle e^1| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \qquad |e_2\rangle\langle e^1| = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$

$|e_1\rangle\langle e^2| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \qquad |e_2\rangle\langle e^2| = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

$\vec{A} = \begin{bmatrix}a & b\\c & d\end{bmatrix} = a\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + c\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + d\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

$= a|e_1\rangle\langle e^1| + b|e_1\rangle\langle e^2| + c|e_2\rangle\langle e^1| + d|e_2\rangle\langle e^2|$

$\hat{A} = \sum_{i,j} A^i_j |e_i\rangle\langle e^j|$

$$\hat{H} = (E_0 - \varepsilon)|-\rangle\langle -| + (E_0 + \varepsilon)|+\rangle\langle +|$$

$$= E_0|L\rangle\langle L| - \varepsilon|L\rangle\langle R| \cancel{+E_0} - \varepsilon|R\rangle\langle L| + E_0|R\rangle\langle R|$$

The Identity Operator $\hat{\mathbb{I}}$

$$\hat{\mathbb{I}}|\alpha\rangle = |\alpha\rangle \qquad \cancel{\hat{\mathbb{I}}}\langle\beta|\hat{\mathbb{I}} = \langle\beta| \qquad \hat{A}\hat{\mathbb{I}} = \hat{\mathbb{I}}\hat{A} = \hat{A}$$

$$\hat{\mathbb{I}} \doteq \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & & 0 & 1 \end{bmatrix} \qquad \langle e^i|\hat{\mathbb{I}}|e_j\rangle = \langle e^i|e_j\rangle = \delta^i_j$$

$$= \hat{\mathbb{I}} = \sum_{ij} \delta^i_j |e_i\rangle\langle e^j|$$

For any orthonormal basis $\hat{\mathbb{I}} = \sum_i |e_i\rangle\langle e^i|$ Resolution of the identity

$$\langle\alpha|\beta\rangle = \langle\alpha|\hat{\mathbb{I}}|\beta\rangle = \langle\alpha|\left(\sum_i |e_i\rangle\langle e^i|\right)|\beta\rangle$$

$$= \sum_i \langle\alpha|e_i\rangle\langle e^i|\beta\rangle = \sum_i \alpha_i \beta^i$$

The adjoint or Hermitian Conjugate Operator $\hat{A}$

adjoint $\hat{A}^\dagger$

$$\mathbb{I}(|\alpha\rangle, \hat{A}|\beta\rangle) = \langle\alpha|\hat{A}|\beta\rangle$$

$$\cancel{\mathbb{I}(|\alpha\rangle} \quad \mathbb{I}(\hat{A}^\dagger|\alpha\rangle, |\beta\rangle) = (\hat{A}^\dagger|\alpha\rangle)^\dagger|\beta\rangle$$

$$\langle\alpha|\hat{A} = (\hat{A}^\dagger|\alpha\rangle)^\dagger \qquad (\hat{A}|\alpha\rangle)^\dagger = \langle\alpha|\hat{A}^\dagger \qquad \hat{A}^{\dagger\dagger} = \hat{A}$$

$$(\langle\alpha|\hat{A})^\dagger = \hat{A}^\dagger|\alpha\rangle$$

What do daggers do?

complex numbers $c^\dagger = c^*$

ket: $|\alpha\rangle^\dagger = \langle\alpha|$

bra: $\langle\beta|^\dagger = |\beta\rangle$

Operators: $\hat{A}^\dagger$

Wellness Operator

$\hat{W}|L\rangle = |L\rangle$

$\hat{W}|R\rangle = 2|R\rangle$

$\vec{W} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$(\langle\alpha|\hat{A}|\beta\rangle)^\dagger = \langle\beta|\hat{A}^\dagger|\alpha\rangle$$

$$((b_i+1)(b_i+2)\cdots(b_i+N))^\dagger = ((b_i+N)^\dagger(b_i+N-1)^\dagger\cdots(b_i+1)^\dagger)^{\dagger}$$

$$\langle\alpha|\hat{A}|\beta\rangle^* = \langle\alpha|\hat{A}|\beta\rangle^\dagger = \langle\beta|\hat{A}^\dagger|\alpha\rangle$$

$$A^i_{\ j} = \langle e^i|\hat{A}|e_j\rangle \qquad (A^\dagger)^i_{\ j} = \langle e^i|\hat{A}^\dagger|e_j\rangle = \langle e^j|\hat{A}|e_i\rangle^* = (A^j_{\ i})^*$$

$$\vec{A}^\dagger = A^{T*}$$

<u>Hermitian Operators</u> $\hat{A}^\dagger = \hat{A}$ $\quad \vec{A}^\dagger = \vec{A}$

$$\vec{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \vec{A}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} \Rightarrow \begin{matrix} a, d \text{ are real} \\ c^* = b \end{matrix}$$

Observables are Hermitian Operators

$$\vec{A} = \begin{bmatrix} a & z \\ z^* & d \end{bmatrix}$$

Unitary Operators $U^\dagger = U^{-1} \Rightarrow U^\dagger U = \mathbb{I}$

Transformations are Unitary Operators

Hermitian Operators have only real eigenvalues

<u>Postulate II</u> Physically Measurable quantities (observables) of a system are represented by Hermitian Operators

<u>Postulate III</u> (Measurement Possibilities) Given an observable A and associated hermitian operator $\hat{A}$ the only possible results of a measurement of A are the eigenvalues of $\hat{A}$

Phys 137A 18Jul19

Postulate I (States) A given quantum system is described by a Hilbert space. The state of a quantum system is given by a normalized ket in the Hilbert space of the system.

Postulate II (Observables) Physically measurable quantities (observables) of a system are represented by Hermitian operators on the Hilbert space for the system.

Postulate III (Measurement Possibilities) Given an observable A with associated Hermitian Operator $\hat{A}$, the only possible results of measurement of A are the eigenvalues of $\hat{A}$

$$\alpha^i = \langle e^i | \alpha \rangle \Rightarrow |\alpha\rangle \doteq \begin{bmatrix} \langle e^1|\alpha\rangle \\ \langle e^2|\alpha\rangle \\ \vdots \end{bmatrix}$$

$$\beta_i = \langle \beta | e_i \rangle \Rightarrow \langle\beta| \doteq \begin{bmatrix} \langle \beta|e_1\rangle & \cdots \end{bmatrix}$$

$$A^i_j = \langle e^i|\hat{A}|e_j\rangle \qquad \hat{A} \doteq \begin{bmatrix} \langle e^1|\hat{A}|e_1\rangle & \cdots & \\ \vdots & \ddots & \\ \vdots & & \langle e^N|\hat{A}|e_N\rangle \end{bmatrix}$$

$$\vec{W}_{[e]} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \qquad \vec{W}_{[w]} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

An operator $\hat{A}$ has eigenvalue λ if there exists some non-zero ket $|q\rangle$ such that $\hat{A}|q\rangle = \lambda|q\rangle$

Any ket $|q\rangle$ satisfying this is an eigenvector (eigenket, eigenstate) of the operator $\hat{A}$ belonging to eigenvalue λ.

$\mathbb{1}$

The set of all eigenvalues of $\hat{A}$ is the spectrum of $\hat{A}$

$\hat{A} \doteq \overleftrightarrow{A} \qquad |q\rangle \doteq \vec{q}$

$\overleftrightarrow{A}\vec{q} = \lambda\vec{q}$

$$\overleftrightarrow{W}_{[e]} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$

$$\begin{vmatrix} \frac{3}{2}-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2}-\lambda \end{vmatrix}$$

$$\left(\tfrac{3}{2}-\lambda\right)^2 - \tfrac{1}{4} = 0$$

$$\lambda^2 - 3\lambda + \tfrac{9}{4} - \tfrac{1}{4} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda-1)(\lambda-2) = 0$$

$$\lambda = \{1, 2\}$$

λ an eigenvalue iff $\det(\overleftrightarrow{A} - \lambda\overleftrightarrow{\mathbb{I}}) = 0$

To find the spectrum, write $\hat{A}$ as $\overleftrightarrow{A}$ create "characteristic Polynomial" $\det(\overleftrightarrow{A} - \lambda\overleftrightarrow{\mathbb{I}})$ and find the roots

$$(\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\cdots(\lambda-\lambda_n)^{n_n} = 0$$

$\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum

n_n is the degeneracy of λ_n

$\lambda = 1$

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \vec{w}_1 \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{q}_L$$

$\frac{1}{2}(a-b) = 0 \qquad a = b$

$+\frac{1}{2}(-a+b) = 0$

$\lambda = 2$

$$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \vec{w}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{q}_R$$

$a + b = 0$

3 state system

$$\hat{H} \doteq \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_2 \end{bmatrix} = \overset{\leftrightarrow}{H} \qquad \lambda = \{E_1, E_2\}$$

$$\det(\overset{\leftrightarrow}{H} - \lambda \mathbb{1}) = (\lambda - E_1)(\lambda - E_2)^2 \qquad \vec{\psi}_1 =$$

$\lambda = E_1$ with degeneracy 1

$\lambda = E_2$ with degeneracy 2

$$\lambda = E_1 \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & E_2 - E_1 & 0 \\ 0 & 0 & E_2 - E_1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ b(E_2 - E_1) \\ c(E_2 - E_1) \end{bmatrix}$$

$b = c = 0, \; a = a$

$$\vec{\psi}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = E_2 \quad \begin{bmatrix} E_1 - E_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a(E_1 - E_2) \\ 0 \\ 0 \end{bmatrix} \qquad a = 0 \qquad \begin{bmatrix} 0 \\ b \\ c \end{bmatrix}$$

$$\frac{1}{\sqrt{|b|^2 + |c|^2}} \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} \quad \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \leftarrow E_2 \text{ eigenspace}$$

a) The expectation value of a Hermitian operator for any state is real.

b) The eigenvalues of a Hermitian operator are all real

c) Eigenvectors belonging to different eigenvalues for a Hermitian operator are orthogonal.

d) The eigenvectors of a Hermitian operator span the Hilbert space.

a) $\langle Q \rangle = \langle \alpha | \hat{Q} | \alpha \rangle$

$\langle Q \rangle^* = \langle \alpha | \hat{Q}^\dagger | \alpha \rangle = \langle \alpha | \hat{Q} | \alpha \rangle = \langle Q \rangle$

Since $\langle Q \rangle = \langle Q \rangle^*$ it must be real

b) $|q\rangle$ an eigenvector belonging to eigenvalue λ

$\hat{Q}|q\rangle = \lambda |q\rangle$

$\langle \hat{Q} \rangle = \langle q | \hat{Q} | q \rangle = \langle q | \lambda | q \rangle = \lambda \langle q | q \rangle = \lambda \|q\|^2$

$\langle \hat{Q} \rangle$ real then λ is real

c) $\hat{Q}|q_1\rangle = \lambda_1 |q_1\rangle \quad \hat{Q}|q_2\rangle = \lambda_2 |q_2\rangle \quad \lambda_1 \neq \lambda_2 \quad \hat{Q} = \hat{Q}^\dagger$

$\langle q_1 | \hat{Q} | q_2 \rangle = \langle q_1 | (\hat{Q} | q_2 \rangle) = \langle q_1 | \lambda_2 | q_2 \rangle = \lambda_2 \langle q_1 | q_2 \rangle$

$(\langle q_1 | \hat{Q}) | q_2 \rangle = \langle q_1 | \lambda_1 | q_2 \rangle = \lambda_1 \langle q_1 | q_2 \rangle \quad \therefore \langle q_1 | q_2 \rangle = 0$

$\hat{A}$ $\det(\hat{A} - \lambda \mathbb{1})$

$\prod_r (\lambda - \lambda_r)^{N_r}$ so all the λ_r are distinct

$\{\lambda_r\}$ is the spectrum and the index r is used to label distinct eigenvalues. N_r is the degeneracy of λ_r

Theorem

The set of all eigenvectors belonging to the same eigenvalue form a vector subspace of the Hilbert space

$\hat{A}|\alpha\rangle = \lambda|\alpha\rangle$ $\hat{A}|\beta\rangle = \lambda|\beta\rangle$

$|\gamma\rangle = a|\alpha\rangle + b|\beta\rangle$

$\hat{A}|\gamma\rangle = a\lambda|\alpha\rangle + b\lambda|\beta\rangle = \lambda|\gamma\rangle$

For each distinct λ_r there is a vector subspace of λ_r eigenvectors called the λ_r-eigenspace

Phys 137A 22Jul19

Review Session W 7/24 3-5 pm 251 LeConte
Exam F 7/26 10-1 1 LeConte

Observable A

$\Downarrow$

Hermitian Operator $\hat{A}$ and matrix representation $\overleftrightarrow{A}$

$\det(\overleftrightarrow{A} - \lambda \overleftrightarrow{\mathbb{1}}) = \prod_r (\lambda - \lambda_r)^{N_r} = 0$ gives eigen spectrum $\{\lambda_r\}$

with degeneracies $\{N_r\}$

Set of all eigenstates belonging to λ_r: $\hat{A}|\alpha\rangle = \lambda_r|\alpha\rangle$ or $\overleftrightarrow{A}\vec{v}_\alpha = \lambda_r \vec{v}_\alpha$

is the "λ_r-eigenspace"

Double Square Well

Energy $\to \hat{H} \to$ $|-\rangle$ $E_0 - \mathcal{E}$ Energy Eigenbasis

$|+\rangle$ $E_0 + \mathcal{E}$

Wellness $\to \hat{W} \to$ $|L\rangle$ 1 Well Eigenbasis

$|R\rangle$ 2

Three-State System

$$\overleftrightarrow{H} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_2 \end{bmatrix}$$

E_1-eigenspace: $\left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \right\}$

E_2-eigenspace: $\left\{ \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} \right\}$

Observable A Operator $\hat{A}$

1st eigenbasis Labeling

$$\{|a_i\rangle\} \quad i = 1, \ldots, \text{dimension of Hilbert space}$$

$\hat{A}|a_i\rangle = \lambda_i |a_i\rangle$ $\{\lambda_i\}$ possible repeated entries

$$\hat{H} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_2 \end{bmatrix}$$

$|h_1\rangle \doteq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $E_1 = \lambda_1$

$|h_2\rangle \doteq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $E_2 = \lambda_2$

$|h_3\rangle \doteq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $E_2 = \lambda_3$

2nd Eigenbasis Labeling

label by eigenvalue or quantum number. Degeneracy gets an added label

Spectrum $\{\lambda_r\}$ with degeneracies $\{N_r\}$

$\lambda_{r=1} = E_1$ $N_r = 1$ $\{|\lambda_r, n_r\rangle\}$ with $n_r = 1, 2, \ldots, N_r$ for a given λ_r

$\lambda_{r=2} = E_2$ $N_r = 2$

$|E_1, 1\rangle = |h_1\rangle$ or $|E_1\rangle = |h_1\rangle$

$|E_2, 1\rangle = |h_2\rangle$

$|E_2, 2\rangle = |h_3\rangle$

Drthonormality: $\langle q^i | q_j \rangle = \delta^i_j$ $\langle \lambda_r, n | \lambda_s, m \rangle = \delta_{rs} \delta_{nm}$

$\langle h^2 | h_3 \rangle = \delta^2_3 = 0$ $\langle E_2, 1 | E_2, 2 \rangle = \delta_{22} \delta_{12} = 0$

Resolution of the identity

$$\hat{\mathbb{1}} = \sum_i |q_i\rangle\langle q^i| = \sum_r \sum_{n=1}^{N_r} |\lambda_r, n\rangle\langle\lambda_r, n|$$

Expansions

$$|\alpha\rangle = \sum_i c^i |q_i\rangle \quad c^i \equiv \langle q^i | \alpha\rangle$$

$$|\alpha\rangle = \sum_r \sum_{n=1}^{N_r} c_{r,n} |\lambda_r, n\rangle \quad c_{rn} \equiv \langle \lambda_r, n | \alpha \rangle$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a|E_1\rangle + b|E_2, 1\rangle + c|E_2, 2\rangle$$

Eigenbasis $\{|q_i\rangle\}$ or $\{|\lambda_r, n\rangle\}$

Spectral Decomposition

$$\hat{A} = \sum_i \lambda_i |q_i\rangle\langle q^i|$$ In eigenbasis $\overleftrightarrow{A}$ is diagonal with components that are the eigenvalues

$$\overleftrightarrow{H} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_2 \end{bmatrix} \quad \hat{H} = E_1|h^1\rangle\langle h_1| + E_2|h^2\rangle\langle h_2| + E_2|h^3\rangle\langle h_3|$$

$$|L\rangle = \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle) \quad |R\rangle = \frac{1}{\sqrt{2}}(|-\rangle - |+\rangle)$$

$\lambda_L = 1$ $\lambda_R = 2$

$$\hat{W} = 1|L\rangle\langle L| + 2|R\rangle\langle R| \doteq \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 2\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + 2\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ -\frac{1}{2} & 3/2 \end{bmatrix}$$

$$\hat{A}|a_j\rangle = \sum_i \lambda_i |a_i\rangle\langle a_i|a_j\rangle = \sum_i \lambda_i |a_i\rangle \delta^i_j = \lambda_j |a_j\rangle$$

$$\hat{A} = \sum_r \sum_{n=1}^{N_r} \lambda_r |\lambda_r, n\rangle\langle\lambda_r, n| = \sum_r \lambda_r \left(\sum_{n=1}^{N_r} |\lambda_r, n\rangle\langle\lambda_r, n|\right)$$

Position Operator $\hat{X}$ momentum operator $\hat{P}$

Position eigenbasis $\{|x\rangle\}$, $x \in \mathbb{R}$

momentum eigenbasis $\{|p\rangle\}$, $P \in \mathbb{R}$

Rigged Hilbert Space

$$\hat{x}|x_0\rangle = x_0|x_0\rangle \qquad \hat{p}|p_0\rangle = p_0|p_0\rangle$$

$$\langle x|\tilde{x}\rangle = \delta(x-\tilde{x}) \qquad \langle p|\tilde{p}\rangle = \delta(p-\tilde{p})$$

$$\hat{\mathbb{1}} = \int |x\rangle\langle x|\,dx \qquad \hat{\mathbb{1}} = \int |p\rangle\langle p|\,dp$$

$$|\alpha\rangle = \hat{\mathbb{1}}|\alpha\rangle = \int |x\rangle\langle x|\alpha\rangle\,dx = \int \underbrace{(\langle x|\alpha\rangle)}_{\text{wave function}} |x\rangle\,dx$$

$$\psi_\alpha(x) \equiv \langle x|\alpha\rangle$$

$$\phi_\alpha(p) \equiv \langle p|\alpha\rangle$$

$$\tilde{\psi}_\alpha(x) \equiv \hat{x}\psi_\alpha(x) \qquad \langle x|\tilde{\alpha}\rangle = \langle x|\hat{x}|\alpha\rangle = x\langle x|\alpha\rangle = x\,\psi_\alpha(x)$$

$$\langle A\rangle = \langle\alpha|\hat{A}|\alpha\rangle$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ixp/\hbar}$$

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ixp/\hbar}$$

$$\begin{aligned}
\hat{x}\,\psi_\alpha(x) &= \langle x|\hat{x}|\alpha\rangle = \langle x|\hat{x}\,\hat{\mathbb{1}}|\alpha\rangle \\
&= \langle x|\hat{x}\left(\int |\tilde{x}\rangle\langle\tilde{x}|\,d\tilde{x}\right)|\alpha\rangle \\
&= \int \langle x|\hat{x}|\tilde{x}\rangle\langle\tilde{x}|\alpha\rangle\,d\tilde{x} = \int \langle x|\hat{x}|\tilde{x}\rangle\,\psi_\alpha(\tilde{x})\,d\tilde{x} \\
&= \int \tilde{x}\langle x|\tilde{x}\rangle\,\psi_\alpha(\tilde{x})\,d\tilde{x} = \int \tilde{x}\,\delta(x-\tilde{x})\,\psi_\alpha(\tilde{x})\,d\tilde{x} \\
&= x\,\psi_\alpha(x)
\end{aligned}$$

Phys 137A 23 Jul 19

Observable Q Operator $\hat{Q}$ (Hermitian)

Eigenbasis $\{|q_i\rangle\}$ $\hat{Q}|q_i\rangle = \lambda_i |q_i\rangle$ $i = 1, \ldots,$ dimension of Hilbert space

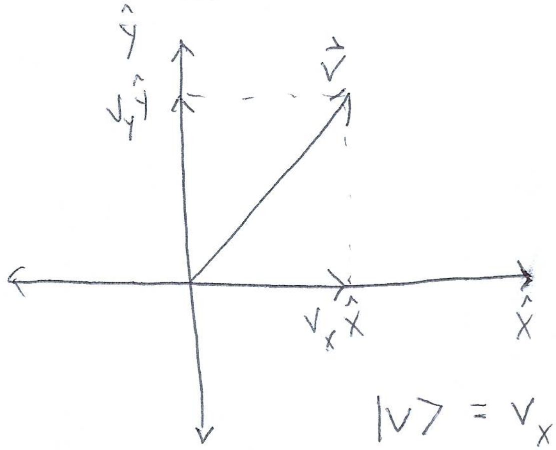
or $\{|\lambda_r, n\rangle\}$ $n = 1, \ldots, N_r$

distinct eigenvalues

degeneracy of eigenvalue λ_r

$$\hat{\mathbb{1}} = \sum_r \sum_{n=1}^{N_r} |\lambda_r, n\rangle\langle\lambda_r, n|$$

$$\hat{Q} = \sum_r \sum_{n=1}^{N_r} \lambda_r |\lambda_r, n\rangle\langle\lambda_r, n| = \sum_r \lambda_r \underbrace{\sum_{n=1}^{N_r} |\lambda_r, n\rangle\langle\lambda_r, n|}_{\text{Projection Operator}}$$



$\vec{V} = V_x\hat{x} + V_y\hat{y}$

$P_x\vec{V} = V_x\hat{x}$ $P_y\vec{V} = V_y\hat{y}$

$|v\rangle = v_x|x\rangle + v_y|y\rangle$

$|v\rangle = |x\rangle\langle x|v\rangle + |y\rangle\langle y|v\rangle$

Projection Operator

$\hat{P} = \hat{P}^\dagger$

$\hat{P}^2 = \hat{P}$ "idempotency"

$\hat{P}$ has eigenvalues 0 and 1

1 eigenspace = the space you are projecting onto

0 eigenspace = the space orthogonal to the space you are projecting onto

For each eigenvalue λ_r, define $\hat{P}_r = \sum_{n=1}^{N_r} |\lambda_r, n\rangle\langle\lambda_r, n|$

is a projection onto the λ_r eigenspace

$$\hat{P}_r^2 = \left(\sum_{n=1}^{N_r} |\lambda_r, n\rangle\langle\lambda_r, n|\right)\left(\sum_{m=1}^{M_r} |\lambda_r, m\rangle\langle\lambda_r, m|\right)$$

$$= \sum_{n,m} |\lambda_r, n\rangle\left(\langle\lambda_r, n|\lambda_r, m\rangle\right)\langle\lambda_r, m| = \sum_{n,m} |\lambda_r, n\rangle \delta_{rr}\delta_{nm} \langle\lambda_r, m|$$

$$= \sum_{n=1}^{N_r} |\lambda_r, n\rangle\langle\lambda_r, n| = \hat{P}_r$$

$$\hat{P}_r \hat{P}_s = 0$$

$r \neq s$

$$\hat{\mathbb{1}} = \sum_r \hat{P}_r \qquad \hat{A} = \sum_r \lambda_r \hat{P}_r$$

$$\overleftrightarrow{\mathbb{1}}_{[q]} = \left[\begin{array}{c|c|c} \mathbb{1} & 0 & 0 \\ \hline 0 & \mathbb{1} & 0 \\ \hline 0 & 0 & \mathbb{1} \end{array}\right] \qquad \overleftrightarrow{A} = \left[\begin{array}{c|c|c} \lambda_1 \mathbb{1} & 0 & 0 \\ \hline 0 & \lambda_2 \mathbb{1} & 0 \\ \hline 0 & 0 & \lambda_3 \mathbb{1} \end{array}\right]$$

$$\overleftrightarrow{H} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_2 \end{bmatrix} \qquad \overleftrightarrow{P}_{E_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\overleftrightarrow{P}_{E_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\overleftrightarrow{P}_{E_2}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} \qquad \overleftrightarrow{P}_{E_1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \qquad \overleftrightarrow{H} = \left[\begin{array}{c|c} E_1 \mathbb{1}_1 & 0 \\ \hline 0 & E_2 \mathbb{1}_2 \end{array}\right]$$

Postulate IV (Measurement Probabilities)

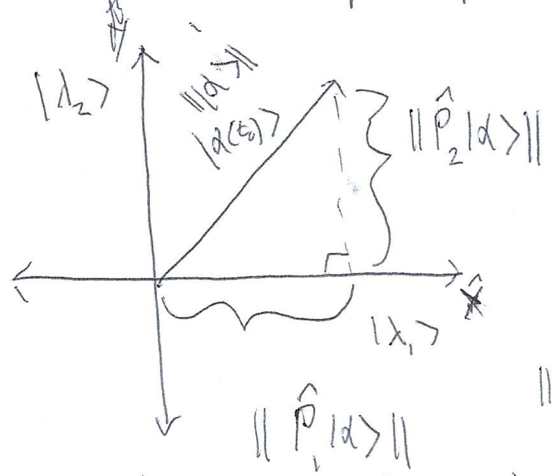
$$P(\lambda_r) = \|\hat{P}_r|\alpha\rangle\|^2 = \langle\alpha|\hat{P}_r|\alpha\rangle = \sum_{n=1}^{N_r} |\langle\lambda_r, n|\alpha\rangle|^2 = \sum_{n=1}^{N_r} |c_{r,n}(t_0)|^2$$

Start with $|\alpha(t_0)\rangle$

Make a measurement of $\hat{A}$ at $t=t_0$

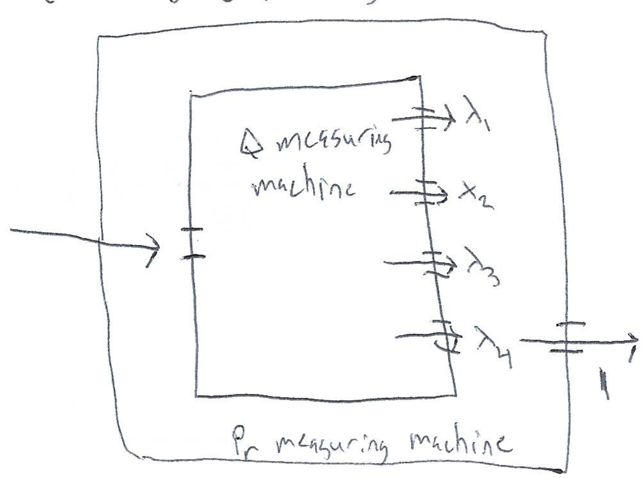
we get result $A = \lambda_r$ with Probability given by

$$P(\lambda_r) = \|\hat{P}_r|\alpha\rangle\|^2 = \langle\alpha|\hat{P}|\alpha\rangle = \sum_{n=1}^{N_r} |\langle\lambda_r, n|\alpha\rangle|^2 = \sum_{n=1}^{N_r} |c_{r,n}|^2$$



$$\||\alpha\rangle\|^2 = \|\hat{P}_1|\alpha\rangle\|^2 + \|\hat{P}_2|\alpha\rangle\|^2 = 1$$

$$\|\hat{P}|\alpha\rangle\|^2 = (\hat{P}_r|\alpha\rangle)^\dagger(\hat{P}_r|\alpha\rangle) = \langle\alpha|\hat{P}^\dagger\hat{P}|\alpha\rangle = \langle\alpha|\hat{P}|\alpha\rangle$$



$$|\alpha\rangle = \sum_r \sum_{n=1}^{N_r} c_{r,n}|\lambda, n\rangle \qquad c_{r,n} = \langle\lambda_r, n|\alpha\rangle$$

$$|\alpha\rangle = a|E_1\rangle + b|E_2, 1\rangle + c|E_2, 2\rangle$$

$$P(E_1) = |a|^2 \qquad P(E_2) = |b|^2 + |c|^2$$

Postulate V (Collapse)

$$|\alpha(T^+)\rangle = N\hat{P}_r|\alpha(T^-)\rangle$$ State changes but then needs to be renormalized

$$N = \langle\alpha(T^-)|\hat{P}_r|\alpha(T^-)\rangle^{-1/2} = \frac{1}{\sqrt{P(\lambda_r)}}$$

Immediately after measurement the state collapses

$$|\alpha(t_0^+)\rangle = \frac{\hat{P}_{\lambda_r}|\alpha(t^-)\rangle}{\sqrt{P(\lambda_r)}}$$

1) Project
2) Normalize

$$|\alpha(t_0)\rangle = \sum_r \sum_{n=1}^{N_r} c_{r,n}|\lambda_r, n\rangle$$

$$P(\lambda_r) = \sum_{n=1}^{N_a} |c_{r,n}|^2 \qquad |\alpha(t_0^+)\rangle = \frac{\sum_{n=1}^{N_r} c_{r,n}|\lambda, n\rangle}{\sqrt{\sum_{n=1}^{\infty}|c_{r,n}|^2}}$$

If λ_r is non-degenerate

$(N_r = 1)$ $\qquad P(\lambda_r) = |c_r|^2 \qquad |\alpha(t_0^+)\rangle = \frac{c_{r,n}}{\sqrt{|c_r|^2}}|\lambda_r\rangle = e^{i\phi}|\lambda_r\rangle$

3- State system

$$|\alpha\rangle \doteq \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad |a|^2 + |b|^2 + |c|^2 = 1$$

Measure energy

<u>decoherence</u>

$$|\alpha\rangle \doteq a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Result E_1: $P(E_1) = |a|^2 \quad |\alpha(t_0^+)\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Result E_2: $P(E_2) = |b|^2 + |c|^2 \quad |\alpha(t_0^+)\rangle = \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} \frac{1}{\sqrt{|b|^2 + |c|^2}}$

Phys 137A 24 Jul 19

"renormalization" is in QFT and "is the worst"

Time Evolution

The dynamics of a quantum system are given by the Hamiltonian Operator.

Schrödinger Picture

$$i\hbar \frac{\partial}{\partial t}|\alpha(t)\rangle = \hat{H}|\alpha(t)\rangle$$

TDSE

Observable H Operator $\hat{H}$ spectrum $\{E_n\}$ Eigenbasis: $\{|n,\mu\rangle\}$

n is our energy quantum number

degeneracy

$$\hat{H}|n,\mu\rangle = E_n|n,\mu\rangle$$

$$|\alpha(0)\rangle = \cancel{|n,\mu\rangle}\ c(0)|n,\mu\rangle$$

$$\hat{H}|\alpha(t)\rangle = \cancel{c(0)E_n}\ c(t)E_n|n,\mu\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\alpha(t)\rangle = i\hbar c'(t)|n,\mu\rangle = c(t)E_n|n,\mu\rangle$$

$$c = Ne^{-iE_nt/\hbar} \qquad |n,\mu\rangle \mapsto e^{-iE_nt/\hbar}|n,\mu\rangle$$

Starting state $|\alpha(0)\rangle$

Expand $|\alpha(0)\rangle = \sum_n \sum_{\mu=1}^{N_n} c_{n,\mu}(0)|n,\mu\rangle$

Evolve $|\alpha(t)\rangle = \sum_n \sum_{\mu=1}^{N_n} c_{n,\mu}(t)|n,\mu\rangle \qquad c_{n,\mu}(t) = e^{-iE_nt/\hbar}c_{n,\mu}(0)$

$$= \sum_n \sum_{\mu=1}^{N_n} e^{-iE_nt/\hbar}c_{n,\mu}(0)|n,\mu\rangle$$

$c_{n,\mu}$ only depends on n, not μ

$$|\alpha(0)\rangle = A_n e^{i\phi_n}|n,\mu\rangle + B_m e^{i\theta_m}|m,\tilde{\mu}\rangle$$

"relative phase" $e^{i(\theta_m - \phi_n)}$

$$|\alpha(t)\rangle = A_n e^{i\phi_n} e^{-iE_n t/\hbar}|n,\mu\rangle + B_m e^{i\theta_m} e^{-iE_m t/\hbar}|m,\tilde{\mu}\rangle$$

"Relative phase" $e^{i\left((\theta_m - \phi_n) - \frac{\Delta E t}{\hbar}\right)}$

<u>Double Square Well</u>

$$|\alpha(0)\rangle = C_-|-\rangle + C_+|+\rangle$$

$$|\alpha(t)\rangle = C_- e^{-i(E_0-\varepsilon)t/\hbar}|-\rangle + C_+ e^{-i(E_0+\varepsilon)t/\hbar}|+\rangle$$

$$= e^{-iE_0 t/\hbar}\left(C_- e^{i\varepsilon t/\hbar}|-\rangle + C_+ e^{-i\varepsilon t/\hbar}|+\rangle\right)$$

$$|\alpha(0)\rangle = |L\rangle = \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle)$$

$$P_R(t) = |\langle R|\alpha(t)\rangle|^2 \qquad |R\rangle = \frac{1}{\sqrt{2}}(|-\rangle - |+\rangle)$$

$$\left| e^{-iE_0 t/\hbar}\left(\overset{\frac{1}{\sqrt{2}}}{C_-} e^{i\varepsilon t/\hbar}\overset{\frac{1}{\sqrt{2}}}{\langle R|-\rangle} + \overset{\frac{1}{\sqrt{2}}}{C_+} e^{-i\varepsilon t/\hbar}\overset{-\frac{1}{\sqrt{2}}}{\langle R|+\rangle}\right)\right|^2$$

$$= \left|\frac{e^{i\varepsilon t/\hbar} - e^{-i\varepsilon t/\hbar}}{2i} i\right|^2 = \sin^2\frac{\varepsilon t}{\hbar}$$

$$\frac{\varepsilon t}{\hbar} = \frac{\pi}{2} \Rightarrow t = \frac{\pi\hbar}{2\varepsilon}$$

$$\Delta t \Delta E = \pi\hbar$$

<u>Commutators</u>

$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ if $[\hat{A},\hat{B}] = 0$ $\hat{A}\hat{B} = \hat{B}\hat{A}$ we say they commute

$[\hat{B},\hat{A}] = -[\hat{A},\hat{B}]$ $\hat{A}\hat{B} = \hat{B}\hat{A} + [\hat{A},\hat{B}]$ Commutator is linear

$[\hat{A},\hat{A}] = 0$ $[\hat{A},\hat{A}^n] = 0$ $[\hat{A}, f(\hat{A})] = 0$

$[a\hat{A}, b\hat{B} + c\hat{C}] = [a\hat{A}, b\hat{B}] + [a\hat{A}, c\hat{C}]$ $[a\hat{A} + b\hat{B}, c\hat{C}] = [aA, c\hat{C}] + [b\hat{B}, c\hat{C}]$

$[\hat{A},\hat{B}\hat{C}] = \hat{B}[\hat{A},\hat{C}] + [\hat{A},\hat{B}]\hat{C}$ $[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$

$$\left[\hat{A},[\hat{B},\hat{C}]\right] + \left[\hat{B},[\hat{C},\hat{A}]\right] + \left[\hat{C},[\hat{A},\hat{B}]\right] = 0$$

$[\hat{x},\hat{p}] = i\hbar\hat{\mathbb{1}} \Rightarrow$ x and p canonically conjugate

$\cancel{[\hat{x},\hat{p}]}$ $\quad [\hat{x}, f(\hat{p})] = i\hbar f'(\hat{p}) \qquad [g(\hat{x}),\hat{p}] = i\hbar g'(\hat{x})$

$$\hat{T} = \frac{\hat{p}^2}{2m} \qquad [\hat{T},\hat{x}] = \left[\frac{\hat{p}^2}{2m},\hat{x}\right] = -\left[\hat{x},\frac{\hat{p}^2}{2m}\right] = \frac{-1}{2m}[\hat{x},\hat{p}^2] \rightarrow -i\hbar\frac{dT}{dp}$$

$$= \frac{-1}{2m}\hat{p}[\hat{x},\hat{p}] - \frac{1}{2m}[\hat{x},\hat{p}]\hat{p}$$

$$= \frac{-i\hbar}{2m}\hat{p} - \frac{i\hbar}{2m}\hat{p} = -\frac{i\hbar}{m}\hat{p}$$

Classical Time

$$H(x,p) \qquad \overbrace{\frac{dp}{dt}}^{F} = \frac{-\partial H}{\partial x} \qquad \overbrace{\frac{dx}{dt}}^{V} = \frac{\partial H}{\partial p}$$

$$H = \frac{p^2}{2m} + V(x) \qquad F = -\frac{dV}{dx} \qquad v = \frac{p}{m}$$

$$a = \frac{F}{m} \Rightarrow F = ma$$

$\cancel{A(x,p)}$ $\quad A(x,p;t)$

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial x} + \frac{\partial A}{\partial t}$$

Poisson Bracket

$$\{A,B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}$$

$$= \{A,H\} + \frac{\partial A}{\partial t}$$

$|\alpha(t)\rangle \qquad A, \hat{A} \qquad \underline{\hat{A}(t)}$

$$\langle\hat{A}\rangle = \langle\alpha|\hat{A}|\alpha\rangle \qquad i\hbar\frac{d|\alpha\rangle}{dt} = \hat{H}|\alpha\rangle \qquad -i\hbar\frac{d\langle\alpha|}{dt} = \langle\alpha|\hat{H}^{*}$$

$$\frac{d\langle\hat{A}\rangle}{dt} = \left(\frac{d\langle\alpha|}{dt}\right)\hat{A}|\alpha\rangle + \langle\alpha|\frac{d\hat{A}}{dt}|\alpha\rangle + \langle\alpha|\hat{A}\left(\frac{d|\alpha\rangle}{dt}\right)$$

$$= \frac{i}{\hbar}\langle\alpha|\hat{H}\hat{A}|\alpha\rangle + \langle\alpha|\frac{d\hat{A}}{dt}|\alpha\rangle + \langle\alpha|\hat{A}\frac{-i}{\hbar}\hat{H}|\alpha\rangle = \frac{i}{\hbar}\langle\alpha|HA - AH|\alpha\rangle + \left\langle\frac{dA}{dt}\right\rangle$$

$$= \left\langle\frac{i}{\hbar}[\hat{H},\hat{A}]\right\rangle + \left\langle\frac{d\hat{A}}{dt}\right\rangle$$

$$\{x, p\} = 1 \qquad [\hat{x}, \hat{p}] = i\hbar \hat{\mathbb{1}}$$

$$\{\cdot, \cdot\} \to \frac{1}{i\hbar}[\hat{\cdot}, \hat{\cdot}]$$

$$\frac{d\langle Q\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{Q}, \hat{H}]\rangle + \left\langle\frac{dQ}{dt}\right\rangle = \frac{i}{\hbar}\langle[\hat{H}, \hat{Q}]\rangle + \left\langle\frac{dQ}{dt}\right\rangle$$

$$\frac{d\langle x\rangle}{dt} = \left\langle\frac{i}{\hbar}[\hat{H}, \hat{x}]\right\rangle + \left\langle\frac{d\hat{x}}{dt}\right\rangle \qquad [\hat{H}, \hat{x}] = [\hat{T}, \hat{x}] + [V(\hat{x}), \hat{x}]$$

$$= \left\langle\frac{i}{\hbar}\frac{-i\hbar}{m}\hat{p}\right\rangle = \frac{\langle p\rangle}{m} \qquad = [\hat{T}, \hat{x}] = -\frac{i\hbar\hat{p}}{m}$$

$$\frac{d\langle p\rangle}{dt} = \left\langle\frac{i}{\hbar}[\hat{H}, \hat{p}]\right\rangle + \left\langle\frac{\partial\hat{p}}{\partial t}\right\rangle \qquad [\hat{H}, \hat{p}] = [\hat{T}, \hat{p}] + [V(\hat{x}), \hat{p}]$$

$$= \left\langle\frac{i}{\hbar} i\hbar\frac{\partial V}{\partial x}\right\rangle = -\left\langle\frac{\partial V}{\partial x}\right\rangle \qquad = [V(\hat{x}), \hat{p}]$$

Phys 137A 25 Jul 19

$$i\hbar \frac{\partial}{\partial t}|\alpha(t)\rangle = \hat{H}|\alpha(t)\rangle \qquad \frac{d\langle \hat{Q}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$$

$$[\hat{x},\hat{p}] = i\hbar\hat{\mathbb{1}} \Rightarrow \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

$$\sigma_Q = \sqrt{\langle Q^2\rangle - \langle Q\rangle^2} \qquad \langle(\hat{Q} - \langle\hat{Q}\rangle\mathbb{1})^2\rangle = \sigma_Q^2$$

Generalized Uncertainty Principle

$$\sigma_A \sigma_B \geq \left|\left\langle \frac{1}{2i}[\hat{A},\hat{B}]\right\rangle\right|$$

$$|\alpha\rangle \quad \langle A\rangle \quad \langle B\rangle \qquad |a\rangle = \hat{A}|\alpha\rangle - \langle A\rangle|\alpha\rangle = (\hat{A} - \langle A\rangle\hat{\mathbb{1}})|\alpha\rangle$$

$$|b\rangle = \hat{B}|\alpha\rangle - \langle B\rangle|\alpha\rangle = (\hat{B} - \langle B\rangle\hat{\mathbb{1}})|\alpha\rangle$$

$$\langle a| = \langle\alpha|(\hat{A} - \langle A\rangle\hat{\mathbb{1}})$$

$$\langle a|a\rangle = \langle\alpha|(\hat{A} - \langle A\rangle\hat{\mathbb{1}})(\hat{A} - \langle A\rangle\hat{\mathbb{1}})|\alpha\rangle = \langle\alpha|(\hat{A} - \langle A\rangle\hat{\mathbb{1}})^2|\alpha\rangle = \sigma_A^2$$

$$\sigma_A^2\sigma_B^2 = \langle a|a\rangle\langle b|b\rangle \geq |\langle a|b\rangle|^2 \qquad |\langle a|b\rangle| \geq \mathrm{Re}\langle a|b\rangle$$

$$|\vec{v}|^2|\vec{w}|^2 \geq (\vec{v}\cdot\vec{w})^2 \qquad |\langle a|b\rangle| \geq \mathrm{Im}\langle a|b\rangle$$

$$\langle a|a\rangle\langle b|b\rangle \geq (\mathrm{Im}\langle a|b\rangle)^2$$

$$\mathrm{Im}\langle a|b\rangle = \frac{\langle a|b\rangle - \langle a|b\rangle^*}{2i} = \frac{1}{2i}(\langle a|b\rangle - \langle b|a\rangle)$$

$$= \frac{1}{2i}\langle\alpha|((\hat{A} - \langle A\rangle\hat{\mathbb{1}})(\hat{B} - \langle B\rangle\hat{\mathbb{1}}) - (\hat{B} - \langle B\rangle\hat{\mathbb{1}})(\hat{A} - \langle A\rangle\hat{\mathbb{1}}))|\alpha\rangle$$

$$= \frac{1}{2i}\langle\alpha|(\hat{A}\hat{B} - \langle A\rangle\hat{\mathbb{1}}\hat{B} - \hat{A}\langle B\rangle\hat{\mathbb{1}} + \langle A\rangle\langle B\rangle\hat{\mathbb{1}}\hat{\mathbb{1}} - \hat{B}\hat{A} + \langle B\rangle\hat{\mathbb{1}}\hat{A} + \hat{B}\langle A\rangle\hat{\mathbb{1}} - \langle B\rangle\langle A\rangle\hat{\mathbb{1}}\hat{\mathbb{1}})|\alpha\rangle$$

$$= \frac{1}{2i}\langle\alpha|[\hat{A},\hat{B}]|\alpha\rangle \qquad \sigma_A\sigma_B \geq \left|\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right|$$

$$\sigma_x \sigma_p \geq \left| \left\langle \frac{1}{2i}[\hat{x},\hat{p}] \right\rangle \right| = \left| \left\langle \frac{1}{2i} i\hbar \hat{\mathbb{1}} \right\rangle \right| = \frac{\hbar}{2}\langle \hat{\mathbb{1}} \rangle = \frac{\hbar}{2} \leq \sigma_x \sigma_p$$

$\hat{X}$

$\hat{T}$ $\quad [\hat{x},\hat{T}] = \frac{1}{2m}[\hat{x},\hat{p}^2] = \frac{i\hbar 2\hat{p}}{2m} = \frac{i\hbar}{m}\hat{p}$

$$\sigma_x \sigma_T \geq \left| \left\langle \frac{1}{2i} \frac{i\hbar}{m} \hat{p} \right\rangle \right| = \frac{\hbar}{2m} |\langle \hat{p} \rangle|$$

Time is its own special/separate thing. It is not an observable

$\Delta t \quad \Delta E \Delta t \geq \frac{\hbar}{2} \qquad \frac{d\langle Q \rangle}{dt} = \left\langle \frac{i}{\hbar}[\hat{H},\hat{Q}] \right\rangle$

ΔE

$$\frac{\Delta Q}{\Delta t} = \frac{-2}{\hbar}\left\langle \frac{1}{2i}[\hat{H},\hat{Q}] \right\rangle$$

$\frac{\Delta Q}{\Delta t} \simeq \frac{\sigma_Q}{\Delta t} \leq \frac{2}{\hbar}\sigma_H \sigma_Q \qquad \sigma_H \Delta t \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$

Active Transformation

Time-Translation

$$\cancel{i\hbar |\alpha t} \quad i\hbar \frac{\partial}{\partial t}|\alpha(t)\rangle = \hat{H}|\alpha(t)\rangle$$

$$\frac{\Delta}{\Delta t}|\alpha(t)\rangle = \cancel{\hat{H}\alpha} - \frac{i\hat{H}}{\hbar}|\alpha(t)\rangle$$

$|\alpha(t)\rangle = e^{-i\hat{H}t/\hbar}|\alpha(0)\rangle \qquad \boxed{e^{-i\hat{H}t/\hbar} \text{ Time Translation Operator}}$

$$e^{-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'} = U(t,t_0)$$

Energy $\hat{H}$ Time Translation $U(t) = e^{-i\hat{H}t/\hbar}$

~~position~~ $\hat{X}$ ~~Translation~~

momentum $\hat{p}$ $\hat{J}(x) = e^{-i\hat{p}x/\hbar}$

Translation

Active Transformations are carried out via unitary operators

$$\hat{U}^\dagger \hat{U} = \hat{\mathbb{1}}$$

$$\hat{U} = e^{-i\hat{H}t/\hbar} \qquad \hat{U}^\dagger = e^{i\hat{H}t/\hbar} \qquad \hat{U}^\dagger\hat{U} = \hat{\mathbb{1}}$$

Unitarity preserves Normalization

$$|\alpha\rangle \qquad |\beta\rangle \equiv \hat{U}|\alpha\rangle$$

$$\langle\beta|\beta\rangle = \langle\alpha|\hat{U}^\dagger)(\hat{U}|\alpha\rangle) = \langle\alpha|\hat{\mathbb{1}}|\alpha\rangle = \langle\alpha|\alpha\rangle = 1$$

Collapse is a nonunitary operation

Projections are not unitary

Entropy doesn't change over unitary transformation

$$\hat{U} = e^{-i\hat{H}t/\hbar} \qquad \overleftrightarrow{U} = e^{-i\hat{H}t/\hbar}$$

Energy Eigenbasis $\{|n,\mu\rangle\}$

$$U^i_j = \langle e^i|\hat{U}|e_j\rangle \qquad \hat{U}|n,\mu\rangle = e^{-i\hat{H}t/\hbar}|n,\mu\rangle$$

$$\hat{H}|n,\mu\rangle = E_n|n,\mu\rangle \qquad f(\hat{H})|n,\mu\rangle = f(E_n)|n,\mu\rangle$$

$$\hat{U}|n,\mu\rangle = e^{-iE_nt/\hbar}|n,\mu\rangle \qquad \overleftrightarrow{U} = \begin{bmatrix} e^{-iE_1t/\hbar} & 0 & 0 & \cdots & 0 \\ 0 & e^{-iE_2t/\hbar} & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & & & & e^{-iE_nt/\hbar} \end{bmatrix}$$

Active Transformation, $\hat{U}$

$$|\alpha\rangle \Rightarrow \hat{U}|\alpha\rangle$$

$$\langle\beta| \Rightarrow \langle\beta|\hat{U}^\dagger$$

$$\hat{A} \Rightarrow \hat{U}\hat{A}\hat{U}^\dagger$$

Phys 137A 29Jul19

Active Transformations

Unitary operator U,

$|\alpha'\rangle = \hat{U}|\alpha\rangle \quad \langle\beta'| = \langle\beta|\hat{U}^\dagger \qquad \hat{A}' = \hat{U}\hat{A}\hat{U}^\dagger$

similarity transformation

Time reversal

Charge conjugation

Parity

Parity Operator $\hat{\Pi}$ $\quad \hat{\Pi}|x\rangle = |-x\rangle \quad$ $\hat{\Pi}$ is hermitian and unitary

$\hat{\Pi}\hat{X}\hat{\Pi}^\dagger = -\hat{X} \qquad \hat{\Pi}|p\rangle = |-p\rangle$

$\hat{\Pi}\hat{p}\hat{\Pi}^\dagger = -\hat{p}$

$|\alpha\rangle, \psi_\alpha(x) = \langle x|\alpha\rangle, \quad |\alpha'\rangle = \hat{\Pi}|\alpha\rangle \qquad \langle x|\alpha'\rangle = \psi_{\alpha'}(x) = \psi_\alpha(-x)$

$\hat{\Pi}^2 = \hat{\mathbb{1}}$

Since $\hat{\Pi}$ is hermitian, there is a parity observable

Eigenvalues of $\hat{\Pi}$

$\hat{\Pi}|\alpha\rangle = \lambda|\alpha\rangle \qquad \hat{\Pi}^2|\alpha\rangle = \hat{\Pi}(\hat{\Pi}|\alpha\rangle) = \lambda^2|\alpha\rangle$

$\hat{\Pi}^2|\alpha\rangle = \hat{\mathbb{1}}|\alpha\rangle \quad \lambda^2 = 1, \quad \lambda = \pm 1$

$\lambda = 1$, parity is even

$\lambda = -1$, parity is odd

$\psi_\alpha(x) = \pm \psi_\alpha(-x)$

eigenfunctions

even parity	odd parity
symmetric ψ	antisymmetric ψ
$\lambda = 1$	$\lambda = -1$

Passive Transformations

States aren't changing ; only descriptions are changing

Change of Basis

kets, bras, operators don't change but their matrix representations do.

"Old" basis $\{|e_i\rangle\}$ "New" basis $\{|f_i\rangle\}$

$\langle e^i|e_j\rangle = \delta^i_j \qquad \langle f^i|f_j\rangle = \delta^i_j$

Change of basis matrix $\overleftrightarrow{C}$

$|f_i\rangle = \sum_i C^i_i |e_j\rangle \qquad C^i_j = \langle e^i|f_j\rangle$ $\quad$ ith row, jth column $= \langle$old bra | new ket$\rangle$

$$\overleftrightarrow{C} = \begin{bmatrix} \langle e^1|f_1\rangle & \cdots & \langle e^1|f_N\rangle \\ \vdots & \ddots & \vdots \\ \langle e^N|f_1\rangle & \cdots & \langle e^N|f_N\rangle \end{bmatrix} = \begin{bmatrix} \vec{f}_{1,[e]} & \cdots & \vec{f}_{N,[e]} \end{bmatrix}$$

Old basis : Energy basis $|-\rangle, |+\rangle$ $\quad$ <u>DSW</u> $\quad |L\rangle = \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle)$

New basis : Well basis $|L\rangle, |R\rangle$ $\quad |R\rangle = \frac{1}{\sqrt{2}}(|-\rangle - |+\rangle)$

$$C_{E\to W} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \cancel{\begin{bmatrix} \langle L|L\rangle \\ \langle \end{bmatrix}} = \begin{bmatrix} \langle -|L\rangle & \langle -|R\rangle \\ \langle +|L\rangle & \langle +|R\rangle \end{bmatrix}$$

$\overleftrightarrow{C}$ is a unitary matrix

$$\overleftrightarrow{C}^\dagger \overleftrightarrow{C} = \overleftrightarrow{\mathbb{1}} \qquad (C^\dagger C)^i_{\ j} = \delta^i_{\ j}$$

$$\sum_k (C^\dagger)^i_{\ k} C^k_{\ j} = \sum_k (C^k_{\ i})^* C^k_{\ j} = \sum_k (\langle e^k | f_i\rangle)^* \langle e^k | f_j\rangle$$

$$= \sum_k \langle f^i | e^k\rangle\langle e^k | f_j\rangle = \cancel{\sum} \langle f^i | \left(\sum_k |e_k\rangle\langle e^k|\right) | f_j\rangle = \langle f^i | f_j\rangle = \delta^i_{\ j}$$

$$\alpha^i_{[f]} = \langle f^i | \alpha\rangle = \sum_j (C^\dagger)^i_{\ j} \alpha^j_{[e]}$$

$$\vec{\psi}_{\alpha,[f]} = \overleftrightarrow{C}^\dagger \vec{\psi}_{\alpha,[e]}$$

$$\beta_{i,[f]} = \langle \beta | f_i\rangle = \sum_j \beta_{j,[e]} C^j_{\ i}$$

$$A^i_{\ j[f]} = \langle f^i | \hat{A} | f_j\rangle = \sum_{kl} (C^\dagger)^i_{\ k} A^k_{\ l[e]} C^l_{\ j} \qquad \overleftrightarrow{A}_{[f]} = \overleftrightarrow{C}^\dagger \overleftrightarrow{A}_{[e]} \overleftrightarrow{C}$$

Rotate objects then rotate observer/coordinate

<u>DSW</u>

$$\overleftrightarrow{C}_{[e]\to w} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \overleftrightarrow{C}_{w\to e} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\overleftrightarrow{C}^\dagger \overleftrightarrow{C} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \overleftrightarrow{\mathbb{1}} \qquad \cancel{\hat{U}_{[e]}(t) = \begin{bmatrix} e^{-iE_1 t/\hbar} & 0 & \cdots & 0 \\ 0 & & & \\ 0 & & & \\ \vdots & & & \end{bmatrix}}$$

$$\hat{U}_{[e]}(t) \doteq \begin{bmatrix} e^{-iE_1 t/\hbar} & 0 \\ 0 & e^{-iE_2 t/\hbar} \end{bmatrix} = e^{-iE_0 t/\hbar}\begin{bmatrix} e^{i\epsilon t/\hbar} & 0 \\ 0 & e^{-i\epsilon t/\hbar} \end{bmatrix}$$

$$\hat{U}_{[w]}(t) \doteq \overleftrightarrow{C}^\dagger \overleftrightarrow{U}_{[e]} \overleftrightarrow{C} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} e^{-iE_0 t/\hbar}\begin{bmatrix} e^{i\epsilon t/\hbar} & 0 \\ 0 & e^{-i\epsilon t/\hbar} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{e^{-iE_0 t/\hbar}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{i\epsilon t/\hbar} & e^{i\epsilon t/\hbar} \\ e^{-i\epsilon t/\hbar} & -e^{-i\epsilon t/\hbar} \end{bmatrix} = \frac{e^{-iE_0 t/\hbar}}{2}\begin{bmatrix} e^{i\epsilon t/\hbar} + e^{-i\epsilon t/\hbar} & e^{i\epsilon t/\hbar} - e^{-i\epsilon t/\hbar} \\ e^{i\epsilon t/\hbar} - e^{-i\epsilon t/\hbar} & e^{i\epsilon t/\hbar} + e^{-i\epsilon t/\hbar} \end{bmatrix} = \cancel{e^{-iE_0 t/\hbar}[}$$

$$= e^{-iE_0 t/\hbar}\begin{bmatrix} \cos\frac{\epsilon t}{\hbar} & i\sin\frac{\epsilon t}{\hbar} \\ i\sin\frac{\epsilon t}{\hbar} & \cos\frac{\epsilon t}{\hbar} \end{bmatrix}$$

Similarity transformations

Trace doesn't change

Eigenvalues don't change

Determinants don't change

Simple Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \qquad E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad \cancel{n \in \mathbb{Z}} \quad n = 0, 1, 2, 3, \ldots$$

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}\, x \qquad \hat{H} = \frac{1}{2m}\left(\hat{p}^2 + (m\omega\hat{x})^2\right) = \cancel{\tfrac{1}{2m}(}$$

$$u^2 + v^2 = (u+iv)(u-iv)$$

$$= u^2 + ivu - i\cancel{uv} + v^2$$

$$= u^2 + v^2 - i[u, v]$$

$$\cancel{\hat{H} = \tfrac{1}{2m}(\hat{p}}$$

Ladder Operators

$$\hat{a}_- \equiv \frac{1}{\sqrt{2\hbar\omega m}}\left(i\hat{p} + m\omega\hat{x}\right)$$ lowering operator, annihilation operator

$$\hat{a}_+ \equiv \frac{1}{\sqrt{2\hbar\omega m}}\left(-i\hat{p} + m\omega\hat{x}\right) = \hat{a}_-^\dagger$$ raising operator, creation operator

$$\hat{x} = \sqrt{\frac{\hbar}{2\omega m}}\left(\hat{a}_+ + \hat{a}_-\right) \qquad \hat{p} = i\sqrt{\frac{\hbar\omega m}{2}}\left(\hat{a}_+ - \hat{a}_-\right) \qquad \hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\hat{\mathbb{1}}\right)$$

$$[\hat{a}_-, \hat{a}_+] = \hat{\mathbb{1}}$$

Phys 137A 30 Jul 19

The Simple Harmonic Oscillator

$$V(x) = \frac{1}{2} m\omega^2 x^2 \qquad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

$$\hat{a}_- \equiv \frac{1}{\sqrt{2\hbar\omega m}}\left(i\hat{p} + m\omega\hat{x}\right) \qquad \hat{a}_+ \equiv \frac{1}{\sqrt{2\hbar\omega m}}\left(-i\hat{p} + m\omega\hat{x}\right)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2\omega m}}\left(\hat{a}_+ + \hat{a}_-\right) \qquad \hat{p} = i\sqrt{\frac{\hbar\omega m}{2}}\left(\hat{a}_+ - \hat{a}_-\right)$$

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\hat{\mathbb{1}}\right) \qquad \hat{a}_+ = \hat{a}_-^\dagger$$

claim: If $|\alpha\rangle$ is an eigenket of $\hat{H}$ with eigenvalue E_α then $\hat{a}_\pm|\alpha\rangle$ is an eigenket of $\hat{H}$ with eigenvalue $E_\alpha \pm \hbar\omega$

$\hat{a}_\pm$ is not Hermitian nor Unitary and may not result in a normalized state.

$$[\hat{H}, \hat{a}_+] = \left[\hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\hat{\mathbb{1}}\right), \hat{a}_+\right]$$

$$= \hbar\omega[\hat{a}_+\hat{a}_-, \hat{a}_+] + \frac{\hbar\omega}{2}[\hat{\mathbb{1}}, \hat{a}_+]$$

$$= \hbar\omega\hat{a}_+[\hat{a}_-, \hat{a}_+] + \hbar\omega[\hat{a}_+, \hat{a}_+]\hat{a}_- = \hbar\omega\hat{a}_+$$

$$[\hat{H}, \hat{a}_-] = -\hbar\omega\hat{a}_-$$

$$\hat{H}\left(\hat{a}_\pm|\alpha\rangle\right) = \hat{H}\hat{a}_\pm|\alpha\rangle = \left(\hat{a}_\pm\hat{H} + [\hat{H}, \hat{a}_\pm]\right)|\alpha\rangle$$

$$= \hat{a}_\pm(\hat{H}|\alpha\rangle) \pm \hbar\omega\hat{a}_\pm|\alpha\rangle = \hat{a}_\pm E_\alpha|\alpha\rangle \pm \hbar\omega\hat{a}_\pm$$

$$= (E_\alpha \pm \hbar\omega)(\hat{a}_\pm|\alpha\rangle)$$

$\vdots$

$\hat{a}_+^2|\alpha\rangle$

$\hat{a}_+|\alpha\rangle$

$|\alpha\rangle$

$\hat{a}_-|\alpha\rangle$ Defines "Ground state"

$\hat{a}_-^2|\alpha\rangle$

$\vdots$

$\hat{a}_-^n|\alpha\rangle$

0

$\Psi \sim E_\alpha$

Lowest rung $|0\rangle$

$$\boxed{\hat{a}_-|0\rangle = 0}$$ or some negative eigenvalue

$$\hat{H}|0\rangle = \hbar\omega\left(\hat{a}_+\hat{a}_- + \tfrac{1}{2}\hat{\mathbb{1}}\right)|0\rangle$$
$$= \hbar\omega\hat{a}_+\hat{a}_-|0\rangle + \frac{\hbar\omega}{2}\hat{\mathbb{1}}|0\rangle$$
$$= \frac{\hbar\omega}{2}|0\rangle$$

Ground state $|0\rangle$ with $E_0 = \frac{1}{2}\hbar\omega$

This ladder is the only ladder

$|1\rangle \propto \hat{a}_+|0\rangle$, $|2\rangle \propto \hat{a}_+|1\rangle$, ...

$|n\rangle = N\hat{a}_+^n|0\rangle$ $\quad \langle n|\hat{H}|n\rangle = E_n$

$\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$ $\quad \hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle$

$$|n\rangle = \frac{1}{\sqrt{n!}}\hat{a}_+^n|0\rangle, \quad E_n = \hbar\omega\left(n + \tfrac{1}{2}\right)$$

$\langle n|m\rangle = \delta_{n,m}$ $\quad \hat{\mathbb{1}} = \sum_i |n\rangle\langle n|$

$\hat{N} \equiv \hat{a}_+\hat{a}_-$ $\quad \hat{N}|n\rangle = \hat{a}_+\hat{a}_-|n\rangle = \hat{a}_+\sqrt{n}|n-1\rangle = \sqrt{n}\sqrt{n-1+1}\,|n-1+1\rangle = n|n\rangle$

number operator

$\hat{N} \equiv \hat{a}_+\hat{a}_-$, $\hat{N}|n\rangle = n|n\rangle$

$$\langle n|\hat{p}|n'\rangle = i\sqrt{\frac{\hbar\omega m}{2}}\langle n|\hat{a}_+ - \hat{a}_-|n'\rangle$$

$$= i\sqrt{\frac{\hbar\omega m}{2}}\left(\langle n|\hat{a}_+|n'\rangle - \langle n|\hat{a}_-|n'\rangle\right)$$

only nonzero if $n = n'-1$

only nonzero if $n = n'+1$

$$\hat{p} \doteq \overleftrightarrow{P} = i\sqrt{\frac{\hbar\omega m}{2}}\begin{bmatrix} 0 & \langle 0|\hat{a}_-|1\rangle & 0 & \cdots & \cdots & 0 \\ \langle 1|\hat{a}_+|0\rangle & 0 & \langle 1|\hat{a}_-|2\rangle & 0 & \cdots & \\ 0 & -\langle 1|\hat{a}_-|2\rangle & 0 & \langle 2|\hat{a}_-|3\rangle & & \\ 0 & 0 & -\langle 2|\hat{a}_-|3\rangle & 0 & \ddots & \\ \vdots & \vdots & 0 & & \ddots & \\ 0 & \vdots & \vdots & & & \ddots \end{bmatrix} = i\sqrt{\frac{\hbar\omega m}{2}}\begin{bmatrix} 0 & -1 & 0 & 0 & \cdots \\ 1 & 0 & -\sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \\ 0 & 0 & \sqrt{3} & 0 & \ddots \\ \vdots & \vdots & 0 & & \ddots \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

$$\hat{T} = \frac{\hat{p}^2}{2m} = \frac{-\hbar\omega m}{2}\frac{1}{2m}(\hat{a}_+ - \hat{a}_-)^2 = \frac{-\hbar\omega m}{4m}\left(\hat{a}_+^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ + \hat{a}_-^2\right)$$

$$= \frac{-\hbar\omega}{4}\left(\hat{a}_+^2 - 2\hat{a}_+\hat{a}_- + \hat{a}_-^2 - \hat{\mathbb{1}}\right)$$

$\hat{a}_+^2$ $\quad n = n'+2$ $\qquad \langle 3|\hat{T}|4\rangle = 0$

$\hat{a}_-^2$ $\quad n = n'-2$

$\hat{a}_+\hat{a}_-$ $\quad n = n'$

$\hat{\mathbb{1}}$ $\quad n = n'$

$$\frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x})|0\rangle = 0$$

$$\left(\frac{i\hbar}{i}\frac{\partial}{\partial x} + m\omega x\right)\psi_0(x) = 0$$

$$\frac{\partial\psi_0(x)}{\partial x} = -\frac{m\omega}{\hbar}x\,\psi_0(x)$$

$$\psi_0 = Ae^{\frac{-m\omega x^2}{2\hbar}}$$

$$|n\rangle = \frac{1}{\sqrt{n!}}\hat{a}_+^n|0\rangle$$

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2\hbar\omega m}}\right)^n\left(-\hbar\frac{\partial}{\partial x} + m\omega x\right)^n\psi_0(x)$$

$|\alpha\rangle$, $\hat{A}$, $\hat{B}$

$$\boxed{\begin{array}{l}\hat{A}|\alpha\rangle = a|\alpha\rangle, \quad \hat{B}|\alpha\rangle = b|\alpha\rangle \\ \text{condition for simultaneous Diagonalization}\end{array}}$$

$$[\hat{A},\hat{B}]|\alpha\rangle = (\hat{A}\hat{B} - \hat{B}\hat{A})|\alpha\rangle = \hat{A}b|\alpha\rangle - \hat{B}a|\alpha\rangle$$

$$= ba|\alpha\rangle - ab|\alpha\rangle = (ba - ab)|\alpha\rangle = 0$$

$\cancel{[\hat{A},\hat{B}]}$ ~~$\neq 0$~~ $\quad [\hat{A},\hat{B}]|\alpha\rangle = 0$

$[\hat{A},\hat{B}] = 0$ if I want a simultaneous Eigenbasis

$\hat{A}$ and $\hat{B}$ are <u>compatible</u>

A complete set of commuting observables (CSCO)

mutually commute

$\{\hat{A}_1, \ldots, \hat{A}_N\}$ $\quad \{|\lambda_i^{(1)}, \lambda_i^{(2)}, \lambda_i^{(3)}, \ldots, \lambda_i^{(N)}\rangle\}$

$$\hat{A}_\alpha|\lambda_i^{(1)}, \lambda_i^{(2)}, \ldots\rangle = \lambda_i^{(\alpha)}|\lambda_i^{(1)}, \lambda_i^{(2)}, \ldots\rangle$$

How do we add degrees of freedom to a quantum system?

$\mathcal{H}_1$ = Hilbert space for 1st degree of freedom $\quad |\alpha\rangle \in \mathcal{H}_1 \quad |\beta\rangle_2 \in \mathcal{H}_2$

$\mathcal{H}_2$ = Hilbertspace for 2nd degree of freedom $\quad |\alpha,\beta\rangle \in \mathcal{H}_{12}$

Combining the degrees of freedom gives $\underset{6D}{\mathcal{H}_{12}} = \underset{2D}{\mathcal{H}_1} \otimes \underset{3D}{\mathcal{H}_2}$ $\quad \equiv |\alpha\rangle_1 \otimes |\beta\rangle_2$

Phys 137A 31Jul19

System 1 Hilbert Space $\mathcal{H}_1$ $\{|e_i\rangle_1\}$ $|\alpha\rangle_1$

System 2 Hilbert Space $\mathcal{H}_2$ $\{|f_k\rangle_2\}$ $|\beta\rangle_2$

System 1 is in $|\alpha\rangle_1$ and system 2 is in $|\beta\rangle_2$

$|\alpha, \beta\rangle$

$\equiv |\alpha\rangle_1 \otimes |\beta\rangle_2$

Combined System Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$

Basis States: ~~$|e_i\rangle$~~ $|e_i, f_k\rangle \equiv |e_i\rangle_1 \otimes |f_k\rangle_2$

N_1 $\{|e_i\rangle_1\}$ $N_2 \{|f_k\rangle_2\}$ $N_1 N_2$ s

$$|\text{arbitrary state}\rangle = \sum_{i,k} c^{i,k} |e_i\rangle_1 \otimes |f_k\rangle_2$$

Simple state or a separable state is a state $|\alpha\rangle_1 \otimes |\beta\rangle_2$

$f(x,y) \leftarrow$ general function of two variables

$\hookrightarrow g(x)(h(y)) \leftarrow$ separable function

If we can't write $|\text{state}\rangle$ as $|\alpha\rangle_1 \otimes |\beta\rangle_2$ then we call $|\text{state}\rangle$ an entangled state

System 1 = proton in ISW with basis $\{|n_p\rangle_p\}$

System 2 = neutron in ISW with basis $\{|n_n\rangle_n\}$

Deuterium = System 1 $\otimes$ System 2 "tensored with"

$|1,2\rangle \leftarrow$ proton in ground state and neutron in first excited state

$$\frac{1}{\sqrt{2}}(|1,2\rangle + |2,2\rangle) = \frac{1}{\sqrt{2}}(|1\rangle_p \otimes |2\rangle_n + |2\rangle_p \otimes |2\rangle_n) = \left(\frac{1}{\sqrt{2}}(|1\rangle_p + |2\rangle_p)\right) \otimes |2\rangle_n$$

$\frac{1}{\sqrt{2}}(|1,2\rangle + |2,1\rangle)$ EPR Paradox

entangled state

Measure proton energy and get "ground state"

The state collapses and after normalization we have $|1,2\rangle$

Kayons have spin and have these properties

Ket bra is a tensor product

$$\langle \alpha, \beta | \gamma, \delta \rangle = \langle \alpha | \gamma \rangle_1 \langle \beta | \delta \rangle_2$$

$$\langle e^i, f^k | e_j, f_l \rangle = \delta_{ij} \delta_{k,l}$$

$$|\gamma\rangle = \sum c_{ij}^{ik} |e_i, f_k\rangle \quad |\delta\rangle = \sum d^{ik} |e_i, f_k\rangle$$

$$\langle \gamma | \delta \rangle = \sum_{ik} (c^{ik})^* d^{jl} \langle e^i, f^k | e_j, f_l \rangle = \sum (c^{ik})^* d^{jl} \langle e^i, f^k | e_j, f_l \rangle$$

3 Dimensions

$\psi(x,y,z)$ or $\psi(\vec{x})$ or $\psi(\vec{r})$

$$\mathcal{H} = L^2(\mathbb{R}^3) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^* \psi \, dx^3 = \text{finite}$$

Probability of finding a particle at position $\vec{x}$ within a small volume d^3x

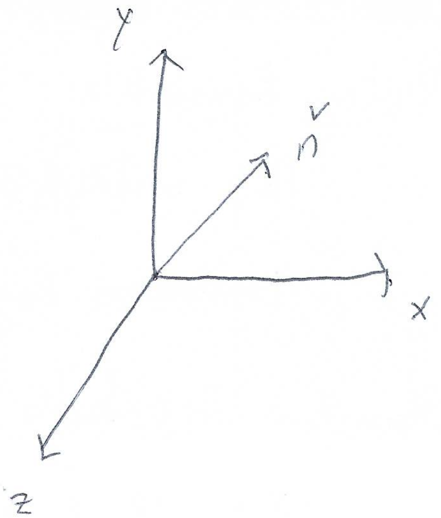
$$dP = |\psi|^2 d^3x \qquad a,b,c \text{ spatial indices}$$

$$X_a = \begin{cases} X_1 = x & a=1 \\ X_2 = y & a=2 \\ X_3 = z & a=3 \end{cases} \qquad P_a = \begin{cases} P_x & a=1 \\ P_y & a=2 \\ P_z & a=3 \end{cases}$$

$\hat{x}, \hat{y}, \hat{z}$ operators

$\check{x}, \check{y}, \check{z}$ unit vectors

$$\check{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$



$$\left.\begin{matrix}\hat{p}_x\\ \hat{p}_y\\ \hat{p}_z\end{matrix}\right\} \quad \hat{\vec{p}} = \sum_a \hat{p}_a \check{x}_a = \hat{p}_x\check{x} + \hat{p}_y\check{y} + \hat{p}_z\check{z}$$

$$\hat{\vec{p}}|\alpha\rangle = \begin{bmatrix}\hat{p}_x|\alpha\rangle\\ \hat{p}_y|\alpha\rangle \\ \hat{p}_z|\alpha\rangle\end{bmatrix}$$

Momentum operator in $\check{n}$ direction

$$\check{n}\cdot\hat{\vec{p}} = \hat{p}_x n_x + \hat{p}_y n_y + \hat{p}_z n_z$$

$$\hat{\vec{p}}^2 = \hat{\vec{p}}\cdot\hat{\vec{p}} = \hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2 = \sum_a \hat{p}_a^2$$

$$\hat{x}_a\psi(\vec{x}) = x_a\psi(\vec{x}) \qquad \hat{\vec{x}}\psi(\vec{x}) = \vec{x}\psi(\vec{x}) = \begin{bmatrix} x\psi(\vec{x})\\ y\psi(\vec{x})\\ z\psi(\vec{x})\end{bmatrix} \qquad \vec{p}_a\psi(\vec{x}) = \frac{\hbar}{i}\partial_a\psi(\vec{x})$$

$$\hat{\vec{p}}_a\psi(\vec{x}) = \frac{\hbar}{i}\frac{\partial}{\partial x_a}\psi(\vec{x}) \qquad \hat{\vec{p}}\psi(\vec{x}) = \begin{bmatrix}\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(\vec{x})\\ \frac{\hbar}{i}\frac{\partial}{\partial y}\psi(\vec{x})\\ \frac{\hbar}{i}\frac{\partial}{\partial z}\psi(\vec{x})\end{bmatrix} \qquad \hat{\vec{p}}\psi(\vec{x}) = \frac{\hbar}{i}\vec{\nabla}\psi(\vec{x})$$

$$|\vec{x}|^2 \cancel{\hat{x}^2}\psi(\vec{x}) = r^2\psi(\vec{x}) \qquad r = \sqrt{x^2+y^2+z^2}$$

$$\hat{p}^2\psi(\vec{x}) = -\hbar^2\nabla^2\psi(\vec{x}) = -\hbar^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right)$$

$$[\hat{x}_a, \hat{x}_b] = 0 \quad [\hat{p}_a, \hat{p}_b] = 0 \quad [\hat{x}_a, \hat{p}_b] = 0 \;\; a \neq b \quad [\hat{x}_a, \hat{p}_b] = i\hbar\,\delta_{ab}$$

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} + V(\vec{x}) \quad \cancel{-\frac{\hbar^2}{2m}}$$

TDSE: $i\hbar\dfrac{\partial\Psi(\vec{x},t)}{\partial t} = -\dfrac{\hbar^2}{2m}\vec{\nabla}^2\Psi(\vec{x},t) + V(\vec{x})\Psi(\vec{x},t)$

TISE: $-\dfrac{\hbar^2}{2m}\nabla^2\psi_n(\vec{x}) + V(\vec{x})\psi_n(\vec{x}) = E_n\psi(\vec{x})$

$$\Psi(\vec{x},t) = \sum_n c_n\psi_n(\vec{x})e^{-iE_nt/\hbar}$$

Position Eigenbasis

$$|\vec{x}\rangle = |x,y,z\rangle = |x\rangle_x \otimes |y\rangle_y \otimes |z\rangle_z$$

Simultaneous Eigenket of the Complete Set of Commuting Observables $\hat{x},\hat{y},\hat{z}$

Momentum Eigenbasis

$$|\vec{p}\rangle = |p_x,p_y,p_z\rangle = |p_x\rangle_x \otimes |p_y\rangle_y \otimes |p_z\rangle_z$$

Simultaneous Eigenket of the complete set of commuting observables $\hat{p}_x,\hat{p}_y,\hat{p}_z$

$$\langle\vec{x}|\vec{x}'\rangle = \langle x|x'\rangle\langle y|y'\rangle\langle z|z'\rangle = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$= \delta^3(\vec{x}-\vec{x}') \qquad \iiint f(\vec{x})\,\delta(\vec{x}-\vec{x}')\,d^3x = f(\vec{x}')$$

$$\langle\vec{p}|\vec{p}'\rangle = \delta^3(\vec{p}-\vec{p}')$$

$$\hat{\mathbb{1}} = \iiint |\vec{x}\rangle\langle\vec{x}|\,d^3x = \iiint |\vec{p}\rangle\langle\vec{p}|\,d^3p$$

$$\Psi(\vec{x},t) = \langle\vec{x}|\alpha(t)\rangle \qquad \Phi(\vec{p},t) = \langle\vec{p}|\alpha(t)\rangle$$

$$\langle\vec{x}|\vec{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}}e^{i\vec{p}\cdot\vec{x}/\hbar} = \left(\frac{1}{\sqrt{2\pi\hbar}}e^{ip_x x/\hbar}\right)\left(\frac{1}{\sqrt{2\pi\hbar}}e^{ip_y y/\hbar}\right)\left(\frac{1}{\sqrt{2\pi\hbar}}e^{ip_z z/\hbar}\right)$$

Phys 137A 01 Aug 19

Infinite Cubical Well

$0 \le x \le a$

$0 \le y \le a$

$0 \le z \le a$

0 inside

∞ outside

Inside well TISE

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) = E\,\psi(x,y,z)$$

$$= -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) = E\,\psi(x,y,z)$$

Separation ansatz

$$\psi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$\langle x,y,z|state\rangle$ $\quad |state\rangle = |x\,state\rangle_x \otimes |y\,state\rangle_y \otimes |z\,state\rangle_z$

$$= \langle x|xstate\rangle\langle y|ystate\rangle\langle z|zstate\rangle$$

$$\cancel{\frac{\hbar^2}{2m}Y} \quad -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) = E\psi = EXYZ$$

$$= -\frac{\hbar^2}{2m}\left(X''YZ + XY''Z + XYZ''\right) = EXYZ$$

$$= -\frac{\hbar^2}{2m}\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) = E$$

$$-\frac{\hbar^2}{2m}\frac{X''}{X} = E_x \qquad -\frac{\hbar^2}{2m}\frac{Y''}{Y} = E_y \qquad -\frac{\hbar^2}{2m}\frac{Z''}{Z} = E_z \qquad E_x + E_y + E_z = E$$

$$X'' = \frac{-2mE_x}{\hbar^2} \qquad Y'' = \frac{-2mE_y}{\hbar^2} \qquad Z'' = \frac{-2mE_z}{\hbar^2}$$

$$X'' = \frac{-2mE_x}{\hbar^2} \qquad Y'' = \frac{-2mE_y}{\hbar^2} \qquad Z'' = -\frac{2mE_z}{\hbar^2}$$

$$X(0) = X(a) = 0 \qquad Y(0) = Y(a) \qquad Z(0) = Z(a)$$

$\cancel{X(x) = A}$

$$X_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad E_x = \frac{n_x^2\pi^2\hbar^2}{2ma^2}$$

$$\psi_{n_x n_y n_z}(\vec{x}) = \left(\frac{2}{a}\right)^{3/2}\sin\left(\frac{n_x\pi x}{a}\right)\sin\left(\frac{n_y\pi y}{a}\right)\sin\left(\frac{n_z\pi z}{a}\right)$$

kets

$$E_{n_x n_y n_z} = \frac{n_x^2\pi^2\hbar^2}{2ma^2} + \frac{n_y^2\pi^2\hbar^3}{2ma^2} + \frac{n_z^2\pi^2\hbar^2}{2ma^2}$$

$$|n_x, n_y, n_z\rangle = |n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle$$

$$\langle n_x, n_y, n_z | n_{x'}, n_{y'}, n_{z'}\rangle = \delta_{xx'}\,\delta_{yy'}\,\delta_{zz'}$$

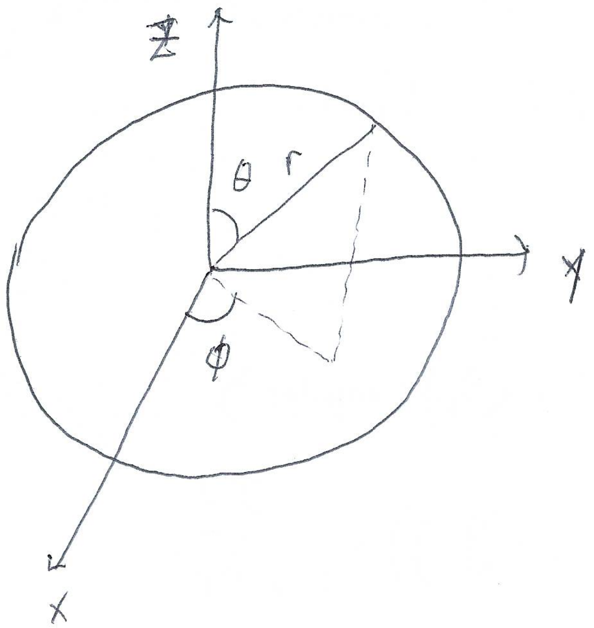
$\cancel{\hat{\mathbb{1}} = \langle n_x, n_y, n_z|}$

This is special

$$\hat{\mathbb{1}} = |n_x, n_y, n_z\rangle\langle n_x, n_y, n_z| \qquad V(\vec{x}) = V(x) + V(y) + V(z)$$

$$\hat{H} = \hat{H}_x + \hat{H}_y + \hat{H}_z = \frac{\hat{p}_x^2}{2m} + V(x) + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m} + V(y) + V(z)$$

$\{\hat{H}_x, \hat{H}_y, \hat{H}_z\}$ We are in a degenerate space



$r = \sqrt{x^2+y^2+z^2}$

θ = polar angle = $\arccos\left(\frac{z}{r}\right)$

ϕ = azimuthal angle = $\arctan\left(\frac{y}{x}\right)$

$\cancel{0 \le r \le \infty}$ $\quad 0 \le r < \infty$

$0 \le \theta \le \pi \qquad 0 \le \phi < 2\pi$

is cyclic

$\cancel{x = r\cos\phi \quad y = r\sin\phi \quad z = r\cos\theta}$

$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

$d\Omega = d\cos\theta\, d\phi = \sin\theta\, d\theta\, d\phi \qquad d^3x = r^2 dr\, d\Omega = r^2 dr\, d\cos\theta\, d\phi$

solid angle element

$= r^2 dr \sin\theta\, d\theta\, d\phi$

$\iint d\Omega = 4\pi \qquad \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\, r^2 \sin\theta\, f(r,\theta,\phi)$

$\delta^{(3)}(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')$

$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$

$\mathcal{H} = \mathcal{H}_{rad} \otimes \mathcal{H}_{angular}$

$|r\rangle \qquad |\theta,\phi\rangle$

$|r,\theta,\phi\rangle = |r\rangle \otimes |\theta,\phi\rangle = |r\sin\theta\cos\phi\rangle \otimes |r\sin\theta\sin\phi\rangle \otimes |r\cos\theta\rangle$

$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f(r)}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r f(r)\right)$

Spherically Symmetric Potentials $V(\vec{x}) = V(r)$

Separate states

$|\text{separable state}\rangle = |\text{radial}\rangle \otimes |\text{angular}\rangle$

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi) = \langle r|\text{radial}\rangle \langle \theta,\phi|\text{angular}\rangle$$

μ = mass

$$\nabla^2\psi = \left(\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right)Y + \frac{R}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}$$

TISE

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + V(r)\psi = E\psi$$

$$\frac{1}{R}\left(\left(\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right) - \frac{2\mu r^2}{\hbar^2}(V(r) - E)\right)$$

$$+ \frac{1}{Y}\left(\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right) = 0$$

Radial $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2\mu r^2}{\hbar^2}(V-E)R = \ell(\ell+1)R$

Angular $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -\ell(\ell+1)Y$

$$\int_0^\infty |R|^2 r^2\,dr = 1 \qquad \iint |Y^*|^2\,d\Omega = 1$$

Normalization Condition

$$\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y}{\partial\theta}\right) + \ell(\ell+1)\sin^2\theta\, Y + \frac{\partial^2 Y}{\partial\phi^2} = 0$$

Separable solutions, separate Y

$$Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$$

$$\frac{1}{\Theta_{\ell m}}\left(\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta_{\ell m}}{d\theta}\right) + \ell(\ell+1)\sin^2\theta\,\Theta\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

Azimuthal: $\frac{d^2\Phi_m}{d\phi^2} = -m^2\Phi_m$ Polar: $\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta_{\ell m}}{d\theta}\right) + \ell(\ell+1)\sin^2\theta\,\Theta_{\ell m} = m^2\Theta_{\ell m}$

$\Phi_m(\phi) = e^{im\phi}$ $\Phi_m(\phi) = \Phi_m(\phi+2\pi)$ $e^{im\phi} = e^{im(\phi+2\pi)}$ $e^{2\pi i m} = 1$

$m \in \mathbb{Z}$

Polar: diverges unless $\ell = 0, 1, 2, 3, \ldots$ and $|m| \leq \ell$

Solutions Associated Legendre Functions $P_\ell^m(\cos\theta)$

Legendre Polynomials $P_\ell(\cos\theta)$

$$\int_{-1}^{1} P_\ell^*(x) P_{\ell'}(x)\,dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

Rodriguez Formula

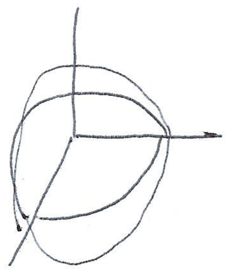
Solutions are the spherical harmonics

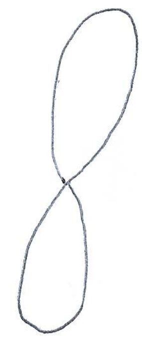
$$Y_\ell^m(\theta,\phi) = \varepsilon\sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}}\sqrt{\frac{2\ell+1}{4\pi}}\,P_\ell^m(\cos\theta)\,e^{im\phi} \qquad \ell = 0, 1, 2, \ldots$$

$$\varepsilon = \begin{cases} (-1)^m & m \geq 0 \\ 1 & m < 0 \end{cases} \qquad m = -\ell, -\ell+1, \ldots, \ell-1, \ell$$

$Y_\ell^m(\theta,\phi) = \langle \theta,\phi | \ell m \rangle$ $\quad$ $\langle \ell m | \ell' m' \rangle = \delta_{\ell\ell'}\delta_{mm'} = \int Y_{\ell'}^{m'}(\theta,\phi)^* Y_\ell^m(\theta,\phi)\, d\Omega\, \cancel{d^3x}$

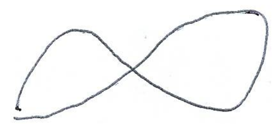
$\hat{1} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |\ell m\rangle\langle \ell m|$ $\quad$ $|\ell m\rangle$



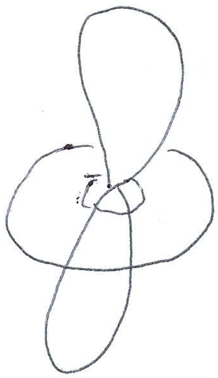


$\ell=1$

$m=0$



$m=1$ $\quad$ $m=1$



$$f(\theta,\phi) = \sum_{\ell m} c_{\ell m} Y_\ell^m(\theta,\phi)$$

$$c_{\ell m} = \langle \ell m | f \rangle = \int Y_\ell^{m*}(\theta,\phi) f(\theta,\phi)\, d\Omega$$

Phys 137A 05Aug19

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi = E\psi \qquad \psi = RY$$

Radial: $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2\mu r^2}{\hbar^2}\left(V(r) - E\right)R = l(l+1)R$

Angular: $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)Y$

Spherical Harmonics

$Y_l^m(\theta, \phi$

$l = 0, 1, 2, 3, \ldots$

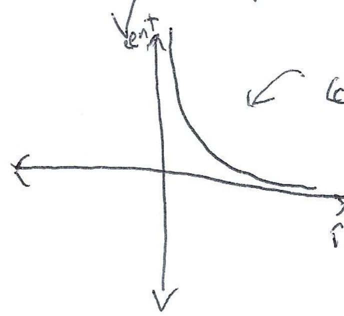
$m = -l, \ldots, l$

$$\frac{\partial^2 Y}{\partial\phi^2} = -m^2 Y$$

Radial $\quad u(r) \equiv rR(r)$

$$-\frac{\hbar^2}{2\mu}\frac{d^2 u_{nl+}}{dr^2}\left(V(r) + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\right)u = E_{nl}u_{nl}$$

Looks identical to a 1d TISE with an "effective potential" $V_{eff}(r) = V(r) + \underbrace{\frac{\hbar^2 l(l+1)}{2mr^2}}_{\text{centrifugal term}}$



$$\int_0^\infty |R|^2 r^2 dr = 1$$

Normalization $\int_0^\infty |u|^2 dr = 1$

$u(0) = 0 \qquad \hat{H}|n\,l\,m\rangle = E_{nl}|nlm\rangle$

$u_{nl}(r)$

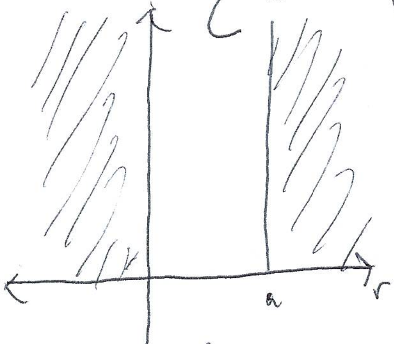
$n = 1, 2, 3, \ldots$

$l = 0, 1, 2, \ldots$

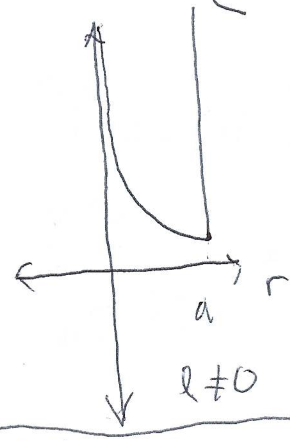
$E_{nl} \qquad \psi_{nlm}(r, \theta, \phi) = \frac{u_{nl}(r)}{r}Y_l^m(\theta, \phi)$

Infinite Spherical Well

$$V(r) = \begin{cases} 0 & 0 < r < a \\ \infty & \text{else} \end{cases} \qquad V_{eff} = \begin{cases} \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} & r \leq a \\ \infty & r > a \end{cases}$$



$l = 0$



$l \neq 0$

$$\boxed{\begin{array}{l} u_{nl}(0) = 0 \\ u_{nl}(a) = 0 \\ \text{Boundary conditions} \end{array}}$$

$$k_{nl} = \sqrt{\frac{2mE_{nl}}{\hbar^2}} \qquad \boxed{\frac{d^2 u_{nl}}{dr^2} = \left(\frac{l(l+1)}{r^2} - k_{nl}^2\right) u_{nl}(r) \quad \underline{\text{Bessel Equation}}}$$

$l = 0$ "s-waves"

$$\frac{d^2 u_{n0}}{dr^2} = -k_{n0}^2 u_{n0} \qquad k_{n0} = \sqrt{\frac{2\mu E_{n0}}{\hbar^2}} \qquad E_{n0} = \frac{\hbar^2 \pi^2 n^2}{2\mu a^2} = \frac{\hbar^2 (\pi n)^2}{2\mu a^2}$$

$$u_{n0}(r) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi r}{a}\right)$$

$$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \qquad \psi_{n00} = \frac{1}{\sqrt{2\pi a}} \frac{\sin\left(\frac{n\pi r}{a}\right)}{r}, \quad E_{n0} = \frac{\hbar^2 (n\pi)^2}{2\mu a^2}$$

$l \neq 0$ Solutions: "spherical Bessel" and "spherical Neuman" functions

$j_l(x)$, $n_l(x)$

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}, \quad j_0(x) = \frac{\sin x}{x}$$

$$n_l(x) = (-x)^{l+1} \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x}, \quad n_0(x) = -\cos x$$

$$u_{n\ell}(r) = A r j_\ell(k_{n\ell} r) + B r n_\ell(k_{n\ell} r)$$

$B = 0$ due to ~~it exploding at 0~~ $u(0) \neq 0$

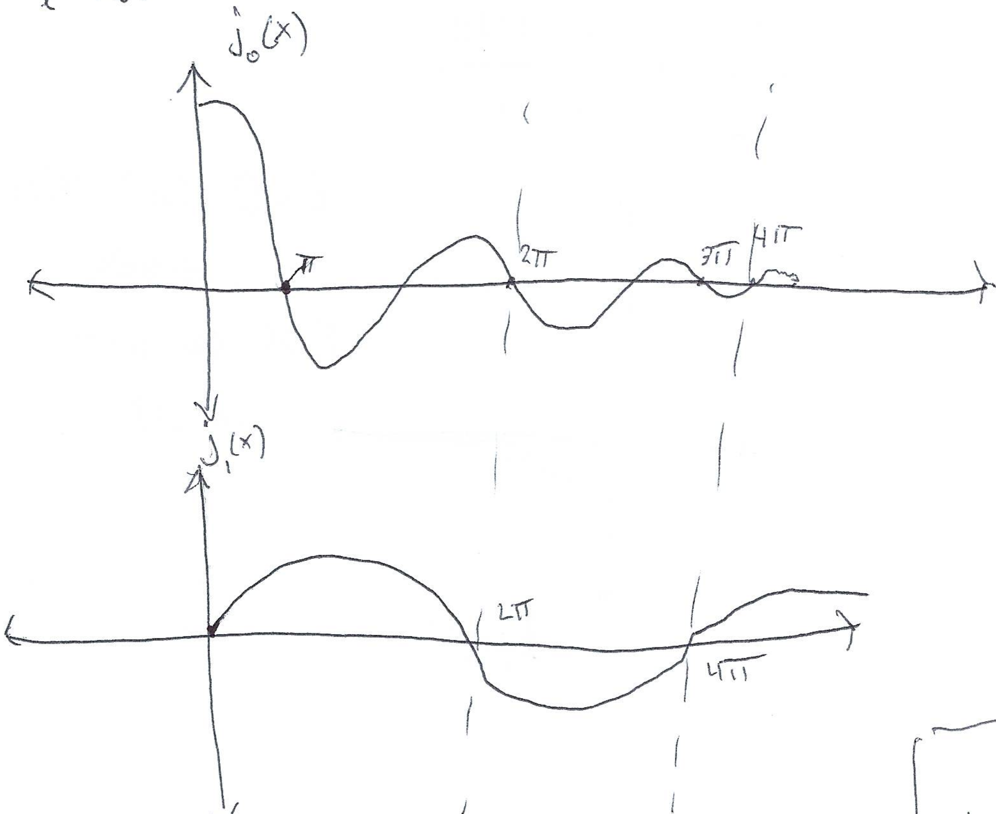
Quantization

$k_{n\ell} a$ is a root of the spherical Bessel Function $j_\ell(x)$

$\cancel{k_{n\ell}} \; k_{n0} = \frac{n\pi}{a}$

$$\boxed{\beta_{n\ell} \equiv \text{the } n\text{th "zero" of } j_\ell(x)}$$

$j_\ell(\beta_{n\ell}) = 0$



$j_0 = \frac{\sin x}{x}$ $\quad \beta_{10} = \pi, \beta_{20} = 2\pi, \beta_{30} = 3\pi, \ldots$

$j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x}$ $\quad \beta_{11} = 4.49, \beta_{21} = 7.73, \ldots$

$$\boxed{\begin{aligned} k_{n\ell} &= \frac{\beta_{n\ell}}{a} \\ E_{n\ell} &= \frac{\hbar^2}{2\mu a^2} \beta_{n\ell}^2 \end{aligned}}$$

Hydrogen Atom

$$\mathcal{H}_{\text{Hydrogen}} = \mathcal{H}_{\text{proton}} \otimes \mathcal{H}_{\text{electron}}$$

$$V = -\frac{e^2}{4\pi\epsilon_0 |\vec{x}_e - \vec{x}_p|}$$

$$= \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{relative}}$$

3 center of mass degrees of freedom

3 relative degrees of freedom

reduced mass

$$\frac{1}{\mu} = \frac{1}{m_p} + \frac{1}{m_e}$$

$m_e c^2 \approx$ ~~511 keV~~ 511,00 keV

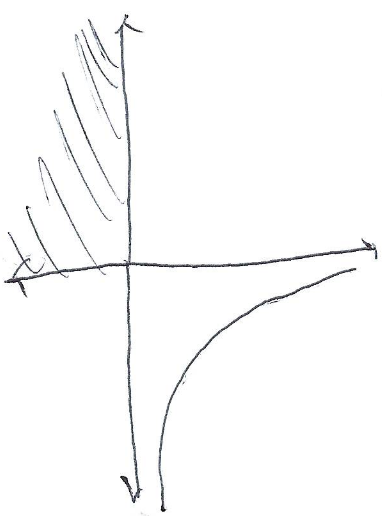
$\mu c^2 \approx 510,72$ keV

Electron mass μ in a fixed Coulomb potential

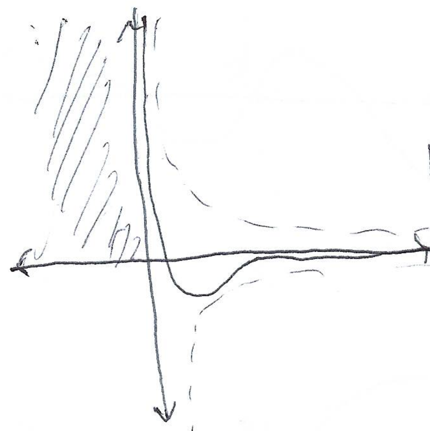
$$\boxed{V(r) = -\frac{e^2}{4\pi\epsilon_0 r}} \qquad V_{\text{eff}}(r) = \frac{-e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}$$

$E < 0$: bound states discrete

$E > 0$ continuum ionized



$l = 0$



$l \neq 0$

$$\frac{-\hbar^2}{2\mu}\frac{d^2u}{dr^2} + \left(\frac{-e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2 l(l+1)}{2\mu r^2}\right)u = Eu$$

$$k \equiv \sqrt{\frac{-2\mu E}{\hbar^2}} \;,\; \rho = kr \;,\; \rho_0 = \frac{e^2\mu}{2\pi\epsilon_0\hbar^2 k}$$

$$\frac{d^2u}{d\rho^2} = \left(1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right)u$$

$$\rho >> 1 \quad \frac{d^2u}{d\rho^2} \simeq u \Rightarrow \underset{(\rho>>1)}{u \approx Ae^{-\rho} + Be^{\rho}} \;,\; B = 0$$

$$\rho << 1 \quad \frac{d^2u}{d\rho^2} = \frac{l(l+1)}{\rho^2}u \quad \underset{(\rho<<1)}{u \approx C\rho^{l+1} + D\rho^{-l}} \;,\; D = 0$$

Ansatz: $u(\rho) = \rho^{l+1}e^{-\rho}v(\rho)$, $v(\rho)$ is a polynomial of finite order

$$\rho\frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + (\rho_0 - 2(l+1))v) = 0$$

$$v(\rho) = \sum_{j=0}^{\infty} c_j\rho^j \qquad c_{j+1} = \frac{2(j+l+1)-\rho_0}{(j+1)(j+2(l+1))}c_j$$

There must be a j_{max} such that $c_{j_{max}} \neq 0$, $c_{j_{max}+1} = 0$

$$0 = (\;\;)c_{j_{max}} \Rightarrow 2(\underbrace{j_{max}+l+1}_{n}) - \rho_0 = 0 \quad \Rightarrow \rho_0 = 2n$$

must be positive integer

$$k = \frac{e^2\mu}{4\pi\epsilon_0\hbar^2}\frac{1}{n}$$

$$\boxed{E_{nl} = -\underbrace{\left(\left(\frac{e}{4\pi\epsilon_0}\right)^2\frac{\mu}{2\hbar^2}\right)}_{= E_1 \text{ the bohr energy } = -13.6\,\text{eV}}\frac{1}{n^2} = \frac{E_1}{n^2} \qquad a \equiv \frac{4\pi\epsilon_0\hbar^2}{e^2\mu} = 0.529\times10^{-10}\,\text{m}}$$

Bohr radius

$$a \equiv \frac{4\pi\epsilon_0\hbar^2}{e^2\mu} = 0.529 \times 10^{-10}\, m \qquad k = \frac{1}{an} \qquad \rho = \frac{r}{an}$$

$$j_{max} = n - l - 1 \Rightarrow n \geq l - 1 \qquad n = 1, 2, 3, \ldots$$

$$l \leq n-1 \qquad l = 0, \ldots, n-1$$

$$R_{nl}(r) = \frac{1}{r}\rho^{l+1}e^{-\rho}v_{nl}(\rho)$$ Laguerre Polynomials

$$\psi_{nlm} = R_{nl}(r)Y_l^m(\theta, \phi) = \langle \vec{x} | n, l, m \rangle$$

n principal quantum number $n = 1, 2, 3, \ldots$

l orbital quantum number $l = 0, 1, 2, \ldots, n-1$

m magnetic quantum number $m = -l, \ldots, l$

~~$\Rightarrow$ There is a $(2l+1$~~

$\Rightarrow$ For a given n, l there are $2l+1$ states

For a given n, there are n^2 states

$$\psi_{100}(\vec{x}) = \frac{1}{\sqrt{\pi a^3}}e^{-r/a} \qquad R_{20} = \frac{1}{\sqrt{\cancel{\pi} a^3 \cdot 2}}\left(1 - \frac{r}{2a}\right)e^{-r/2a}$$

$$R_{21} = \frac{1}{\sqrt{\cancel{\pi} a^3 \cdot 24}}\left(\frac{r}{a}\right)e^{-r/2a}$$

$E_{n\ell} = \frac{E_1}{n^2}$ Gross-structure

Spectroscopic notation

$\ell = 0 \quad 1 \quad 2 \quad 3$

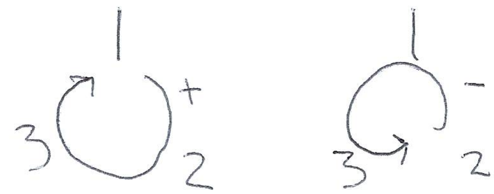
$\quad\;\; s \quad p \quad d \quad f$

"$n=2, \ell=1$" $\Rightarrow$ "$2p$"

$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}$ fine structure constant $= \frac{1}{137.036} = \frac{e^2}{4\pi\epsilon_0 \hbar c}$

$E_{n\ell} = \frac{E_1}{n^2} +$ corrections proportional to α^2

$E_{photon} = \frac{hc}{\lambda}$



Angular Momentum

a, b, c for space indices

$$V_a = \begin{cases} V_1 = V_x & a=1 \\ V_2 = V_y & a=2 \\ V_3 = V_z & a=3 \end{cases} \qquad \epsilon_{abc} = \begin{cases} 1 & (abc) = (1,2,3),(2,3,1)(3,1,2) \\ -1 & (abc) = (3,2,1),(2,1,3),(1,3,2) \\ 0 & \text{else} \end{cases}$$

$$\vec{L} = \vec{x} \times \vec{p} \qquad L_a = \sum_{bc} \epsilon_{abc} x_b p_c$$

$$\begin{aligned} L_x = L_1 &= \epsilon_{111} x_1 p_1 + \epsilon_{112} x_1 p_2 + \epsilon_{113} x_1 p_3 \\ &+ \epsilon_{121} x_2 p_1 + \epsilon_{122} x_2 p_2 + \epsilon_{123} x_2 p_3 \\ &+ \epsilon_{131} x_3 p_1 + \epsilon_{132} x_3 p_2 + \epsilon_{133} x_3 p_3 \\ &= \epsilon_{123} x_2 p_3 + \epsilon_{132} x_3 p_2 = x_2 p_3 - x_3 p_2 = y p_z - z p_y \end{aligned}$$

$$L_x = y p_z - z p_y \qquad L_y = z p_x - x p_z \qquad L_z = x p_y - y p_x$$

$$\hat{L}_a = \varepsilon_{abc} \hat{X}_b p_c \qquad \hat{\vec{L}} = \begin{bmatrix} \hat{L}_x \\ \hat{L}_y \\ \hat{L}_z \end{bmatrix} = \begin{bmatrix} \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{bmatrix}$$

$$\hat{L}^2 = \hat{\vec{L}} \cdot \hat{L} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \sum_a L_a^2$$

Phys 137A 06 Aug 19

Angular Momentum

$$\vec{L} = \vec{x} \times \vec{p} \Rightarrow \hat{\vec{L}} = \hat{\vec{x}} \times \hat{\vec{p}} \qquad \hat{L}_a = \sum_{b,c} \varepsilon_{abc} \hat{x}_b \hat{p}_c = \begin{cases} \hat{y}\hat{p}_z - \hat{z}\hat{p}_y & a=1 \\ \hat{z}\hat{p}_x - \hat{x}\hat{p}_z & a=2 \\ \hat{x}\hat{p}_y - \hat{y}\hat{p}_x & a=3 \end{cases}$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \sum_a \hat{L}_a \hat{L}_a$$

$$\vec{x}\psi(\vec{x}) = \vec{x}\,\psi(\vec{x}) = x\psi(\vec{x})\check{x} + y\,\psi(\vec{x})\check{y} + z\psi(\vec{x})\check{z}$$

$$\hat{\vec{p}}\psi(x) = \frac{\hbar}{i}\vec{\nabla}\psi(x) = \frac{\hbar}{i}\left(\frac{\partial\psi}{\partial x}\check{x} + \frac{\partial\psi}{\partial y}\check{y} + \frac{\partial\psi}{\partial z}\check{z}\right)$$

$$\hat{\vec{x}}\psi(\vec{x}) = r\,\psi(\vec{x})\,\check{r}$$

$$\hat{\vec{p}}\,\psi(\vec{x}) = \left(\frac{\hbar}{i}\frac{\partial\psi}{\partial r}\right)\check{r} + \left(\frac{\hbar}{i}\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)\check{\theta} + \left(\frac{\hbar}{i}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\hat{\phi}$$

$$\hat{\vec{L}}\,\psi(\vec{x}) = \frac{\hbar}{i}\left(\check{\phi}\frac{\partial}{\partial\theta} - \check{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)\psi(\vec{x})$$

$$\hat{L}_x\psi = \frac{\hbar}{i}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)\psi(\vec{x})$$

$$\hat{L}_y\psi = \frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)\psi(\vec{x})$$

$$\hat{L}_z\psi = \frac{\hbar}{i}\frac{\partial\psi}{\partial\phi}$$

$$\hat{L}^2\psi(\vec{x}) = -\hbar\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\psi(\vec{x})$$

$$\hat{L}^2 \psi = \hbar^2 \ell(\ell+1)\psi$$

$$\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\phi^2} = -\ell(\ell+1)\sin^2\theta\,\psi$$

$$\hat{L}^2 Y_\ell^m(\theta,\phi) = \hbar^2 \ell(\ell+1) Y_\ell^m(\theta,\phi)$$

$$\hat{L}_z Y_\ell^m(\theta,\phi) = \hbar m_\ell Y_\ell^m(\theta,\phi)$$

Spherical Harmonics are simultaneous eigenfunctions of $\hat{L}^2$ with $\hbar^2\ell(\ell+1)$ and $\hat{L}_z$ with $\hbar m_\ell$

Spherical symmetry

$\{\hat{H}, \hat{L}^2, \hat{L}_z\}$ is a mutually commuting set

Commutation Relations

$$[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z] = \cancel{[\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] - [y p_z, x p_z]}$$

$$= [\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] - [\hat{y}\hat{p}_z, \hat{x}\hat{p}_z] + [\hat{z}\hat{p}_y, \hat{z}\hat{p}_x] + [\hat{z}\hat{p}_y, \hat{x}\hat{p}_z]$$

$$= i\hbar \hat{p}_z$$

$$[L_x, L_y] = i\hbar \hat{L}_z$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

$$[\hat{L}_a, \hat{L}_b] = \sum_a i\hbar \varepsilon_{abc} \hat{L}_c$$

This describes a Lie Algebra su_2

$$[\hat{L}^2, \hat{L}_a] = 0$$

$$[\hat{L}^2, \hat{L}_z] = [\hat{L}_x\hat{L}_x, \hat{L}_z] + [\hat{L}_y\hat{L}_y, \hat{L}_z] + [\hat{L}_z\hat{L}_z, \hat{L}_z]$$

$$= \hat{L}_x[\hat{L}_x, \hat{L}_z] + [\hat{L}_x, \hat{L}_z]\hat{L}_x + \hat{L}_y[\hat{L}_y, \hat{L}_z] + [\hat{L}_y, \hat{L}_z]\hat{L}_y$$

$$= -i\hbar \hat{L}_x\hat{L}_y - i\hbar\hat{L}_y\hat{L}_x + i\hbar\hat{L}_y\hat{L}_x + i\hbar\hat{L}_x\hat{L}_y = 0$$

$$\boxed{\hat{L}_\pm \equiv \hat{L}_x \pm i\hat{L}_y}$$

Ladder Operators

$$\boxed{\hat{L}_+^\dagger = \hat{L}_-}$$

$$[\hat{L}^2, \hat{L}_\pm] = 0 \qquad [\hat{L}_z, \hat{L}_\pm] = \pm\hbar \hat{L}_\pm$$

$$[\hat{L}_+, \hat{L}_-] = 2\hbar L_z$$

$$\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-) \qquad \hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-) \qquad \hat{L}^2 = \hat{L}_+\hat{L}_- - \hbar\hat{L}_z + \hat{L}_z^2$$

$$= \hat{L}_-\hat{L}_+ + \hbar\hat{L}_z + \hat{L}_z^2$$

Given a Hermitian Operator $\hat{A}$, a ladder ~~operator~~ $\hat{R}$ Operator $\hat{R}$ for $\hat{A}$ is an ~~operator~~ operator satisfying $[\hat{A}, \hat{R}] = r\hat{R}$

Let $|a\rangle$ be an eigenket of $\hat{A}$ with eigenvalue a $\quad \hat{A}|a\rangle = a|a\rangle$

Then, $\hat{R}|a\rangle$ is either an eigenket of $\hat{A}$ with eigenvalue $a+r$ or 0

$$[\hat{A}, \hat{R}] = r\hat{R} \qquad \hat{A}|a\rangle = a|a\rangle$$

$$\downarrow \quad \swarrow$$

$$\hat{A}(\hat{R}|a\rangle) = (a+r)(\hat{R}|a\rangle)$$

$$\hat{A}(\hat{R}|a\rangle) = \hat{R}\hat{A}|a\rangle + [\hat{A},\hat{R}]|a\rangle = a\hat{R}|a\rangle + r\hat{R}|a\rangle = (a+r)(\hat{R}|a\rangle)$$

Positive Definite Operators

$\hat{Q}$ is positive definite if $\langle Q\rangle \geq 0$ for any state

$$\langle\alpha|\hat{Q}|\alpha\rangle \geq 0$$

1) For any operator $\hat{A}$, the products $\hat{A}^\dagger\hat{A}$ and $\hat{A}\hat{A}^\dagger$ are positive definite

$$\langle\alpha|\hat{A}^\dagger\hat{A}|\alpha\rangle = (\hat{A}|\alpha\rangle)^\dagger\hat{A}|\alpha\rangle = \|\hat{A}|\alpha\rangle\|^2 \geq 0$$

2) If $\hat{A}$ is Hermitian, $\hat{A}^2$ is positive definite

3) A Hermitian operator is positive definite iff all of its eigenvalues are nonnegative

4) A sum of positive definite operators is positive definite

Goal: Find the simultaneous eigenstates of $\hat{L}^2$ and $\hat{L}_z$

$|\lambda,\mu\rangle \qquad \hat{L}^2|\lambda,\mu\rangle = \lambda|\lambda,\mu\rangle \qquad \hat{L}_z|\lambda,\mu\rangle = \mu|\lambda,\mu\rangle$

$[L] = [\hbar] = J\cdot s$

$[L_z] = [\hbar] \qquad [L^2] = [\hbar^2]$

1) The eigenvalue λ is real with units of $\hbar^2$

The eigenvalue μ is real with units of $\hbar$

$\hat{L}_z^2$ is positive definite

$\hat{L}^2$ is positive definite

$$\langle\lambda,\mu|\hat{L}^2|\lambda,\mu\rangle = \lambda = \underbrace{\langle\lambda,\mu|\hat{L}_x^2|\lambda,\mu\rangle + \langle\lambda,\mu|\hat{L}_y^2|\lambda,\mu\rangle}_{\geq 0} + \underbrace{\langle\lambda,\mu|\hat{L}_z^2|\lambda,\mu\rangle}_{\mu^2}$$

$$\lambda \geq \mu^2$$

2) For any eigenstate $|\lambda, \mu\rangle$ $\lambda \geq 0$ $|\mu| \leq \sqrt{\lambda}$

$$\hat{L}^2(\hat{L}_\pm |\lambda,\mu\rangle) = \lambda(\hat{L}_\pm |\lambda,\mu\rangle)$$

$$\hat{L}_z(\hat{L}_\pm |\lambda,\mu\rangle) = (\mu \pm \hbar)(\hat{L}_\pm |\lambda,\mu\rangle)$$

3) $\hat{L}_+ |\lambda,\mu\rangle$ is either 0 or a simultaneous eigenket with $(\lambda, \mu+\hbar)$

$\hat{L}_- |\lambda,\mu\rangle$ is either 0 or a simultaneous eigenket with $(\lambda, \mu-\hbar)$

4) For a given ladder of eigenstates with $\hat{L}^2$ eigenvalue λ there must be a top rung $|\lambda, \mu_{max}\rangle$ such that trying to raise ~~$\hat{L}_+$~~ $\hat{L}_+ |\lambda, \mu_{max}\rangle = 0$ and a bottom rung such that $\hat{L}_- |\lambda, \mu_{min}\rangle = 0$ at a bottom rung $|\lambda, \mu_{min}\rangle$

$|\mu_{max} + \hbar| > \sqrt{\lambda}$

$|\mu_{min} - \hbar| > \sqrt{\lambda}$

$\mu_{max} \equiv \hbar l$ $\mu_{min} \equiv \hbar \tilde{l}$

$$\hat{L}^2 |\lambda, \mu_{max}\rangle = \lambda |\lambda, \mu_{max}\rangle = (\hat{L}_-\hat{L}_+ + \hbar \hat{L}_z + \hat{L}_z^2)|\lambda, \mu_{max}\rangle$$

$$= (\hbar(\hbar l) + (\hbar l)^2)|\lambda, \mu_{max}\rangle$$

$$\lambda|\lambda, \mu_{max}\rangle = \hbar^2(l(l+1))|\lambda, \mu_{max}\rangle$$

5) $\lambda = \hbar^2 l(l+1)$ $= \hbar^2 \tilde{l}(\tilde{l}-1)$

6) $\hbar l = \hbar \tilde{l} + N\hbar$ top = bottom + N raisings

$l(l+1) = \tilde{l}(\tilde{l}-1)$ $l = \tilde{l} + N$

$l = \frac{N}{2}$

7) The spectrum of $\hat{L}^2$ is $\hbar^2 l(l+1)$ with $l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$

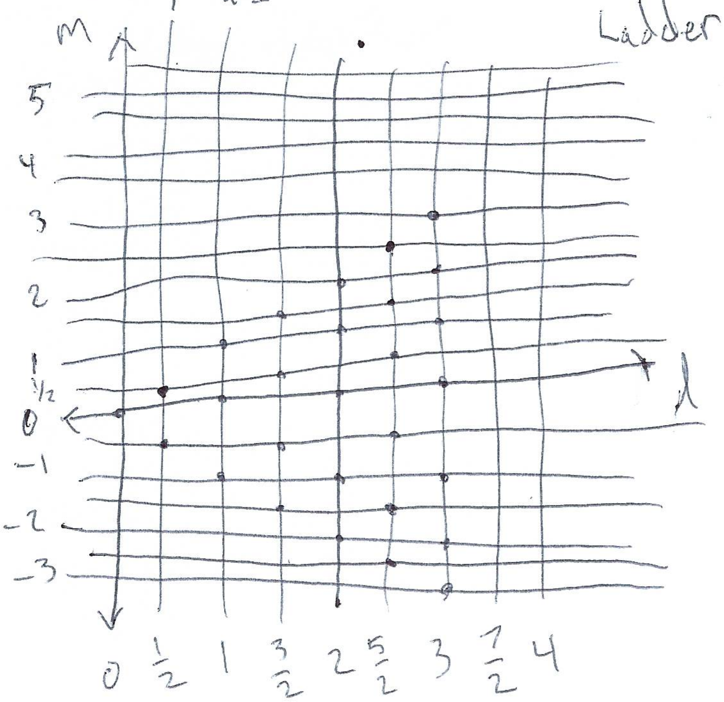
8) Given l, the spectrum of $\hat{L}_z$ is $\hbar m$, m ranges from $-l$ to $+l$ in integer steps

$2l+1$ states per ladder

Phys 137A 07Aug19

$$[\hat{L}_a, \hat{L}_b] = \sum_c i\hbar \varepsilon_{abc} \hat{L}_c$$

$$[\hat{L}^2, \hat{L}_a] = 0$$



Simultaneous Eigenkets

$$\hat{L}^2 |l,m\rangle = \hbar^2 l(l+1) |l,m\rangle$$

$$\hat{L}_z |l,m\rangle = \hbar m |l,m\rangle$$

$l \in \{0, \frac{1}{2}, 1, \ldots\}$ $m \in \{-l, \ldots, l\}$

$\hat{L}_+ \uparrow\downarrow \hat{L}_-$

$$\hat{L}_\pm |l m\rangle = \hbar\sqrt{l(l+1) - m(m\pm)}\,|l\, m\pm 1\rangle$$

Orbital Angular Momentum

$l = 0, 1, 2, \ldots$

"Spin Angular Momentum" as an internal degree of freedom

Spin $[\hat{S}_a, \hat{S}_b] = \sum i\hbar \varepsilon_{abc} \hat{S}_c$

$[\hat{S}^2, \hat{S}_a] = 0$ $|s\, m\rangle$

$$\hat{S}^2 |s m\rangle = \hbar^2 s(s+1) |s m\rangle \qquad \hat{S}_z |s m\rangle = \hbar m |s m\rangle$$

$$\hat{S}_\pm |s m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\,|s\, m+1\rangle$$

$$\mathcal{H} = \mathcal{H}_{space} \otimes \mathcal{H}_{spin}$$

$|state\rangle = |space\rangle \otimes |spin\rangle = |space, spin\rangle$

simple/separable states

$|particle\ with\ spin\rangle = \sum c\,|space\rangle \otimes |spin\rangle = \sum_{m_s} c\,|space\rangle \otimes |s, m_s\rangle$

s is fixed by particle

Hydrogen $\underbrace{|n\, l\, m_l\rangle}_{\text{Orbital angular momentum}} \otimes \underbrace{|\tfrac{1}{2}\, m_s\rangle}_{\text{electron spin}} = |n\, l\, m_l\, m_s\rangle$

Particles symmetric over interchange

Bose-Einstein statistics   Bosons

Tend to accumulate in same state

Particles antisymmetric over interchange

Fermi-Dirac statistics   Fermions

Particles try not to be in same state

Obeys Pauli Exclusion Principle

Spin Statistics Theorem

Particles with integer spin are Bosons

Particles with half integer spin are Fermions

Spin 0 particles - scalar bosons   Higgs only currently known individual

- Mesons

Spin 1/2 particles - Leptons, quarks, Baryons

Spin 1 particles - vector bosons   Force carrying particles

spin 3/2 particles - no fundamental ones, only composites

spin 2 particles Graviton

Spin-s system

$m_s = -s, \ldots, s$ $\qquad |s\, m_s\rangle$ ≠ $2s+1$ dimensional Hilbert Space $\mathcal{H}_s = \mathbb{C}^{2s+1}$

Basis $\{|s\, s\rangle, |s\, s-1\rangle, \ldots, |s, -(s-1)\rangle, |s\, -s\rangle\}$

The matrix representation of a spin state is a $(2s+1)$ element column vector called a spinor χ

$$|s, s\rangle \doteq \chi_s = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad |s, s-1\rangle \doteq \chi_{s-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad |s, -s\rangle \doteq \chi_{-s} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$|\alpha\rangle = \sum_{m_s=-s}^{s} C_{m_s} |s\, m_s\rangle \doteq \begin{bmatrix} C_s \\ \vdots \\ \vdots \\ C_{-s} \end{bmatrix} = \begin{bmatrix} \langle s\, s|\alpha\rangle \\ \vdots \\ \vdots \\ \langle s\, -s|\alpha\rangle \end{bmatrix}$$

$$|\beta\rangle = \sum_{-s}^{s} C_{m_s} |space\rangle \otimes |spin\rangle \doteq \begin{bmatrix} C_s \psi_s(\vec{x}) \\ \vdots \\ C_{-s}\psi_{-s}(\vec{x}) \end{bmatrix} = \sum_{m_s} C_{m_s} \psi_{m_s}(\vec{x})\, \chi_{m_s}$$

$$\psi_{m_s}(\vec{x}) \equiv \langle \vec{x} | space_{m_s}\rangle$$

Spin-0 system

$|0\,0\rangle$ $\qquad \mathcal{H}_0 = \mathbb{C}$ $\qquad |\alpha\rangle = C_0 |0\,0\rangle$ $\qquad \chi_0 = [1]$

so $m_s = 0$

$\hat{Q} \doteq [q]$ $\qquad$ Spin 0 is boring

Spin 1/2 system

$s = \frac{1}{2} \quad m_s = \frac{1}{2}, -\frac{1}{2} \quad \mathcal{H}_{1/2} = \mathbb{C}^2 \quad \{|\tfrac{1}{2}\tfrac{1}{2}\rangle, |\tfrac{1}{2}-\tfrac{1}{2}\rangle\}$ standard basis

$|\frac{1}{2}\,\frac{1}{2}\rangle$: "spin up" $|\uparrow\rangle$ $|\frac{1}{2}\,-\frac{1}{2}\rangle$: "spin down" $|\downarrow\rangle$

$|\alpha\rangle = c_+|\uparrow\rangle + c_-|\downarrow\rangle$ with $|c_+|^2 + |c_-|^2 = 1$

$c_+ = \langle\uparrow|\alpha\rangle \qquad c_- = \langle\downarrow|\alpha\rangle$

$|\alpha\rangle \doteq \mathcal{X}_\alpha = \begin{bmatrix} c_+ \\ c_- \end{bmatrix}, \quad \mathcal{X}_\uparrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathcal{X}_\downarrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Operators $\hat{A} \doteq \overleftrightarrow{A} = \begin{bmatrix} \langle\uparrow|\hat{A}|\uparrow\rangle & \langle\uparrow|\hat{A}|\downarrow\rangle \\ \langle\downarrow|\hat{A}|\uparrow\rangle & \langle\downarrow|\hat{A}|\downarrow\rangle \end{bmatrix} = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix}$

$\hat{A}$ Hermitian

A_{++}, A_{--} are real and $A^{*}_{+-} = A_{-+}$

$\overleftrightarrow{A} = \begin{bmatrix} a+d & b-ic \\ b+ic & a-d \end{bmatrix}, \quad a,b,c,d \in \mathbb{R}$

$= a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Pauli-spin matrices $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\vec{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} = \sigma_x \check{x} + \sigma_y \check{y} + \sigma_z \check{z}$

$\hat{A} = a\mathbb{1} + \vec{b}\cdot\vec{\sigma} = \begin{bmatrix} a+b_z & b_x - ib_y \\ b_x + ib_y & a-b_z \end{bmatrix} \qquad a = \frac{1}{2}\,\mathrm{tr}\,\overleftrightarrow{A}$

$b = \frac{1}{2}\,\mathrm{tr}(\vec{\sigma}A)$

Phys 137A 08Aug19

Spin 1/2 System

$$|s=\tfrac{1}{2}, m_s=\tfrac{1}{2}\rangle = |\uparrow\rangle \doteq \chi_+ = \begin{bmatrix}1\\0\end{bmatrix}$$

$$|s=\tfrac{1}{2}, m_s=-\tfrac{1}{2}\rangle = |\downarrow\rangle \doteq \chi_- = \begin{bmatrix}0\\1\end{bmatrix}$$

Pauli Spin Matrices

$$\sigma_x = \begin{bmatrix}0&1\\1&0\end{bmatrix} \quad \sigma_y = \begin{bmatrix}0&-i\\i&0\end{bmatrix} \quad \sigma_z = \begin{bmatrix}1&0\\0&-1\end{bmatrix} \qquad \vec{\sigma} = \begin{bmatrix}\sigma_x\\\sigma_y\\\sigma_z\end{bmatrix} = \sigma_x\check{x} + \sigma_y\check{y} + \sigma_z\check{z}$$

$$\check{n} \doteq \begin{bmatrix}\sin\theta\cos\phi\\\sin\theta\sin\phi\\\cos\theta\end{bmatrix}$$

$$\sigma_n = \check{n}\cdot\vec{\sigma} = \sin\theta\cos\phi\,\sigma_x + \sin\theta\sin\phi\,\sigma_y + \cos\theta\,\sigma_z$$

$$\doteq \begin{bmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{bmatrix}$$

$$\hat{A} \doteq a\mathbb{1} + \vec{b}\cdot\vec{\sigma} = \begin{bmatrix}a+b_z & b_x-ib_y\\ b_x+ib_y & a-b_z\end{bmatrix}$$

$$a = \tfrac{1}{2}\,\mathrm{tr}\,\hat{A} \qquad \vec{b} = \tfrac{1}{2}\,\mathrm{tr}(\vec{\sigma}\hat{A})$$

- σ_n has eigenvalues ± 1 $\qquad (\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\mathbb{1} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$
- traceless $\mathrm{tr}\,\sigma_n = 0$ $\quad \det\sigma_n = -1$ $\quad \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \sigma_n^2 = \mathbb{1}$
- $\sigma_a\sigma_b = \delta_{ab}\mathbb{1} + \sum_c i\varepsilon_{abc}\sigma_c$
- $[\sigma_a, \sigma_b] = \sum_c 2i\varepsilon_{abc}\sigma_c$ $\qquad \{\sigma_a, \sigma_b\} = \sigma_a\sigma_b + \sigma_b\sigma_a = 2\delta_{ab}\mathbb{1}$

$$\hat{S}^2 \doteq \frac{3}{4}\hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{3}{4}\hbar^2 \mathbb{1}$$

$$\hbar^2\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right) = \frac{3}{4}\hbar^2$$

$$\hat{S}_z \doteq \begin{bmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{bmatrix} = \frac{\hbar}{2}\sigma_z$$

$$\hat{S}_+|\uparrow\rangle = 0$$

$$\hat{S}_+|\downarrow\rangle = \hbar\sqrt{s(s+1) - m_s(m_s+1)}\,|\uparrow\rangle = \hbar|\uparrow\rangle$$

$$\hat{S}_+ \doteq \begin{bmatrix} \langle\uparrow|\hat{S}_+|\uparrow\rangle & \langle\uparrow|\hat{S}_+|\downarrow\rangle \\ \langle\downarrow|\hat{S}_+|\uparrow\rangle & \langle\downarrow|\hat{S}_+|\downarrow\rangle \end{bmatrix} = \begin{bmatrix} 0 & \hbar \\ 0 & 0 \end{bmatrix} = \hbar\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\hat{S}_- = \hat{S}_+^\dagger \doteq \begin{bmatrix} 0 & 0 \\ \hbar & 0 \end{bmatrix}$$

$$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) \doteq \begin{bmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{\hbar}{2}\sigma_x$$

$$\hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-) \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{\hbar}{2}\sigma_y$$

$$\hat{S}^2 \doteq \frac{3\hbar^3}{4}\mathbb{1} \quad \hat{S}_x = \frac{\hbar}{2}\sigma_x \quad \hat{S}_y = \frac{\hbar}{2}\sigma_y \quad \hat{S}_z = \frac{\hbar}{2}\sigma_z$$

$$\hat{\vec{S}} \doteq \frac{\hbar}{2}\vec{\sigma}$$

$$\hat{S}_n = \check{n}\cdot\hat{\vec{S}} \doteq \frac{\hbar}{2}(\check{n}\cdot\vec{\sigma}) = \frac{\hbar}{2}\sigma_n = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix}$$

$\hat{S}_n$ eigenvalues

$$\det\left[\hat{S}_n - \lambda\hat{\mathbb{1}}\right] = \begin{vmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}\sin\theta e^{-i\phi} \\ \frac{\hbar}{2}\sin\theta e^{i\phi} & -\frac{\hbar}{2}\cos\theta - \lambda \end{vmatrix}$$

$$= \left(\frac{\hbar}{2}\cos\theta - \lambda\right)\left(-\frac{\hbar}{2}\cos\theta - \lambda\right) - \left(\frac{\hbar}{2}\sin\theta e^{i\phi}\right)\left(\frac{\hbar}{2}\sin\theta e^{-\phi}\right)$$

$$= -\frac{\hbar^2}{4}\cos^2\theta + \lambda^2 - \frac{\hbar^2}{4}\sin^2\theta = \lambda^2 - \frac{\hbar^2}{4}, \quad \lambda = \pm\frac{\hbar}{2}$$

$$\lambda = \left\{-\frac{\hbar}{2}, \frac{\hbar}{2}\right\}$$

S_z eigenbasis

spin up: $|\uparrow\rangle \doteq \chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ spin down: $|\downarrow\rangle \doteq \chi_-^{(z)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

S_x eigenbasis

spin right $\left(\frac{\hbar}{2}\right)$ $|\rightarrow\rangle \doteq \chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

spin left $\left(-\frac{\hbar}{2}\right)$ $|\leftarrow\rangle \doteq \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$S_x\chi_\pm^{(x)} = \pm\frac{\hbar}{2}\chi_\pm^{(x)}$ $\quad \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \pm\frac{\hbar}{2}\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} b \\ a \end{bmatrix} = \pm\begin{bmatrix} a \\ b \end{bmatrix}$

S_y eigenbasis

spin in $\left(\frac{\hbar}{2}\right)$ $|\otimes\rangle \doteq \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

spin out $\left(-\frac{\hbar}{2}\right)$ $|\odot\rangle \doteq \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

S_n eigenbasis

$\pm\frac{\hbar}{2}$ state $|\check{n},\pm\rangle \doteq \chi_{\pm}^{(n)}$

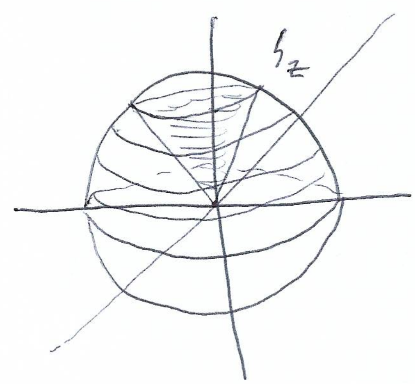
$$\begin{bmatrix} a \\ b \end{bmatrix} \to \begin{bmatrix} b \\ a \end{bmatrix} \to \begin{bmatrix} b^* \\ a^* \end{bmatrix} \to \begin{bmatrix} b^* \\ -a^* \end{bmatrix}$$

$$\chi_+^{(n)} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{bmatrix}$$

$$\chi_-^{(n)} = \begin{bmatrix} \sin\frac{\theta}{2} e^{-i\phi} \\ -\cos\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{i\phi} \end{bmatrix}$$

phase convention

$|-\check{n},+\rangle = (\text{phase})|+\check{n},-\rangle$



Cone of uncertainty

$|\uparrow\rangle$

$$\hat{S}^2 = \frac{3\hbar^2}{4} \qquad \langle\vec{S}\rangle = \begin{bmatrix} 0 \\ 0 \\ \frac{\hbar}{2} \end{bmatrix}$$

$$S_z = \frac{\hbar}{2}$$

Spin Polarization Vector

$$\langle\hat{\vec{S}}\rangle = \begin{bmatrix} \langle S_x\rangle \\ \langle S_y\rangle \\ \langle S_z\rangle \end{bmatrix} = \frac{\hbar^2}{2}\begin{bmatrix} \chi^\dagger\sigma_x\chi \\ \chi^\dagger\sigma_y\chi \\ \chi^\dagger\sigma_z\chi \end{bmatrix}$$

$$\langle\hat{A}\rangle = \chi^\dagger \overleftrightarrow{A}\chi$$

$$|\check{n},+\rangle \doteq \chi_+^{(n)} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{bmatrix}$$

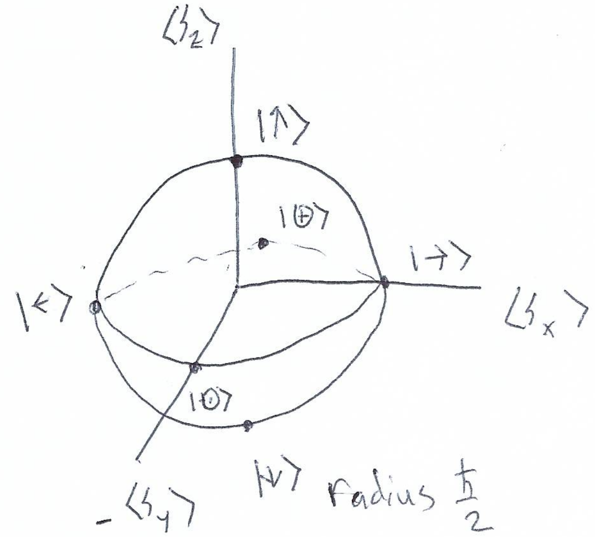
$$\langle\vec{S}\rangle = \frac{\hbar}{2}\check{n} = \frac{\hbar}{2}\begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \qquad |a|^2 + |b|^2 = 1$$

$$|a| = \cos\frac{\theta}{2} \qquad |b| = \left|\sin\frac{\theta}{2}\right|$$

$$\begin{bmatrix} \cos\frac{\theta}{2} e^{i\phi_+} \\ \sin\frac{\theta}{2} e^{i\phi_-} \end{bmatrix}$$

For <u>any state</u> in the spin $\frac{1}{2}$ system there exists $\check{n}$ such that $|\text{state}\rangle \propto |\check{n},+\rangle$



$\boxed{\text{Bloch sphere}}$

Given a charged object with angular momentum $\vec{L}$ classically, the magnetic dipole moment vector is

$$\vec{\mu} = \gamma \vec{L} \quad , \; \gamma = \text{gyromagnetic ration}$$

$$[\gamma] = \frac{Hz}{T}$$

uniform sphere of mass m charge q $\quad \gamma = \frac{q}{2m}$

$\gamma = g\left(\frac{q}{2m}\right) \quad g =$ g-factor, dimensionless

relativity and quantum mechanics $\quad g_{electron} = 2$

QFT prediction $g_{electron} = 2.0023193043636\ (15)$

Measurement $\quad g_{electron} = 2.00231930436146\ (56)$

Phys 137A 12 Aug 19

Vermouth carpano antic formula
Ledger's

Midterm 2 Postulates
Ladder Operators

$\vec{\mu} = \gamma \vec{L}$

spin dipole
moment vector

$\gamma_e = 1.761 \times 10^{11}\ Hz/T$

gyromagnetic ratio of electron
frequency per unit magnetic field

Bohr magneton

$$\mu_B \equiv \frac{e\hbar}{2m_e} = \frac{\hbar}{2}\gamma = 5.79 \times 10^{-5} \frac{eV}{T}$$

Energy per unit magnetic field

$$U = -\vec{\mu} \cdot \vec{B} \Rightarrow \hat{H} = -\hat{\vec{\mu}} \cdot \vec{B}$$
$$= -\gamma B_0 \hat{S}_n$$

Spin in a Field

$\hat{\vec{\mu}} = \gamma \hat{\vec{S}}$ $\vec{B} = B_0 \check{n}$ $\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B} = -\gamma B_0 \hat{S}_n \doteq \frac{-\gamma B_0 \hbar}{2} \sigma_n = -\mu_B B_0 \sigma_n$

$\hat{S}_n \doteq \frac{\hbar}{2} \sigma_n$

$|\check{n}, \pm\rangle$ $\langle \hat{\vec{S}} \rangle = \pm \frac{\hbar}{2}$

energy eigenstates

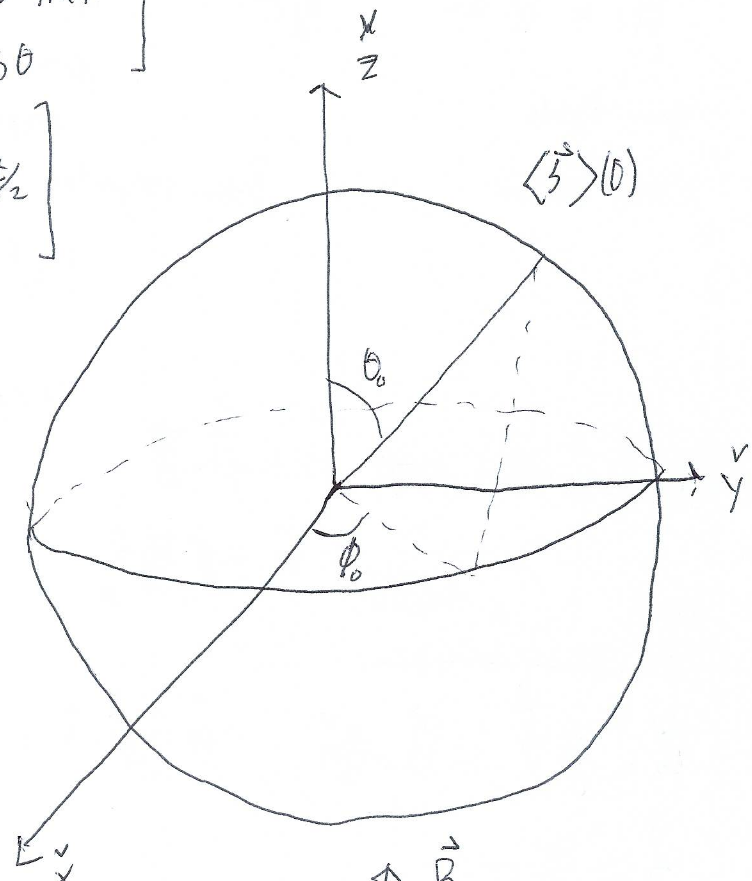
$E_{\pm} = -\gamma B_0 (\pm \frac{\hbar}{2})$ $E_{\pm} = \mp \frac{\gamma \hbar B_0}{2} = \mp \mu_B B_0$

$|\check{n}, \pm\rangle$

Let $\vec{B} = B_0 \check{z}$

$|\uparrow\rangle \quad E = \dfrac{-\gamma\hbar B_0}{2} \qquad |\downarrow\rangle \quad E = \dfrac{\gamma\hbar B_0}{2}$

$$\chi(0) = e^{i\alpha}\begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{bmatrix} \qquad \langle \vec{S} \rangle = \frac{\hbar}{2}\begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$



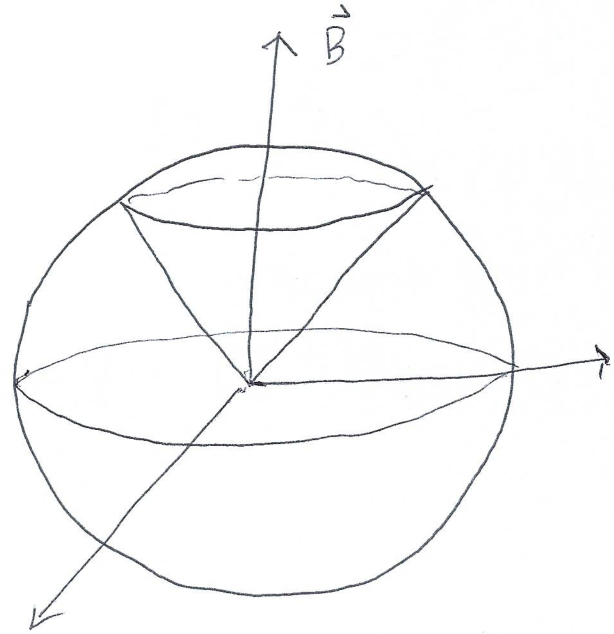
$$\cancel{\hat{\mathcal{U}}(t) = e^{iEt}} \quad \hat{U}(t) = e^{-i\frac{\hat{H}t}{\hbar}} = \begin{bmatrix} e^{i\gamma B_0 t/2} & 0 \\ 0 & e^{-i\gamma B_0 t/2} \end{bmatrix}$$

$$\cancel{\chi(t) = e^{i\alpha}\begin{bmatrix} \cos\frac{\theta}{2}\sin\theta \\ \\ \end{bmatrix}}$$

$$\chi(t) = e^{i\alpha}\begin{bmatrix} \cos\frac{\theta}{2} e^{i\gamma B_0 t/2} \\ \sin\frac{\theta}{2} e^{i\phi} e^{-i\gamma B_0 t/2} \end{bmatrix}$$

$$= e^{i\alpha} e^{\frac{i\gamma B_0 t}{2}} \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} e^{-i\gamma B_0 t} \end{bmatrix}$$

$$= e^{i\alpha} e^{\frac{i\gamma B_0 t}{2}} \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i(\phi - \gamma B_0 t)} \end{bmatrix}$$



$$\theta(t) = \theta_0 \qquad \phi(t) = \phi_0 - \gamma B_0 t$$

$$\langle \vec{S}(t) \rangle = \frac{\hbar}{2}\begin{bmatrix} \sin\theta\cos(\phi - \gamma B_0 t) \\ \sin\theta\sin(\phi - \gamma B_0 t) \\ \cos\theta \end{bmatrix}$$

Larmor precession

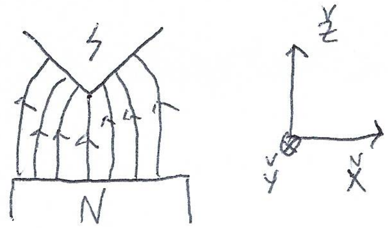
$\omega_{Larmor} = \gamma B_0$

Hopf fibration

$U = -\vec{\mu} \cdot \vec{B} \qquad \vec{F} = -\vec{\nabla} U$

Stern-Gerlach Apparatus

$$\vec{B} = (B_0 + \alpha z)\check{z} - \alpha x \check{x}$$



$$\vec{F} = -\alpha(\mu_x \check{x} - \mu_z \check{z})$$

$$\hat{H} = \frac{\hat{p}^2}{2m} - \hat{\vec{\mu}} \cdot \vec{B} = \underbrace{\frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m}}_{\text{Free particle behavior in x and y doF}} + \frac{\hat{p}_z^2}{2m} - \gamma(B_0 + \alpha\hat{z})\hat{S}_z + \underbrace{\gamma\alpha\hat{x}\hat{S}_x}$$

let α be small

This term can be treated as a perturbation

Acts like expectation value $\langle \hat{S}_x \rangle$

It precesses around x axis very fast so $\langle \hat{x} \rangle$ averages to zeroish

So we can ignore it

$$\hat{H}_{SG} = \frac{\hat{p}_z^2}{2m} - \gamma(B_0 + \alpha\hat{z})\hat{S}_z$$

creates entanglement between position and spin due to $\hat{z}\hat{S}_z$

$$\hat{H} \doteq \begin{bmatrix} \frac{\hat{p}_z^2}{2m} - \mu_B B_0 - \mu_B \alpha \hat{z} & 0 \\ 0 & \frac{\hat{p}_z^2}{2m} + \mu_B B_0 + \mu_B \alpha \hat{z} \end{bmatrix}$$

$$\hat{U}(t) \doteq \begin{bmatrix} e^{\frac{-i\left(\frac{\hat{p}_z^2}{2m} - \mu_B B_0 - \mu_B \alpha \hat{z}\right)t}{\hbar}} & 0 \\ 0 & e^{-i\left(\frac{\hat{p}_z^2}{2m} + \mu_B B_0 + \mu_B \alpha \hat{z}\right)t/\hbar} \end{bmatrix}$$

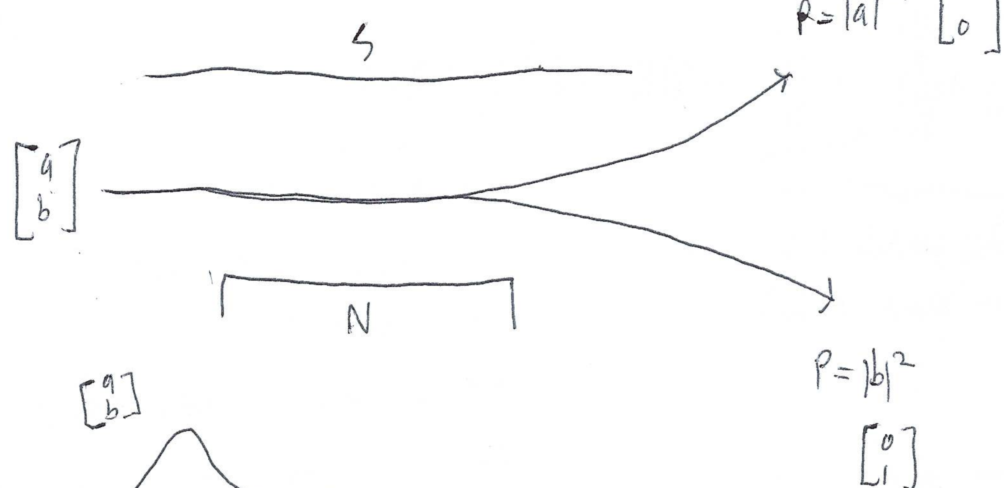
$$\hat{U}_{SG}(t) = \begin{bmatrix} e^{i\mu_B \alpha \hat{z} t/\hbar} & 0 \\ 0 & e^{-i\mu_B \alpha \hat{z} t/\hbar} \end{bmatrix}$$

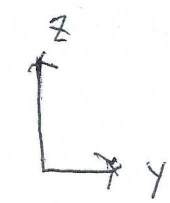
Detexify.com texexchange

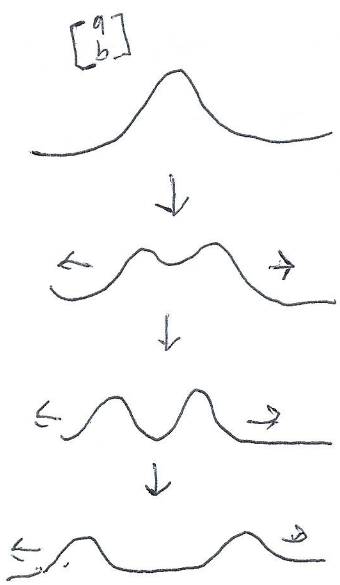
$$|\text{initial}\rangle = |\text{space}\rangle \otimes |\text{spin}\rangle \doteq \psi_0(z) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\psi_0(z) \\ b\psi_0(z) \end{bmatrix}$$

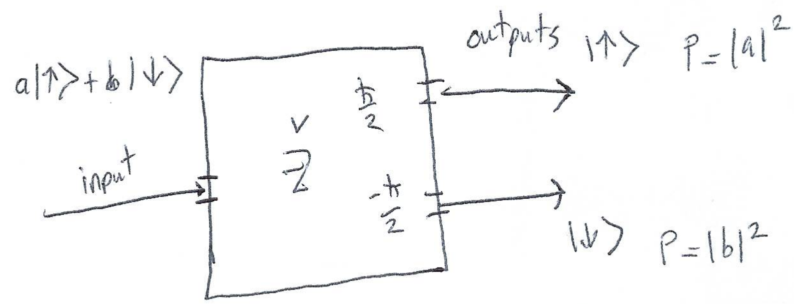
Later Time $\begin{bmatrix} a\,\psi_0(z)\, e^{i\mu_B \alpha \hat{z} t/\hbar} \\ b\,\psi_0(z)\, e^{-i\mu_B \alpha \hat{z} t/\hbar} \end{bmatrix}$

$e^{i\mu_B \alpha \hat{z} t/\hbar}$ is a boost operator



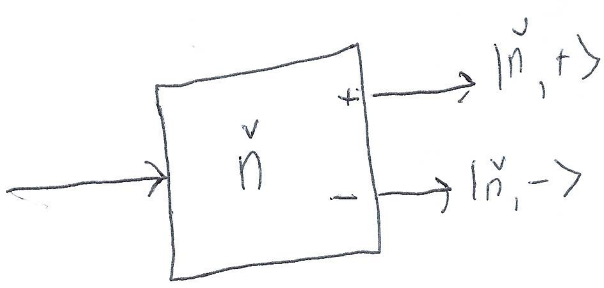






a z oriented

Stern Gerlach



I want to build $\chi_{\check{n}}$

1) Find $\langle \vec{S} \rangle = \frac{\hbar}{2}\check{n}$

2) Set up an $\check{n}$ - SG device

3) Block $-\frac{\hbar}{2}$ output beam

$e^{+ix\hat{p}/\hbar}$ = space translation operator

$e^{-i\hat{H}t/\hbar}$ = time translation operator

$e^{-i\theta\check{n}\cdot\vec{L}/\hbar}$ = rotation operator $= \hat{U}(\check{n}, \theta)$

$$\hat{U}(\check{n}, 2\pi) = (-1)^{2s}\hat{\mathbb{1}}$$

Phys 137A 13Aug19

Addition of Angular Momentum

angular momentum $\hat{\vec{L}}$

$$[\hat{L}_a, \hat{L}_b] = \sum_c i\hbar \varepsilon_{abc} \hat{L}_c \quad [\hat{L}^2, \hat{L}_a] = 0 \quad |l, m_l\rangle \quad \mathcal{H}_L$$

Spin angular Momentum $\hat{\vec{S}}$

$$[\hat{S}_a, \hat{S}_b] = \sum_c i\hbar \varepsilon_{abc} \hat{S}_c \quad [\hat{S}^2, \hat{S}_a] = 0 \quad |s, m_s\rangle \quad \mathcal{H}_S$$

Total Angular Momentum

$$\hat{\vec{L}} + \hat{\vec{S}} = \hat{\vec{J}} \quad \mathcal{H}_J = \mathcal{H}_L \otimes \mathcal{H}_S$$

$\cancel{|l, m_l\rangle}$

The Uncoupled Basis

$$|l, s, m_l, m_s\rangle \equiv |l, m_l\rangle \otimes |s, m_s\rangle$$

$$[\hat{S}_a, \hat{L}_b] = 0$$

Complete Set of Commuting Observables $\{\hat{L}^2, \hat{S}^2, \hat{L}_z, \hat{S}_z\}$

$$\hat{L}^2 |l, s, m_l, m_s\rangle = \hbar^2 l(l+1) |l, s, m_l, m_s\rangle$$

$$\hat{S}^2 |l, s, m_l, m_s\rangle = \hbar^2 s(s+1) |l, s, m_l, m_s\rangle$$

$$\hat{L}_z |l, s, m_l, m_s\rangle = \hbar m_l |l, s, m_l, m_s\rangle$$

$$\hat{S}_z |l, s, m_l, m_s\rangle = \hbar m_s |l, s, m_l, m_s\rangle$$

$|n=2, l=1, m_l=1, m_s=-\frac{1}{2}\rangle$

2p, spin down electron

$|1, \frac{1}{2}, 1, -\frac{1}{2}\rangle$

$l \quad s \quad m_l \quad m_s$

<u>The Coupled Basis</u>

$$[\hat{J}_a, \hat{J}_b] = [\hat{L}_a + \hat{S}_a, \hat{L}_b + \hat{S}_b] = [L_a, L_b] + [L_a\ S_b] + [S_a, L_b] + [S_a, S_b]$$

$$= \sum_c i\hbar \varepsilon_{abc} \hat{L}_c + \sum_c i\hbar \varepsilon_{abc} \hat{S}_c = \sum_c i\hbar \varepsilon_{abc} (\hat{L}_c + \hat{S}_c) = \sum_c i\hbar \varepsilon_{abc} \hat{J}_c$$

$\hat{L}^2 \quad \hat{S}^2 \quad \hat{J}^2 \quad \hat{L}_z \quad \hat{S}_z \quad \hat{J}_z$

$$\hat{J}^2 = (\hat{\vec{L}} + \hat{\vec{S}})^2 = (\hat{\vec{L}} + \hat{\vec{S}}) \cdot (\hat{\vec{L}} + \hat{\vec{S}}) = \vec{L}^2 + 2\hat{\vec{L}} \cdot \hat{\vec{S}} + \vec{S}^2$$

$$\hat{\vec{L}} \cdot \hat{\vec{S}} = \frac{\hat{J}^2 - \hat{L}^2 - \hat{S}^2}{2}$$

$$[\hat{J}^2, \hat{L}_z] \neq 0 \qquad [\hat{J}^2, \hat{S}_z] \neq 0$$

Coupled states

$|l, s; j\ m_j\rangle$

$$\hat{L}^2 |l, s; j, m_j\rangle = \hbar^2 l(l+1) |l, s; j m_j\rangle$$

$$\hat{S}^2 |l, s; j, m_j\rangle = \hbar^2 s(s+1) |l, s; j, m_j\rangle$$

$$\hat{J}^2 |l, s; j\ m_j\rangle = \hbar^2 j(j+1) |l, s; j, m_j\rangle$$

$$\hat{J}_z |l, s; j\ m_j\rangle = \hbar m_j |l, s; j\ m_j\rangle$$

Clebsch - Gordan Coefficients

$$|l,s;j,m_j\rangle = \sum_{l'=0}^{\infty}\sum_{s'=0}^{\infty}\sum_{m_l=-l'}^{l'}\sum_{m_s=-s'}^{s'} C_{l',s',m_l,m_s}\,|l',s',m_l,m_s\rangle$$

$$C_{l's'm_lm_s} = \langle l',s',m_l,m_s|l,s;j,m_j\rangle$$

$$\Rightarrow l=l' \text{ or } C=0$$

$$s=s' \text{ or } C=0$$

$$|l,s;j,m_j\rangle = \sum_{m_l=-l}^{l}\sum_{m_s=-s}^{s} C^{lsj}_{m_lm_sm_j}\,|l,s,m_l,m_s\rangle$$

$$C^{lsj}_{m_lm_sm_j} \equiv \langle l,s;j,m_j|l,s,m_l,m_s\rangle = \langle j\,m_j|l\,s\,m_l\,m_s\rangle$$

phase convention real

$$= \langle l,s,m_l,m_s|j,m_j\rangle$$

$$\hat{J}_z = \hat{L}_z + \hat{S}_z \quad \cancel{\text{unless } m_j = m_l + m_s}$$

$$m_j = m_l + m_s \text{ or } C=0$$

$$|l,s;jm_j\rangle = \sum_{m_l=-l}^{l} C^{lsj}_{m_lm_sm_j}|l,s,m_l,m_s\rangle \quad \text{with } m_j - m_l = m_s$$

and $|m_s| \le s$

$|2,1;3,0\rangle$
$l\ s\ j\ m_j$

m_l	m_s
$\cancel{-2}$	$\cancel{2}$
-1	1
0	0
1	-1
$\cancel{2}$	$\cancel{-2}$

$$|2,1;3,0\rangle = C^{213}_{1-10}|211-1\rangle + C^{213}_{000}|2100\rangle + C^{213}_{-110}|21-11\rangle$$

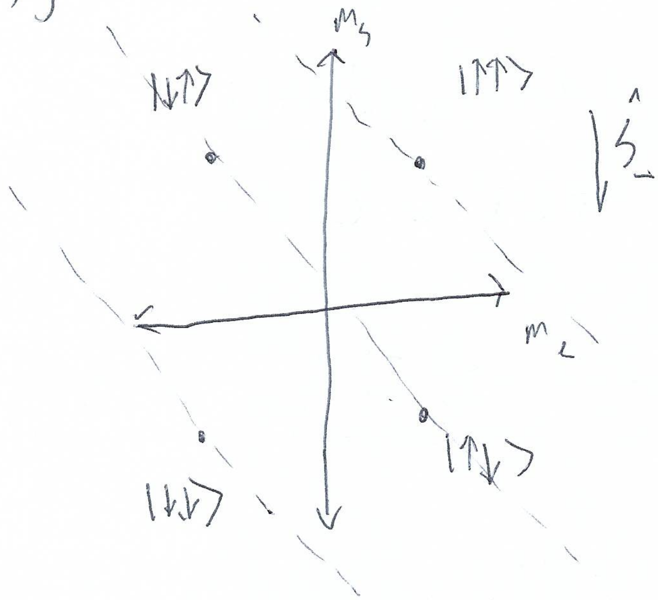
$$= \frac{1}{\sqrt{5}}|211-1\rangle + \sqrt{\frac{3}{5}}|2100\rangle + \frac{1}{\sqrt{5}}|21-11\rangle$$

$|l\,s\,m_l\,m_s\rangle = \sum_j C^{lsj}_{m_l m_s m_j} |l, s; j\, m_j\rangle \qquad m_j = m_l + m_s$

if $l+s$ is an integer then so is j

if $l+s$ is a half integer then so is j

$|l-s| \le j \le l+s$



$l = s = \frac{1}{2}$

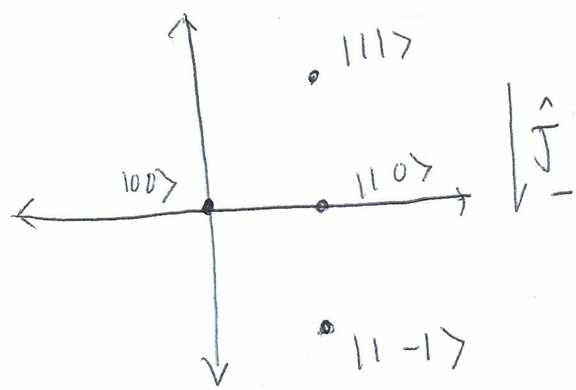
Un Coupled Basis

$|\uparrow\uparrow\rangle$

$|\uparrow\downarrow\rangle$

$|\downarrow\uparrow\rangle$

$|\downarrow\downarrow\rangle$



Coupled Basis $\left|\frac{1}{2}-\frac{1}{2}\right| \le j \le \frac{1}{2}+\frac{1}{2}$

$\Rightarrow j = 0, 1$

$|0\,0\rangle$ "singlet" state

$|1\,-1\rangle\ |1\,0\rangle\ |1\,1\rangle$ "triplet" states

$|1\,1\rangle = |\uparrow\uparrow\rangle$

$|1\,-1\rangle = |\downarrow\downarrow\rangle$

$|1\,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$|0\,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

$|\alpha\rangle \doteq \begin{bmatrix} a \\ b \end{bmatrix}_z$ How to prepare $|\alpha\rangle$

1. Find an operator $\hat{A}$ that has $|\alpha\rangle$ as an eigenstate
2. Make a measurement device for $\hat{Q}$
3. Pass random states through device
4. Block off all other measurement results.

$|\alpha\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \doteq |\uparrow\rangle$ $\quad \hat{S}_z \nearrow |\uparrow\rangle$, $\searrow |\downarrow\rangle /$

$$\langle \vec{S} \rangle = \begin{bmatrix} \langle S_x \rangle \\ \langle S_y \rangle \\ \langle S_z \rangle \end{bmatrix}$$

For arbitrary $|\alpha\rangle$ in a $\frac{1}{2}$ spin system

$\langle \vec{S} \rangle$ for $|\alpha\rangle$

$\langle \vec{S} \rangle$ provides $\check{n}$ in direction of $\langle \vec{S} \rangle$ or (θ, ϕ)

$S_{\check{n}} = (S)_{(\theta,\phi)}$ block off $-\frac{\hbar}{2}$ results

Physics 137A 24 Jun 19
Discussion

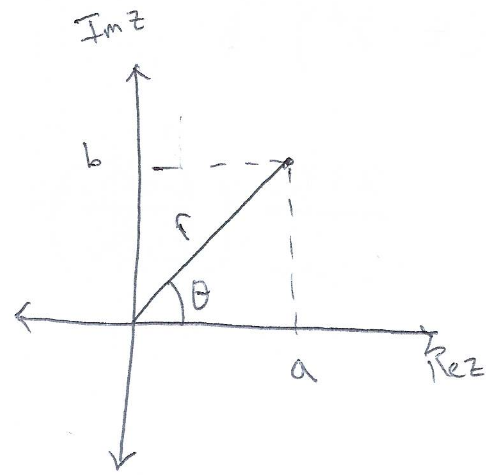
15m mini lecture
20m Problem(s)
15m Homework Questions

Nathaniel Leslie
nathaniel_leslie@berkeley.edu
office hours: 105 LeConte Time TBD

Complex Numbers

$z = a + bi$ Cartesian

$z = re^{i\theta}$ Polar



$$\boxed{\text{Euler's Identity: } e^{i\theta} = \cos\theta + i\sin\theta}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Taylor Series: Any infinitely differentiable function $f(x)$ satisfies

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{[n]}(a)}{n!} (x-a)^n$$

Cartesian

$a = r\cos\theta$

$b = r\sin\theta$

$a = \operatorname{Re} z$

$b = \operatorname{Im} z$

Polar

$r = \sqrt{a^2 + b^2}$

$\theta = \arctan\left(\frac{b}{a}\right)$

$r = |z|$ (magnitude)

$\theta = \arg z$ (phase)

Complex Conjugation

$z^* = a - bi$ Cartesian

$z^* = re^{-i\theta}$ Polar

$z^* z = |z|^2 = r^2 \quad , \quad r = \sqrt{z^* z}$

$\frac{1}{2}(z + z^*) = a$

$\frac{1}{2i}(z - z^*) = b$

$$\frac{a+bi}{c+di}\left(\frac{c-di}{c-di}\right) = \frac{ac + bd + bci - adi}{c^2 + d^2}$$

Phys 137A 26 Jun 19
Discussion

$$P(a<x<b) = \int_a^b dx\, |\Psi(x,t)|^2 \leq 1$$

$$P(-\infty<x<\infty) = \int_{-\infty}^{\infty} dx\, |\Psi(x)|^2 = 1 \quad \text{(Normalization)}$$

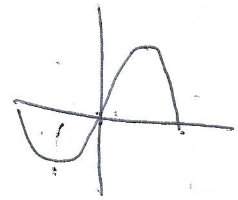
$$\langle x \rangle = \sum_i x_i p_i$$

Outcomes: x with probability p_i

$$\langle x \rangle = E[x] = \int x |\Psi|^2 dx$$

$$\langle x^2 \rangle = \int x^2 |\Psi|^2 dx$$

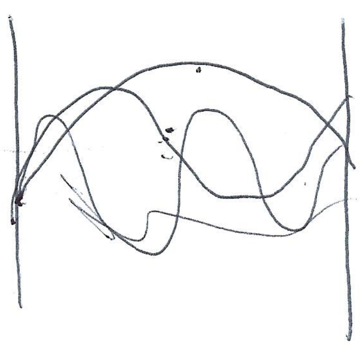
$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = E\Psi$$



$\Psi(x) = A\sin kx + B\cos kx$, $k = \sqrt{\frac{2mE}{\hbar^2}}$ $\qquad \frac{\pi n\hbar}{a\sqrt{2m}}$

$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi \qquad x\sqrt{\frac{2mE}{\hbar^2}} = \cancel{2}\pi n$

$\psi(0) = \psi(a) = 0 \qquad E = \frac{\cancel{2}\pi n\hbar}{a\sqrt{2m}} = \frac{\pi n\hbar}{a\sqrt{2m}}$



$$E = \frac{\pi n\hbar}{a\sqrt{2m}}$$

$\cancel{\psi(x) = A\sin\sqrt{\frac{2mE}{\hbar^2}} = A\sin\sqrt{\frac{2m}{\hbar^2}\left(\frac{\pi n\hbar}{x\sqrt{2m}}\right)^{1/2}} = A\sin\left(\frac{\pi n}{\hbar x}\right)}$

$1 = A^2\int \sin^2\left(\frac{\pi n}{\hbar x}\right)dx = \frac{A^2}{2}\int 1-\cos\left(\frac{2\pi n}{\hbar x}\right)dx$

$\cancel{= A^2} \quad = \frac{A^2}{2}(b-a)$

$\psi(x) = A\sin\left(\sqrt{\frac{2m}{\hbar^2}}\sqrt{\frac{\pi n\hbar}{a\sqrt{2m}}}\,x\right) \qquad \sin^2 kx$

$= A\sin\left((2m)^{1/4}\right. \qquad \frac{1-\cos 2kx}{2}$

$$\psi(x) = A\sin kx + B\cos kx, \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\sin ka = 0$$

$$ka = \pi n = a\sqrt{\frac{2mE}{\hbar^2}} \qquad \left(\frac{\pi n\hbar}{a\sqrt{2m}}\right)^2 = E \Rightarrow E = \frac{\pi^2 n^2\hbar^2}{2ma^2}$$

$$\psi(x) = A\sin\sqrt{\frac{2m}{\hbar^2}}\left(\frac{\pi n\hbar}{\sqrt{2m}}\right) = A\sin\frac{\pi n}{a}$$

$$1 = A^2\int \sin^2\left(\frac{\pi n x}{a}\right)dx = \frac{A^2}{2}\int 1 - \cos\left(\frac{2\pi n}{a}x\right)dx$$

$$= \frac{A^2}{2}\left(a - \frac{a}{2\pi n}\sin\left(\frac{2\pi n}{a}x\right)\Big|_0^a\right) = \frac{A^2}{2}a = 1 \Rightarrow A = \sqrt{\frac{2}{a}}e^{i\phi}$$

$$\langle x\rangle = \frac{2}{a}\int_0^a x\sin^2\left(\frac{\pi n x}{a}\right)dx = \frac{a}{2} - \frac{1}{a}\int_0^a x\cos\left(\frac{2\pi n x}{a}\right)dx$$

$$\langle x^2\rangle = \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{\pi n x}{a}\right)dx$$

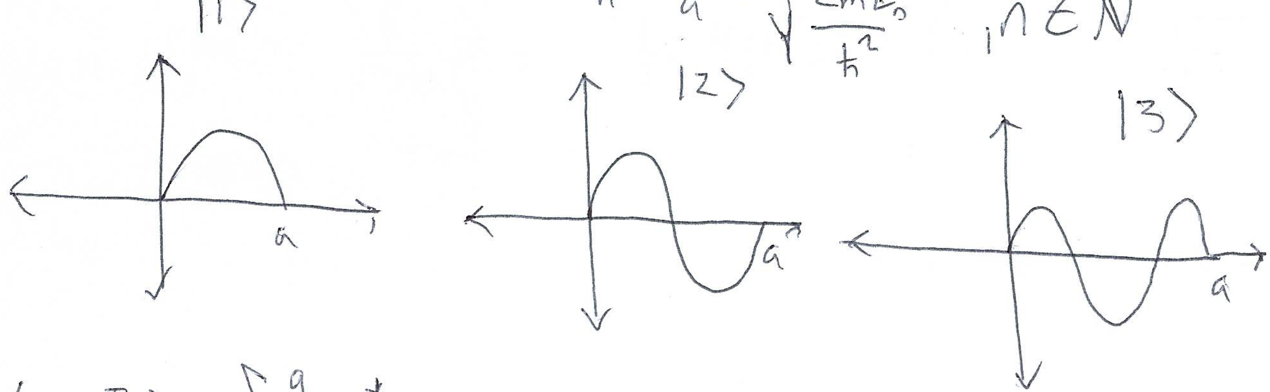
Phys 137A 01Jul19
Discussion

nathaniel_leslie@berkeley.edu

$|n\rangle \to \Psi_n(x) = \sqrt{\frac{2}{a}} \sin k_n x$

$k_n = \frac{n\pi}{a} = \sqrt{\frac{2mE_n}{\hbar^2}}$, $n \in \mathbb{N}$

a)



b) $\langle n|\Psi\rangle = \int_0^a \Psi_n^*(x)\Psi(x) = \lambda_a$ $\quad \Psi(x) = \sum_{n=1}^{\infty} \lambda_n \Psi_n(x)$

$\langle n|\Psi\rangle = \sqrt{\frac{2}{a}} \int_0^a \sin k_n x\, \Psi(x)\, dx = \sqrt{\frac{2}{a}} \int_0^a \sin k_n x \sum_{m=1}^{\infty} \lambda_m \Psi_m(x)\, dx$

$= \int_0^a \Psi_n^* \sum_{m=0}^{\infty} \lambda_m \Psi_m\, dx = \int_0^a \sum_{m=0}^{\infty} \lambda_m \Psi_n^* \Psi_m\, dx = \int_0^a \lambda_n |\Psi_n|^2 dx = \lambda_a$

c) $\Psi = \frac{1}{\sqrt{a}}$

$\int_0^a \frac{1}{\sqrt{a}} \sqrt{\frac{2}{\sqrt{a}}} \sin \frac{n\pi x}{a}\, dx = \frac{\sqrt{2}}{2a} \int_0^a$

$\frac{\sqrt{2}}{a} \int \sin \frac{n\pi x}{a}\, dx = \frac{a}{n\pi} \frac{\sqrt{2}}{a} \cos\left(\frac{n\pi x}{a}\right)\Big|_0^a \quad (-1-1)$

only $n = 1, 3, 5, \ldots$

$+\frac{2\sqrt{2}}{n\pi}$ $\qquad +\frac{2\sqrt{2}}{\pi}$

d)

$$\lambda_n = \frac{2\sqrt{2}}{n\pi} \quad , n = 1, 3, 5, \ldots$$

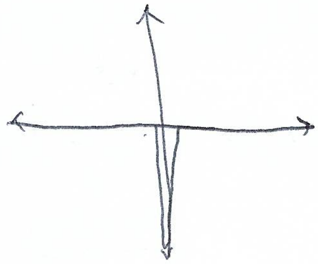
$$\left|\frac{2\sqrt{2}}{n\pi}\right|^2 = P(E_n) = |\langle n|\Psi\rangle|^2 = \begin{cases} \dfrac{8}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

e)

$$\langle E \rangle = \sum_{n=0}^{\infty} E_n P(E_n)$$

$$= \sum_{n=0}^{\infty} \frac{n^2\hbar^2\pi^2}{2ma^2} \frac{8}{\pi^2 n^2} = \sum_{\substack{n=0 \\ \text{odd}}}^{\infty} \frac{4\hbar^2}{ma^2} = \cancel{\frac{4\hbar^2}{ma^2}} = \infty$$

Phys 137A 03Jul19
Discussion



$$\int_I \delta(x-x_0) f(x)\,dx = f(x_0) \quad x_0 \in I$$

$$\int_I \delta(x-x_0)\,dx = 1 \quad x_0 \in I$$

$$V(x) = -K\delta(x), \quad K>0$$

$$\delta(x) = \begin{cases} \infty & x=0 \\ 0 & x \neq 0 \end{cases}$$

$$\delta_{nm} = \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}$$

$$\sum_{m \in S} \delta_{mn} = \delta_{nn} = 1$$

$$\sum_{m \in S} \delta_{mn} a_m = a_n \quad n \in S$$

a) TISE

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - K\delta(x)\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = \left(-\frac{2mE}{\hbar^2} \mp \frac{-2mK\delta(x)}{\hbar^2}\right)\psi$$

b) $x<0$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -w^2\psi, \quad w = \sqrt{\frac{2mE}{\hbar^2}}$$

$E>0$
$$\psi = Ae^{iwx} + Be^{-iwx}$$

$E<0$
$$\psi = Ee^{wx} + Fe^{-wx}$$

$x>0$
$$\psi = Ce^{iwx} + De^{-iwx}$$

$E<0$
$$\psi = Ge^{wx} + He^{-wx}$$

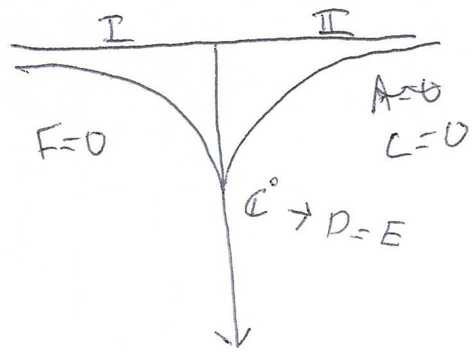
$$-\frac{2mE}{\hbar^2} = w^2 \quad E<0$$
$$-w^2 \quad E>0$$

at $x=0$
$$Ae^{iwx} + Be^{-iwx} = Ce^{iwx} + De^{-iwx}$$

$E>0$
$$\psi = A\cos(wx) + B\sin(wx)$$

$E<0$
$$\psi = Ce^{wx} + De^{-wx}$$

$$\psi(x) = \begin{cases} Ce^{wx} + De^{-wx} & x>0 \quad (\mathrm{II}) \\ Ee^{wx} + Fe^{-wx} & x<0 \quad (\mathrm{I}) \end{cases}$$



$V(x) = -K\delta(x)$

c) at $x=0$

$$De^{-wx} = Ee^{wx}. \qquad D=E=B$$

$$\psi(x) = \begin{cases} Be^{-wx} & x>0 \\ Be^{wx} & x<0 \end{cases}$$

$$\frac{-\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{\partial^2\psi}{\partial x^2}\,dx - k\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)\,dx = \int_{-\epsilon}^{\epsilon}E\,\psi(x)\,dx \longrightarrow 0$$

$$\frac{-\hbar^2}{2m}\frac{\partial\psi}{\partial x}\Big|_{-\epsilon}^{\epsilon} = K\psi(0)$$

Normalize

$$|B|^2 = w \Rightarrow B = \sqrt{w}$$

$$\frac{\partial\psi}{\partial x}\Big|_{+0} - \frac{\partial\psi}{\partial x}\Big|_{-0} = \frac{-2mk}{\hbar^2}\psi(0)$$

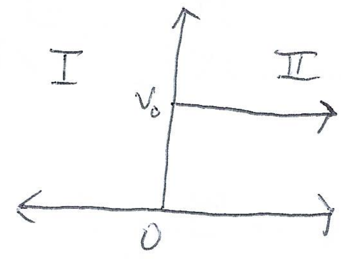
$$\psi'(x) = \begin{cases} -Bwe^{-wx} & x>0 \\ Bwe^{wx} & x<0 \end{cases}$$

$$-Bw - Bw = \frac{-2mK}{\hbar^2}B \qquad w^2 = \frac{-2mE}{\hbar^2} = \frac{m^2K^2}{\hbar^{\cancel{4}}}$$

$$w = \frac{mK}{\hbar^2} \qquad E = \frac{-mK^2}{2\hbar^2}$$

Phys 137A 10Jul19
Discussion

Step Potential



$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x \geq 0 \end{cases}$$

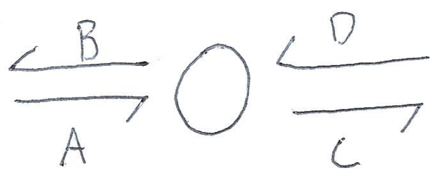
a) $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$

b) for $x < 0$ $\quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \frac{d^2\psi}{dx^2} = \frac{-2mE}{\hbar^2}\psi \;,\; k = \sqrt{\frac{2mE}{\hbar^2}}$

$\psi_I = Ae^{ikx} + Be^{-ikx}$

for $x \geq 0$ $\quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \rightarrow \frac{d^2\psi}{dx^2} = \frac{-2m(E-V_0)}{\hbar^2}\psi \;,\; k' = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

$\psi_{II} = Ce^{ik'x} + De^{-ik'x}$



c) If coming from the left
D = 0 since the incident wave is A, If $D \neq 0$, it would be a wave coming from nowhere, or something would have to cause C to reflect back.

d) ψ and ψ' are continuous

$A + B = C \qquad \psi_I' = Aike^{ikx} - Bike^{-ikx} \qquad \psi_{II}' = ik'Ce^{ik'x}$

$Aik - Bik = Cik'$

$A - B = C\frac{k'}{k} \qquad 2B + C\frac{k'}{k} = C$

$2B = C\left(1 - \frac{k'}{k}\right) \qquad B = \frac{C}{2}\left(1 - \frac{k'}{k}\right)$

$A + \frac{C}{2}\left(1 - \frac{k'}{k}\right) = C \Rightarrow A = \frac{C}{2} + \frac{Ck'}{2k} = \frac{C}{2}\left(1 + \frac{k'}{k}\right)$

$\frac{2A}{1+\frac{k'}{k}} = \frac{2Ak}{k+k'} = C \qquad B = \frac{Ak}{k+k'}\left(1 - \frac{k'}{k}\right) = \frac{A(k-k')}{k+k'}$ *

$$\frac{-\hbar^2}{2m}\left(\frac{\partial\psi}{\partial x}\Big|_{0^+} - \frac{\partial\psi}{\partial x}\Big|_{0^-}\right) = \int_{-\epsilon}^{\epsilon}(E-V)\psi\,dx \to 0$$

e) $j_{inc} = \frac{\hbar k}{m}|A|^2$, $j_{ref} = \frac{\hbar k}{m}|B|^2$ $\quad R = \frac{j_{ref}}{j_{inc}} \quad T = 1-R$

$j_{inc} = \frac{\hbar k}{m}|A|^2 \quad R = \left(\frac{k-k'}{k+k'}\right)^2 = \frac{k^2-2kk'+k'^2}{k^2+2kk'+k'^2} \quad T = \frac{4kk'}{(k+k')^2}$

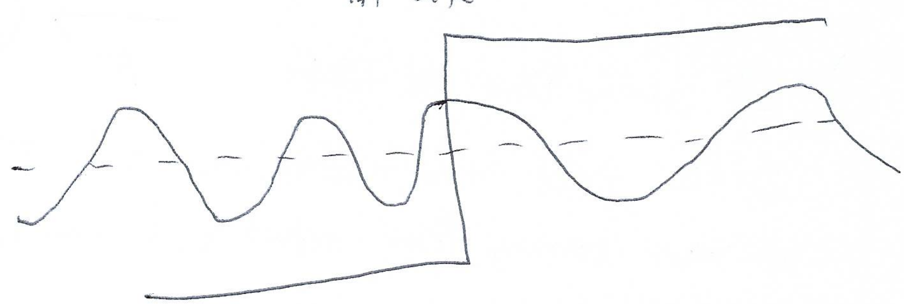
f) $E < V_0 \quad x \geq 0$

$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = -(V_0-E)\psi \quad \frac{d^2\psi}{dx^2} = \frac{2m(V_0-E)}{\hbar^2}\psi = k^2\psi$

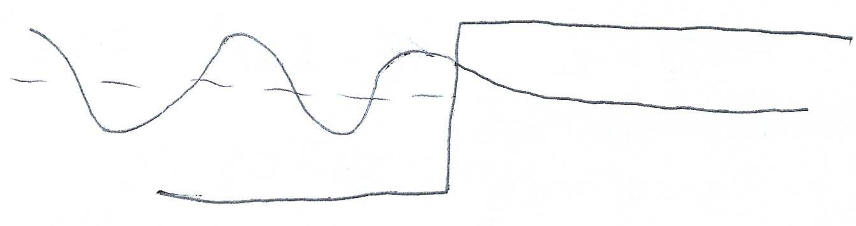
$\psi_{II} = \cancel{Ae^{k}}\, Ce^{kx} + De^{-kx} \quad k = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Ce^{kx} + De^{-kx} & x \geq 0 \end{cases}$$

1st case



2nd case



Phys 137A 15Jul19
Discussion

$|\Psi\rangle = a|0\rangle + b|1\rangle \quad a, b \in \mathbb{C}$

$\langle\Psi|\Psi\rangle = |a|^2 + |b|^2 = 1$

$\{|0\rangle, |1\rangle\} \quad |\Psi\rangle \doteq \begin{bmatrix} a \\ b \end{bmatrix} \quad \langle\Psi| \doteq [a^* \; b^*] \quad |\phi\rangle = c|0\rangle + d|1\rangle$

$\langle\phi|\Psi\rangle = [c^* \; d^*] \begin{bmatrix} a \\ b \end{bmatrix}$

a) $|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \quad |\rightarrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

These states are normalized

$\langle\leftarrow|\rightarrow\rangle = \frac{1}{2}(0) = 0$ $\therefore$ They are orthogonal

$\langle\leftarrow|\rightarrow\rangle = \frac{1}{2}[1 \; -i] \begin{bmatrix} 1 \\ -i \end{bmatrix} = 0$

b) $\hat{J} \doteq \begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix} \quad \hat{J}^\dagger \doteq \begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix}$ $\therefore$ ~~They are Hermitian~~

$\begin{vmatrix} -j & -i/2 \\ i/2 & -j \end{vmatrix} = j^2 - \frac{1}{4} = 0$

$j = \pm\frac{1}{2}$

~~They are not Hermitian since its eigen values are complex~~
They are Hermitian

c)

$\cancel{\begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i/2 \end{bmatrix}}$

$\cancel{\begin{bmatrix} 0 \\ i/2 \end{bmatrix}} \quad j = \frac{1}{2}$

$\begin{bmatrix} \frac{1}{2} & -i/2 \\ i/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ i/2 \end{bmatrix} \quad \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -\frac{1}{2} & -\frac{i}{2} \\ i/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [$

$\frac{a}{2} - \frac{bi}{2} = 0$

$\frac{ia}{2} + \frac{b}{2} = 0$

$b = -ia$

$\begin{bmatrix} -\frac{1}{2} & -\frac{i}{2} \\ i/2 & -1/2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$ia = b$

$|j_1\rangle = a \begin{bmatrix} 1 \\ i \end{bmatrix} \frac{1}{\sqrt{2}}$

$|j_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\langle J\rangle = \langle 0|\hat{J}|0\rangle = [1 \; 0] \begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [1 \; 0] \begin{bmatrix} 0 \\ i/2 \end{bmatrix} = 0$

$\hat{J}|\leftarrow\rangle = \frac{1}{2}|\leftarrow\rangle$

$\hat{J}|\rightarrow\rangle = -\frac{1}{2}|\rightarrow\rangle$

$$|\psi\rangle = a|0\rangle + b|1\rangle \doteq \begin{bmatrix} a \\ b \end{bmatrix}$$

Unitary $U^\dagger U = 1$

$$|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \qquad |\rightarrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$|\leftarrow\rangle + |\rightarrow\rangle = \frac{2}{\sqrt{2}}|0\rangle$$

$$P = |\langle\rightarrow|0\rangle|^2 = \left|\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right|^2 = 1/2$$

$$P = |\langle\leftarrow|0\rangle|^2 = \left|\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right|^2 = \frac{1}{2}$$

Phys 137A 17 Jul 19
Discussion

$$|0\rangle \doteq \begin{bmatrix}1\\0\end{bmatrix} \qquad |1\rangle \doteq \begin{bmatrix}0\\1\end{bmatrix} \qquad \hat{J} \doteq \begin{bmatrix}0 & -i/2\\ i/2 & 0\end{bmatrix}$$

$$|L\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \qquad |R\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$\langle 0|\hat{J}|0\rangle = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & -i/2\\ i/2 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\ i/2\end{bmatrix} = 0$$

$$|\psi\rangle = a|0\rangle + b|1\rangle \doteq \begin{bmatrix}a\\b\end{bmatrix}$$

$$|L\rangle + |R\rangle = \frac{2}{\sqrt{2}}|0\rangle \qquad |0\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)$$

$$|1\rangle = \frac{-i}{\sqrt{2}}(|L\rangle - |R\rangle)$$

$$|\psi\rangle = a\frac{1}{\sqrt{2}}(|L\rangle + i|R\rangle) + b\frac{1}{\sqrt{2}}(|L\rangle - |R\rangle)$$

$$= \frac{a+b}{\sqrt{2}}|L\rangle + \frac{a-b}{\sqrt{2}}|R\rangle \doteq \begin{bmatrix}\frac{a+b}{\sqrt{2}}\\ \frac{a-b}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}c\\d\end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}a+b\\ a-b\end{bmatrix} = U\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}u_1^1 & u_1^2\\ u_2^1 & u_2^2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}au_1^1 + bu_1^2\\ au_2^1 \; bu_2^2\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}(a+b) = au_1^1 + bu_1^2 \Rightarrow u_1^1 = u_1^2 = \frac{1}{\sqrt{2}}$$

$$au_2^1 + bu_2^2 = \frac{1}{\sqrt{2}}(a-b) \Rightarrow u_2^1 = \frac{1}{\sqrt{2}} \quad u_2^2 = \frac{-1}{\sqrt{2}}$$

$$U = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 1 & -1 & 0 & 1\end{array}\right] \rightarrow \left[\begin{array}{cc|cc}2 & 0 & 1 & 1\\ 1 & -1 & 0 & 1\end{array}\right] \Rightarrow \left[\begin{array}{cc|cc}1 & 0 & 1/2 & 1/2\\ 0 & -1 & -\frac{1}{2} & 1/2\end{array}\right] \rightarrow \left[\begin{array}{cc|cc}1 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 1 & \frac{1}{2} & -\frac{1}{2}\end{array}\right]$$

$$U^{-1}JU = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}0 & -i/2\\ i/2 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}-i/2 & i/2\\ i/2 & i/2\end{bmatrix} = \begin{bmatrix}0 & i/2\\ -\frac{i}{2} & 0\end{bmatrix} = \begin{bmatrix}0 & i/2\\ -\frac{i}{2} & 0\end{bmatrix} = J_{L,R}$$

$$J_{L,R} = \begin{bmatrix} 0 & i/2 \\ -i/2 & 0 \end{bmatrix}$$

$$\langle 0 | J_{L,R} | 0 \rangle = \frac{1}{\sqrt{2}}(\langle L| + \langle R|)\, J_{L,R}\, \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)$$

$$\frac{1}{\sqrt{2}}(\langle L|J|L\rangle + \langle L|J|R\rangle + \langle R|J|L\rangle + \langle R|J|R\rangle)$$

$$= 0$$

$a =$ $\qquad \frac{a-bi}{\sqrt{2}} = c \quad , \frac{a+bi}{\sqrt{2}} = d \qquad U\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -i/\sqrt{2} \\ \frac{1}{\sqrt{2}} & i/\sqrt{2} \end{bmatrix} \qquad U^{\dagger}U = \mathbb{I}$$

$$U^{\dagger} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \qquad U^{\dagger}U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = U^{\dagger}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ L & R \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = c\begin{bmatrix} 1 \\ L \\ 1 \end{bmatrix} + d\begin{bmatrix} 1 \\ R \\ 1 \end{bmatrix}$$

Phys 137A 22Jul19
Discussion

$$\hat{H} = \begin{bmatrix} 0 & -iE \\ iE & 0 \end{bmatrix} \qquad \{|x\rangle, |y\rangle\}$$

$$e^{-iEt/\hbar}$$

a) $\begin{vmatrix} -\lambda & -iE \\ iE & -\lambda \end{vmatrix} = \lambda^2 - E^2$

$\lambda = \pm E$

$\lambda = E$

$$\begin{bmatrix} -E & -iE \\ iE & -E \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -Ea - iEb \\ iEa - Eb \end{bmatrix} \qquad \begin{array}{l} a = -ib \\ a = -ib \end{array} \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix} \doteq |E\rangle$$

$\lambda = -E$

$$\begin{bmatrix} E & -iE \\ iE & E \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Ea - ibE \\ iEa + Eb \end{bmatrix} \qquad \begin{array}{l} a = ib \\ a = ib \end{array} \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix} \doteq |-E\rangle$$

$$|\psi\rangle = a\,|\lambda = E\rangle e^{-iEt/\hbar} + b\,|\lambda = -E\rangle e^{iEt/\hbar}$$

$$= a\left(\frac{-i}{\sqrt{2}}|x\rangle + \frac{1}{\sqrt{2}}|y\rangle\right)e^{-iEt/\hbar} + b\left(\frac{i}{\sqrt{2}}|x\rangle + \frac{1}{\sqrt{2}}|y\rangle\right)e^{iEt/\hbar}$$

$$= |x\rangle\left(\frac{bi}{\sqrt{2}}e^{-iEt/\hbar} - \frac{ai}{\sqrt{2}}e^{iEt/\hbar}\right) + |y\rangle\left(\frac{a}{\sqrt{2}}e^{-iEt/\hbar} + \frac{b}{\sqrt{2}}e^{iEt/\hbar}\right)$$

$= \cancel{|x\rangle}$

$$\frac{a}{\sqrt{2}}e^{-iEt/\hbar} = \frac{-b}{\sqrt{2}}e^{iEt/\hbar}$$

$$|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle)$$

$$|L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle)$$

$$\frac{a}{\sqrt{2}} = \frac{-b}{\sqrt{2}}e^{2iEt/\hbar}$$

$$\frac{-a}{b} = e^{2iEt/\hbar} = -1$$

$$\frac{2iEt}{\hbar} = \pi \Rightarrow t = \frac{\pi\hbar}{2iE}$$

$$|\psi(0)\rangle = |x\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-iEt/\hbar}|R\rangle + e^{iEt/\hbar}|L\rangle\right)$$

$$|\psi(t)\rangle = \frac{1}{2}\left(e^{iEt/\hbar}(|x\rangle + i|y\rangle) + e^{-iEt/\hbar}(|x\rangle - i|y\rangle)\right)$$

$$= \frac{1}{2}\left(|x\rangle\left(e^{iEt/\hbar} + e^{-iEt/\hbar}\right) + |y\rangle\left(ie^{iEt/\hbar} - ie^{-iEt/\hbar}\right)\right)$$

$$= |x\rangle \cos\left(\frac{Et}{\hbar}\right) + |y\rangle \sin\left(\frac{Et}{\hbar}\right)$$

c)

$$\frac{Et}{\hbar} = \frac{\pi}{2} \Rightarrow t = \frac{\pi\hbar}{2E}$$

d) $$|x\rangle \cos\left(\frac{E}{\hbar}\left(\frac{\pi\hbar}{4E}\right)\right) + |y\rangle \sin\left(\frac{E}{\hbar}\left(\frac{\pi\hbar}{4E}\right)\right) = \frac{1}{\sqrt{2}}|x\rangle + \frac{1}{\sqrt{2}}|y\rangle$$

e) $\cancel{E} =$ $P(x) = \frac{1}{2}$ $P(y) = \frac{1}{2}$ $P(L) = \frac{1}{2}$ $P(R) = \frac{1}{2}$

$$\hat{H} = \begin{bmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{bmatrix}$$

$$\begin{vmatrix} E_0-\lambda & 0 & A \\ 0 & E_1-\lambda & 0 \\ A & 0 & E_0-\lambda \end{vmatrix} = (E_0-\lambda)\begin{vmatrix} E_1-\lambda & 0 \\ 0 & E_0-\lambda \end{vmatrix} + A\begin{vmatrix} 0 & E_1-\lambda \\ A & 0 \end{vmatrix}$$

$$= (E_0-\lambda)^2(E_1-\lambda) - A^2(E_1-\lambda)$$

$$(E_1-\lambda)\left((E_0-\lambda)^2 - A^2\right) = 0$$

$$(E_0-\lambda)^2 = A^2 \qquad E_0 + \lambda = \pm A$$

$$\cancel{\lambda^2 = E_0 - A^2} \qquad \lambda = -E_0 \pm A$$

$$\cancel{\lambda = \pm(E_0 - A^2)^{1/2}}$$

Phys 137A 24 Jul 19
Discussion

$$\hat{H} \doteq \begin{bmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{bmatrix}$$

$$\begin{vmatrix} 0 & E_1-\lambda \\ A & 0 \end{vmatrix}$$

$$\begin{vmatrix} E_0-\lambda & 0 & A \\ 0 & E_1-\lambda & 0 \\ A & 0 & E_0-\lambda \end{vmatrix} = (E_0-\lambda)\begin{vmatrix} E_1-\lambda & 0 \\ 0 & E_0-\lambda \end{vmatrix} + A \cancel{\begin{vmatrix} E_1-\lambda & 0 \\ 0 & E_0-\lambda \end{vmatrix}}$$

$$\cancel{= (E_1-\lambda)(E_0-\lambda)^2 + A(E_1-\lambda)(E_0-\lambda) = 0}$$

$$\cancel{= (E_1-\lambda)\left((E_0-\lambda)^2 + A(E_0-\lambda)\right)}$$

$$\cancel{= (E_1-\lambda)(E_0-\lambda)(\lambda-(E_0+A))} \qquad \frac{-A \pm \sqrt{A^2}}{2} \qquad \cancel{= (E_0-\lambda) = \frac{-A \pm A}{2}}$$

$$\cancel{\lambda = E_1, E_0, E_0+A} \qquad \cancel{\lambda = E_0, E_0+A}$$

$\lambda = E_0$

$$\begin{bmatrix} 0 & 0 & A \\ 0 & E_1-E_0 & 0 \\ A & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Ac \\ b(E_1-E_0) \\ a \end{bmatrix} \qquad \begin{matrix} a=0 \\ b=0 \\ c=0 \end{matrix} \qquad \cancel{(E_0-\lambda)^2 = -A(E_0-\lambda)}$$

$$E_0-\lambda = -A$$

$$\lambda = E_0 + A$$
$$= E_0$$

$$\cancel{\lambda = \begin{bmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = E_0\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} E_0 a + Ac \\ E_1 b \\ Aa + E_0 c \end{bmatrix} \qquad \begin{matrix} a=1, c=0 \\ b=0 \\ a=0, c=1 \end{matrix}$$

$$(E_0-\lambda)^2(E_1-\lambda) - A^2(E_1-\lambda) = (E_1-\lambda)\left((E_0-\lambda)^2 - A^2\right) = (E_1-\lambda)(\lambda-(E_0-A))(\lambda-(E_0+A))$$

$$\lambda = E_1, E_0-A, E_0+A \qquad E_0-\lambda = \pm A$$

$$\lambda = E_1 \quad \begin{bmatrix} E_0-E_1 & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_0-E_1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a(E_0-E_1) + Ac \\ 0 \\ Aa + c(E_0-E_1) \end{bmatrix}$$

$$|\lambda = E_1\rangle \doteq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad a = \frac{-Ac}{(E_0-E_1)} \qquad a = \frac{-c(E_0-E_1)}{A}$$

$\lambda = E_0 - A$

$$\begin{bmatrix} A & 0 & A \\ 0 & E_1 - E_0 + A & 0 \\ A & 0 & A \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Aa + Ac \\ b(E_1 - E_0 + A) \\ Aa + Ac \end{bmatrix} \quad \begin{matrix} a = -c \\ b = 0 \end{matrix}$$

$$|\lambda = E_0 - A\rangle \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\lambda = E_0 + A$

$$\begin{bmatrix} -A & 0 & A \\ 0 & E_1 + E_0 - A & 0 \\ A & 0 & -A \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -Aa + Ac \\ b(E_1 - E_0 - A) \\ Aa - Ac \end{bmatrix} \quad \begin{matrix} a = c \\ b = 0 \end{matrix}$$

$$|\lambda = E_0 + A\rangle \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \{E_1, E_0 - A, E_0 + A\} \text{ with } \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

b) Start at $|2\rangle$

$P(E_1) = 1 \quad P(E_0 - A) = 0 \quad P(E_0 + A) = 0$ initially ? forever

Start at $|3\rangle \doteq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) = \cancel{\;}$

$$|3\rangle = \frac{1}{\sqrt{2}}\left(|\lambda = E_0 + A\rangle - |\lambda = E_0 - A\rangle \right)$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|\lambda = E_0 + A\rangle e^{-i(E_0+A)t/\hbar} - |\lambda = E_0 - A\rangle e^{-i(E_0 - A)t/\hbar} \right)$$

$$= \frac{1}{\sqrt{2}}\left(|\lambda = E_0 + A\rangle e^{-iAt/\hbar} - |\lambda = E_0 - A\rangle e^{iAt/\hbar} \right)$$

$$= \frac{1}{2}\left(\left(e^{-iAt/\hbar} - e^{iAt/\hbar}\right)|1\rangle + \left(e^{-iAt/\hbar} + e^{iAt/\hbar}\right)|3\rangle \right)$$

$$= -i\sin\left(\frac{At}{\hbar}\right)|1\rangle + \cos\left(\frac{At}{\hbar}\right)|2\rangle$$

Phys 137A 24 Jul 19

MT Review

$M^\dagger = (M^T)^*$ adjoint $\qquad H^\dagger = H$ (Hermitian) observables $\qquad U^\dagger = U^{-1}$ Unitary (change of bases matrices)

$|L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle) \qquad |R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle)$

$|\psi\rangle = a|L\rangle + b|R\rangle = c|x\rangle + d|y\rangle$

$$\begin{bmatrix} c \\ d \end{bmatrix} = a\begin{bmatrix} 1/\sqrt{2} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} + b\begin{bmatrix} 1/\sqrt{2} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \qquad \overleftrightarrow{M}_{LR\to xy} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} | & | \\ L & R \\ | & | \end{bmatrix}$$

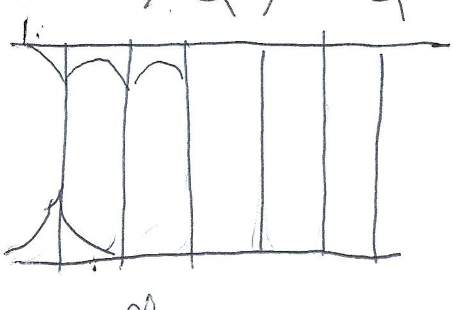
$$\overleftrightarrow{M}_{A\to B} = \begin{bmatrix} | & | \\ e_1 & e_2 \\ | & | \end{bmatrix}_B$$

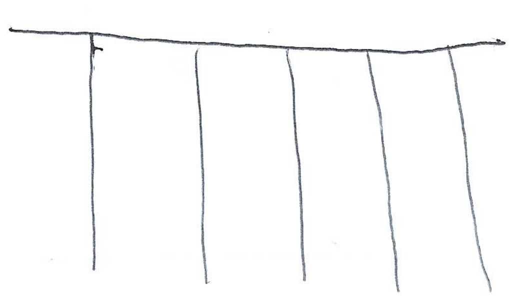
e_1 and e_2 are basis vectors of the A basis expressed in the B basis

$\overleftrightarrow{\Delta}_A \vec{v}_A = \vec{u}_A \qquad \overleftrightarrow{O}_B \vec{v}_B = \vec{u}_B \qquad \cancel{\vec{u}_B = \overleftrightarrow{M}_{A\to B}\, \vec{u}_A}$

$\cancel{\overleftrightarrow{\Delta}_B \vec{v}_B}$

$S^{-1}\Delta S = \tilde{\Delta} \qquad \overleftrightarrow{\Delta}_B = \overleftrightarrow{M}_{A\to B}\, \overleftrightarrow{\Delta}_A\, \overleftrightarrow{M}_{B\to A}$





$$\sum_{n=0}^{\infty} \alpha^n \delta(x-n)$$

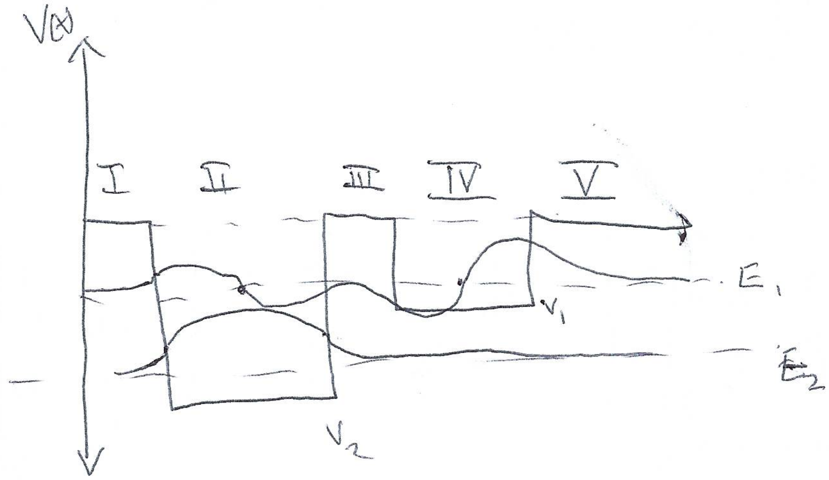
$$\langle \chi_1 | \chi_2 \rangle = \int_{-\infty}^{\infty} dx\, \chi_1^*(x)\, \chi_2(x)$$

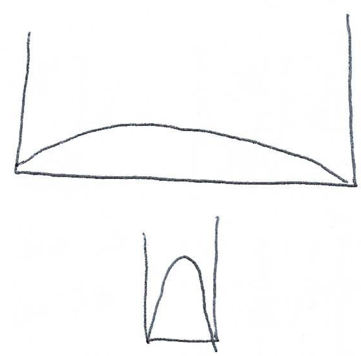
$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$\langle \phi_n | \psi \rangle = \int_{-\infty}^{\infty} \phi_n^* \psi \, dx = C_n$$

$$E_n = \frac{\pi^2 \hbar^2 n^2}{2ma^2}$$

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{else} \end{cases}$$





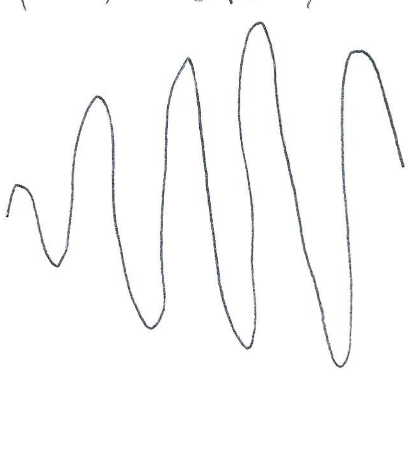
$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = E\psi$$

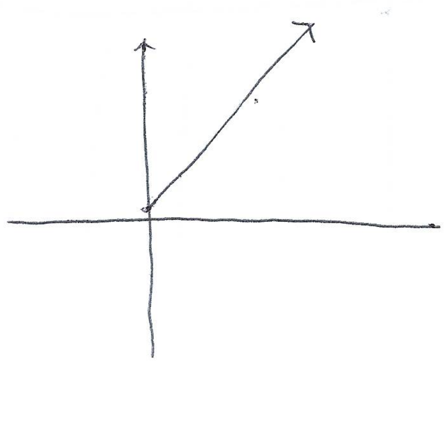
$$\frac{\partial^2 \psi}{\partial x^2} = \underbrace{\frac{-2m(E-V)}{\hbar^2}}\psi$$

$e^{\pm kx}$ $\quad k^2 \quad E < V$ exponential

$e^{\pm ikx}$ $\quad -k^2 \quad E > V$ oscillatory

$Ai(z) + Bi(z)$





Two types of Hilbert Spaces

<u>Wave functions</u>

$|\psi\rangle \doteq \psi(x)$

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dx\, \psi^* \psi$$

<u>n level system</u>

$$|a\rangle = a_1|1\rangle + a_2|2\rangle + \dots + a_n|n\rangle$$

$$\doteq \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\langle b|a\rangle = \begin{bmatrix} b_1^* & b_2^* & \dots & b_n^* \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Normalization $\langle\psi|\psi\rangle = 1$ $\quad \int_{-\infty}^{\infty} dx\, \psi^* \psi = 1 \quad \sum_{i=1}^{\infty} a_i^* a_i = 1$

Orthogonality: $\langle\phi|\psi\rangle = 0 \quad \int_{-\infty}^{\infty} dx\, \phi\psi = 0 \quad \sum_{i=1}^{\infty} b_i^* a_i = 0$

Fourier's Trick

$$a_i = \langle\psi_i|\psi\rangle = \int_{-\infty}^{\infty} \psi_i^* \psi\, dx \quad a_i = \langle i|a\rangle$$

$$|p\rangle \doteq \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

Fourier Transform

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\, \phi(p) e^{ipx/\hbar}$$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx\, \psi(x) e^{-ipx/\hbar}$$

$\psi(x) = \langle x|state\rangle \quad \phi(p) = \langle p|state\rangle$

For drawing wave functions
Smaller difference $E - V_0$, larger amplitude and larger wavelength

Phys 137A 29 Jul 19
Discussion

1) $$[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

a) $\hat{\pi}\psi(x) = \psi(-x)$

b) $\hat{\pi}^2 = \hat{\mathbb{1}}$

$$\hat{\pi}^2|\alpha\rangle = \hat{\pi}(\hat{\pi}|\alpha\rangle) = \hat{\pi}(|-\alpha\rangle) = |\alpha\rangle$$
$$\hat{\mathbb{1}}|\alpha\rangle = |\alpha\rangle = \hat{\pi}^2|\alpha\rangle$$
$$\hat{\pi}^2|\alpha\rangle = \lambda^2|\alpha\rangle \Rightarrow \lambda = \pm 1$$

c) $\psi_{\pm}(x)$ ψ_+ is even $\hat{\pi}\psi_{\pm} = \pm\psi_{\pm}$ $\psi_{\pm}(-x) = \pm\psi_{\pm}(x)$
ψ_- is odd
+ even
− odd

d) $\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(\hat{x})$ $V(\hat{x}) = V(-\hat{x})$

$$[\hat{H},\hat{\pi}] = \left(\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi_-\right)(\psi(-x)) - \psi(-x)\left(-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi_-\right)$$
$$= 0 \therefore \text{commute}$$

e) If $[\hat{H},\hat{\pi}] = 0$, all energy eigenfunctions are even or odd. If they commute, they share an eigenspace
Since parity reflects the wave function and energy and parity commute, reflections don't change the energy so the eigenfunctions can be symmetric or antisymmetric.

2) $|n\rangle$, $n = 0,1,2,\ldots$ $E_n = (n+\frac{1}{2})\hbar\omega$ $\psi(x) = \sum_{n=0}^{\infty} c_n\psi_n(x)$

$|\psi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$ $|\psi\rangle \doteq \begin{bmatrix} c_0 \\ c_1 \\ \vdots \end{bmatrix}$ $\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$
$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

$\hat{\Pi}(a) = \hat{T}(a)\,\hat{\Pi}(0)\,\hat{T}(a)$ ~~Moving~~ Move reflection point to zero, reflect then translate it more in the same direction

1c) $H\psi = E\psi$

$H(\Pi\psi) = \Pi H\psi = \Pi(E\psi) = E\Pi\psi$ Assuming no degeneracy

This is $\Pi\psi = P\psi$ $P = \pm 1$

$\psi(-x) = \pm\psi(x)$

Phys 137A 31 Jul 19

Discussion

$$\begin{bmatrix} 0 & i & \\ +i & 0 & i \\ 0 & -i & 0 \end{bmatrix}$$

$|n\rangle$, $n = 0, 1, 2, 3, \ldots$ $\quad E_n = (n + \frac{1}{2})\hbar\omega$

a) $\hat{A} = \begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & & \vdots \\ 0 & & \ddots & \vdots \\ 0 & \cdots & & E_n \end{bmatrix} = \hat{H} = \begin{bmatrix} \frac{1}{2} & & & \\ & \frac{3}{2} & & \\ & & \frac{5}{2} & \\ & & & \ddots \end{bmatrix}$

b)

$$\hat{A}_+ = \begin{bmatrix} 0 & \sqrt{1+1} & 0 & 0 & \cdots & 0 \\ \sqrt{1+1} & 0 & \sqrt{2+1} & & & \vdots \\ & \sqrt{2+1} & \ddots & & & \vdots \\ \vdots & & & \ddots & & \\ \vdots & & & & & \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 & \cdots \\ \sqrt{2} & 0 & \sqrt{3} & 0 & \\ 0 & \sqrt{3} & 0 & 2 & \\ 0 & 0 & 0 & 0 & \\ \vdots & & & & \ddots \end{bmatrix}$$

$$\hat{a}_+ = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & & \\ 0 & \sqrt{2} & 0 & & \\ 0 & 0 & \sqrt{3} & & \\ \cdots & & & \ddots & \end{bmatrix} \qquad \hat{a}_- = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \cdots & & & & \ddots \end{bmatrix}$$

$$\hat{a}_+\hat{a}_- = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & & \\ 0 & & \sqrt{3} & 0 & \\ \vdots & & & 2 & \\ \vdots & & & & \end{bmatrix}$$

$$\hat{a}_-\hat{a}_+ = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\hat{1}$$

$$= \left(\hat{a}_+\hat{a}_- - \frac{1}{2}\hat{1}\right)\hbar\omega = \hat{H} = \begin{bmatrix} \frac{3\hbar\omega}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{5\hbar\omega}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{7\hbar\omega}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{9\hbar\omega}{2} & 0 \end{bmatrix}$$

$$\hat{a}^{\dagger}+\hat{a} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 & \ddots \end{bmatrix}$$

$$\hat{a}^{\dagger}\hat{a} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & \ddots \end{bmatrix}$$

$$\left(\hat{a}^{\dagger}\hat{a}+\tfrac{1}{2}\right)\hbar\omega = \hat{H}$$

$$[\hat{a},\hat{a}^{\dagger}] = \hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}$$

$$\hat{a}\hat{a}^{\dagger} = \begin{bmatrix} 0 & 1 \\ & 0 & \sqrt{2} \\ & & 0 & \sqrt{3} \\ & & & 0 & 2 \\ & & & & 0 & \sqrt{5} \\ & & & & & 0 & \sqrt{6} \\ & & & & & & 0 & \sqrt{7} \\ & & & & & & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 & 0 \\ & \sqrt{2} & 0 \\ & & \sqrt{3} & 0 \\ & & & 2 & 0 \\ & & & & \sqrt{5} & 0 \\ & & & & & \sqrt{6} & 0 & \ddots \end{bmatrix} = \begin{bmatrix} 1 \\ & 2 \\ & & 3 \\ & & & 4 \\ & & & & \ddots \end{bmatrix}$$

$$[\hat{a},\hat{a}^{\dagger}] = \begin{bmatrix} 1 \\ & 2 \\ & & 3 \\ & & & 4 \\ & & & & \ddots \end{bmatrix} - \begin{bmatrix} 0 \\ & 1 \\ & & 2 \\ & & & 3 \\ & & & & 4 \\ & & & & & 5 & \ddots \end{bmatrix} = \hat{1}$$

$$\hat{H} = (\hat{a}\hat{a}^{\dagger} + K)\hbar\omega$$

$$2\sqrt{\frac{m\omega}{2\hbar}}\,\hat{X} = \hat{a}+\hat{a}^{\dagger} \Rightarrow \hat{x} = \frac{\sqrt{\hbar}}{\sqrt{2m\omega}}(\hat{a}+\hat{a}^{\dagger})$$

$$\hat{x} = \sqrt{\frac{h}{2m\omega}} \begin{bmatrix} 0 & 1 \\ 1 & 0 & \sqrt{2} \\ & \sqrt{2} & 0 & \sqrt{3} \\ & & \sqrt{3} & 0 & 2 \\ & & & 2 & 0 & \ddots \\ & & & & & 0 \end{bmatrix} \qquad \hat{p} \doteq i\sqrt{\frac{2}{\hbar m\omega}} \begin{bmatrix} 0 & -1 \\ +1 & 0 & -\sqrt{2} \\ & \cancel{-\sqrt{2}} & 0 & \sqrt{3} \\ & & \sqrt{3} & 0 & -2 \\ & & & \cancel{\sqrt{4}} & 0 & \sqrt{5} \\ & & & & 2 & 0 & -\sqrt{6} \\ & & & & & & 0 \end{bmatrix}$$

Phys 137A 07 Aug 19

Discussion

$$|\psi\rangle_A = \alpha|0\rangle + \beta|1\rangle$$

$$|\Psi^-\rangle_{BC} = \frac{1}{\sqrt{2}}\left(|0\rangle_B|1\rangle_C + |1\rangle_B|0\rangle_C\right)$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$|ab\rangle = |a\rangle_{\text{particle 1}} \otimes |b\rangle_{\text{particle 2}}$$

$$\langle ab| = \langle a|_{\text{particle 1}} \otimes \langle b|_{\text{particle 2}}$$

b) $$|\Psi\rangle_{ABC} = |\psi\rangle_A |\Psi^-\rangle_{BC}$$

$$= \frac{1}{\sqrt{2}}(\alpha|0\rangle_A + \beta|1\rangle_A)(|0\rangle_B|1\rangle_C - |1\rangle_B|0\rangle_C)$$

$$= \frac{1}{\sqrt{2}}\left(\alpha(|0\rangle_A|0\rangle_B|1\rangle_C - \alpha|0\rangle_A|1\rangle_B|0\rangle_C + \beta|1\rangle_A|0\rangle_B|1\rangle_C - \beta|1\rangle_A|1\rangle_B|0\rangle_C\right)$$

$$= \frac{1}{\sqrt{2}}(\alpha|001\rangle - \alpha|010\rangle + \beta|101\rangle - \beta|110\rangle)$$

c) $\cancel{|\Psi\rangle_{ABC} =}$

$$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Phi^-\rangle) \qquad |01\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle + |\psi^-\rangle)$$

$$|10\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle - |\psi^-\rangle \qquad |11\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle - |\Phi^-\rangle)$$

$$|\Psi\rangle_{ABC} = \frac{1}{2}\left(\alpha(|\psi^+\rangle - |\psi^-\rangle)|1\rangle - \alpha(|\psi^+\rangle + |\psi^-\rangle)|0\rangle + \beta(|\psi^+\rangle - |\psi^-\rangle)|1\rangle - \beta(|\Phi^+\rangle - |\Phi^-\rangle)|0\rangle\right)$$

$$= \frac{1}{2}\left(|\Phi^+\rangle(-\beta|0\rangle + \alpha|1\rangle) + |\Phi^-\rangle(\beta|0\rangle + \alpha|1\rangle + |\psi^+\rangle(\alpha|0\rangle + \beta|1\rangle) + |\psi^-\rangle(-\alpha|0\rangle - \beta|1\rangle)\right)$$

d) For $|\Phi^+\rangle$ $\quad P(|\Phi^+\rangle \otimes |0\rangle) = \dfrac{|\beta|^2}{4}$

$$P(|\Phi^+\rangle \otimes |1\rangle) = \frac{|\alpha|^2}{4}$$

$$P(|\Phi^+\rangle) = \frac{|\alpha|^2 + |\beta|^2}{4} = \frac{1}{4}$$

$\hat{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ bit flip gate

$\hat{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ phase flip gate

e)

Measured Bell state	Charlie's qubit	operation needed to be performed
$\lvert\Phi^+\rangle$	$\beta\lvert 0\rangle - \alpha\lvert 1\rangle$	$\hat{X}\hat{Z}$
$\lvert\Phi^-\rangle$	$\beta\lvert 0\rangle + \alpha\lvert 1\rangle$	$\hat{X}$
$\lvert\Psi^+\rangle$	$\alpha\lvert 0\rangle - \beta\lvert 1\rangle$	$\hat{Z}$
$\lvert\Psi^-\rangle$	$\alpha\lvert 0\rangle + \beta\lvert 1\rangle$	$\hat{I}$

Phys 137A 12Aug19
Discussion

$$\hat{J}^2 |j,m\rangle = j(j+1)\hbar^2 |j,m\rangle$$
$$\hat{J}_z |j,m\rangle = m\hbar |j,m\rangle$$

a) $|j=1, m=1\rangle$

$$\cancel{J_x = \begin{bmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{bmatrix}} \qquad J_x = \begin{bmatrix} 0 & -\frac{\hbar}{2} & 0 \\ \frac{\hbar}{2} & 0 & -\frac{\hbar}{2} \\ 0 & \frac{\hbar}{2} & 0 \end{bmatrix}$$

b) $$\cancel{J_+ = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\hbar}{2} & 0 & 0 \\ 0 & \frac{\hbar}{2} & 0 \end{bmatrix} \qquad J_- = \begin{bmatrix} 0 & \frac{\hbar}{2} & 0 \\ 0 & 0 & -\frac{\hbar}{2} \\ 0 & 0 & 0 \end{bmatrix}}$$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \qquad \hat{J}_y = \begin{bmatrix} 0 & \frac{\hbar}{2} & 0 \\ \frac{\hbar}{2} & 0 & \frac{\hbar}{2} \\ 0 & \frac{\hbar}{2} & 0 \end{bmatrix}$$

$$J_+ = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\hbar}{2} & 0 & 0 \\ 0 & \frac{\hbar}{2} & 0 \end{bmatrix} \qquad \cancel{J_- =} \; J_- = \begin{bmatrix} 0 & -\frac{\hbar}{2} & 0 \\ 0 & 0 & -\frac{\hbar}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$J_+ J_- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{\hbar^2}{4} & 0 \\ 0 & 0 & -\frac{\hbar^2}{2} \end{bmatrix} \qquad J^2 |j,m\rangle =$$

$$j(j+1)\hbar^2 - m\hbar^2 - m^2\hbar^2$$
$$3\hbar^2 - m\hbar^2 - m^2\hbar^2$$

standard basis: $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$

a) $$\hat{J}_z \doteq \begin{bmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{bmatrix} \qquad \hat{J}^2 \doteq 2\hbar^2 \hat{1}$$

b) $$\hat{J}_+ \doteq \begin{bmatrix} 0 & c_1 & 0 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \hat{J}_- \doteq \begin{bmatrix} 0 & 0 & 0 \\ c_1^* & 0 & 0 \\ 0 & c_2^* & 0 \end{bmatrix}$$

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + J_z^2$$

$$J_- J_+ = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i[J_x, J_y]$$

$$= J_x^2 + J_y^2 - \hbar J_z$$

$$J_- J_+ = \hat{J}^2 - \hat{J}_z^2 - \hbar J_z$$

$$\begin{bmatrix} 0 & 0 & 0 \\ c_1^* & 0 & 0 \\ 0 & c_2^* & 0 \end{bmatrix} \begin{bmatrix} 0 & c_1 & 0 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \overset{0}{\cancel{|c_1|^2}} & 0 & 0 \\ 0 & |c_1|^2 & 0 \\ 0 & 0 & |c_2|^2 \end{bmatrix} = 2\hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{bmatrix} \Rightarrow c_1 = \hbar\sqrt{2} = c_2$$

$$\frac{1}{2}(\hat{J}_+ + \hat{J}_-) \doteq \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & -\lambda & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & \frac{\hbar}{\sqrt{2}} \\ \frac{\hbar}{\sqrt{2}} & -\lambda \end{vmatrix} - \frac{\hbar}{\sqrt{2}} \begin{vmatrix} \frac{\hbar}{\sqrt{2}} & \frac{\hbar}{\sqrt{2}} \\ 0 & -\lambda \end{vmatrix}$$

$$= \lambda\left(-\lambda^2 + \frac{\hbar^2}{2}\right) + \frac{\hbar^2}{2}\lambda$$

$$= \left(\frac{2\hbar^2}{2} - \lambda^2\right)\lambda = \left(\hbar^2 - \lambda^2\right)\lambda$$

$\lambda = 0$ $\qquad\qquad\qquad\qquad$ $\lambda = 0, \hbar, -\hbar$

$$\begin{bmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\lambda = \pm\hbar$

$$\begin{bmatrix} \mp\hbar & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & \mp\hbar & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & \mp\hbar \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0} \Rightarrow V_{\pm} = \frac{1}{2} \begin{bmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{bmatrix}$$

$$|1,\pm1\rangle_x = \frac{1}{2}\left(|1,1\rangle + \pm\sqrt{2}|1,0\rangle + |1,-1\rangle\right)$$

$$|1,0\rangle_x = \frac{1}{\sqrt{2}}\left(|1,1\rangle - |1,-1\rangle\right)$$

$$|1,1\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|1,1\rangle_x + |1,-1\rangle_x\right) + |1,0\rangle_x\right)$$

$$= \frac{1}{2}\left(|1,1\rangle_x + \sqrt{2}|1,0\rangle_x + |1,-1\rangle_x\right)$$

Phys 137A 14Aug14
Discussion

Final Review

Change of Basis

n-state system

Any state $|\alpha\rangle$ can be represented as a vector

reverse scripts

$$|\alpha\rangle \doteq \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_{[e]} \qquad |\alpha\rangle = a_1|e^1\rangle + a_2|e^2\rangle + \cdots + a_n|e^n\rangle$$

$$= b_1|f^1\rangle + b_2|f^2\rangle + \cdots + b_n|f^n\rangle$$

$$|\alpha\rangle \doteq \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}_{[f]}$$

$$\begin{bmatrix} \overleftrightarrow{C}_{e\to f} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_e = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}_f$$

$\overleftrightarrow{C}_{e\to f}$

$$\overleftrightarrow{C}_{e\to f} = \begin{bmatrix} \langle f_1|e^1\rangle & \langle f_1|e^2\rangle & \cdots & \langle f_1|e^n\rangle \\ \langle f_2|e^1\rangle & \cdots & & \langle f_2|e^n\rangle \\ \vdots & & & \vdots \\ \langle f_n|e^1\rangle & \cdots & & \langle f_n|e^n\rangle \end{bmatrix} \qquad \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \langle e^1|\alpha\rangle \\ \langle e^2|\alpha\rangle \\ \vdots \\ \langle e^n|\alpha\rangle \end{bmatrix}$$

$$\hat{O} \doteq \overleftrightarrow{O} = \begin{bmatrix} \langle e^1|\hat{O}|e_1\rangle & \langle e^1|\hat{O}|e_2\rangle & \cdots & \langle e^1|\hat{O}|e_n\rangle \\ \vdots & & \ddots & \vdots \\ \langle e^n|\hat{O}|e_1\rangle & \cdots & & \langle e^n|\hat{O}|e_n\rangle \end{bmatrix} \doteq \begin{bmatrix} \langle f^1|\hat{O}|f_1\rangle & \cdots & \langle f^1|\hat{O}|f_n\rangle \\ \vdots & \ddots & \vdots \\ \langle f^n|\hat{O}|f_1\rangle & \cdots & \langle f^n|\hat{O}|f_n\rangle \end{bmatrix}$$

$$\overleftrightarrow{\Delta}_f = \overleftrightarrow{C}_{e\to f}\ \overleftrightarrow{\Delta}_e\ \overleftrightarrow{C}_{f\to e}$$

$$[a_1^* \ \cdots \ a_n^*]_e \doteq \langle a|$$

Final 2016 1e)

Sketch $l=1$ effective potential of Hydrogen

$$V_{eff}(r) = V(r) + \frac{\hbar^2 l(l+1)}{2\mu r^2}$$ holds for all spherically symmetric Potentials $V(r)$

$$V(r) = -\frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r}$$, Z = # protons = atomic number

3D SE

$$\left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r)\right)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

$\downarrow$ separation of variables $\qquad \psi(r,\theta,\phi) = R_{nl}(r)\,Y_l^m(\theta,\phi)$

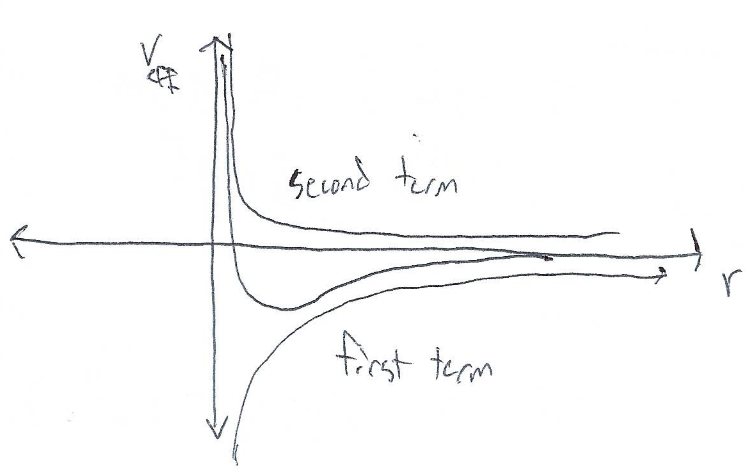
$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)R(r) + \frac{1}{2\mu r^2}\hbar^2 l(l+1)R(r) + V(r)R(r) = ER$$

radial equation

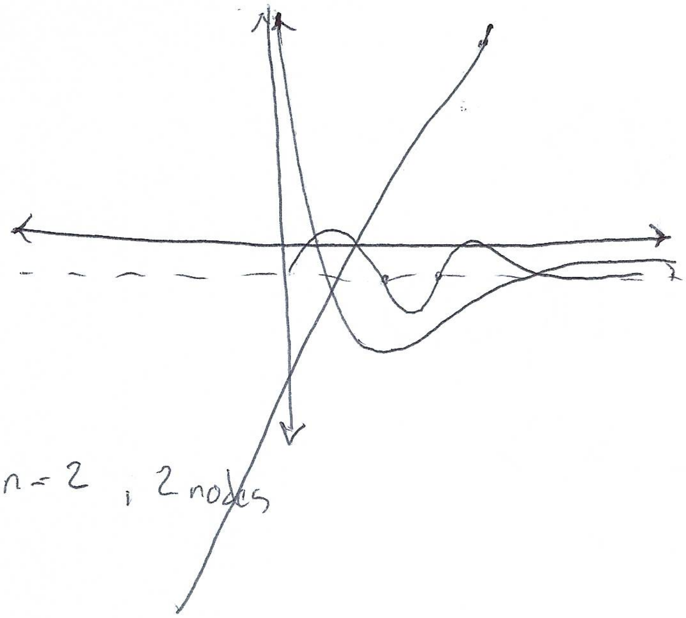
$$u(r) = R(r)\,r$$

$$\underbrace{-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}u(r)}_{\text{1D Schrödinger}} + \underbrace{\frac{\hbar^2 l(l+1)}{2\mu r^2}u(r) + V(r)u(r)}_{V_{eff}(r)\,u(r)} = Eu(r)$$

$$V_{eff} = -k\frac{e^2}{r} + \frac{\hbar^2}{\mu r^2}$$



Sketch 3_p wave function



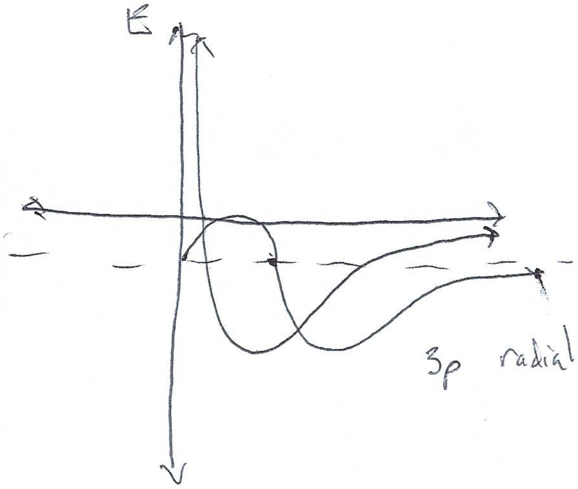
n = 2 , 2 nodes

$$E_n = \frac{E_1}{n^2} \qquad E_1 = -13.6\,eV$$

E must be < 0 for bound states

n	l	m
1	0	
2	0,1	
3	0,1,2	

l = 1 has lowest state n = 2



3p radial wave

Final review.pdf

Show that for $H = -\gamma \vec{L} \cdot \vec{B}$, $\vec{B}$ constant

then $\frac{d}{dt}\langle \vec{L} \rangle = \langle \vec{\mu} \times \vec{B} \rangle$ $\quad \vec{\mu} = \gamma \vec{L}$

Actually 3 equations

$$\frac{d\langle L_x \rangle}{dt} = \langle (\vec{\mu} \times \vec{B})_x \rangle = \langle \mu_y B_z - \mu_z B_y \rangle = \gamma \langle L_y B_z - L_z B_y \rangle$$

$$\frac{d\langle \hat{Q} \rangle}{dt} = \frac{i}{\hbar}\langle [H, Q] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle$$

$$\frac{d\langle L_x \rangle}{dt} = \frac{i}{\hbar}\langle [H, L_x] \rangle = \frac{i}{\hbar}\langle [-\gamma \vec{L} \cdot \vec{B}, L_x] \rangle$$

$$= \frac{-i\gamma}{\hbar}\langle [L_x B_x + L_y B_y + L_z B_z, L_x] \rangle$$

$$\cancel{= \frac{-i\gamma}{\hbar}\langle L_x B_x L_x + L_x L_x B_x + L_y B_y L_x}$$

$$= \frac{-i\gamma}{\hbar}\langle [L_x B_x, L_x] + [L_y B_y, L_x] + [L_z B_z, L_x] \rangle$$

$$= \frac{-i\gamma}{\hbar}\langle -i\hbar B_y L_z + i\hbar B_z L_y \rangle = \gamma \langle L_y B_z - L_z B_y \rangle$$

Assume doing SG experiment, select for only $|\uparrow\rangle$

No Can we flip the state to $|\downarrow\rangle$? (How?)

$|\uparrow_z\rangle$ put in magnetic field in x direction

$$\hat{H} = -\gamma \hat{\vec{S}} \cdot \hat{\vec{B}} = \gamma \hat{S}_x B_x$$