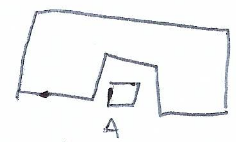
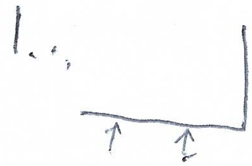
22 Jan 19 110A

Class before gets out at ~ 9

Enter back from LC direction

Panasonic RZ570 WUXGA

7R 

CB 4x4 

Only seats behind A have obstructed vision

Fourier optics pushed aside and neglected

copy HW for reference

Math Background

Scalars, vectors, and tensors

Example: Scalar potential V

$V(\vec{r},t)$ is a scalar field

$V(\vec{r},t) = V(x,y,z,t) \equiv V(r,\theta,\phi,t)$

~~Cartesian~~ Cartesian and curvilinear coordinate systems

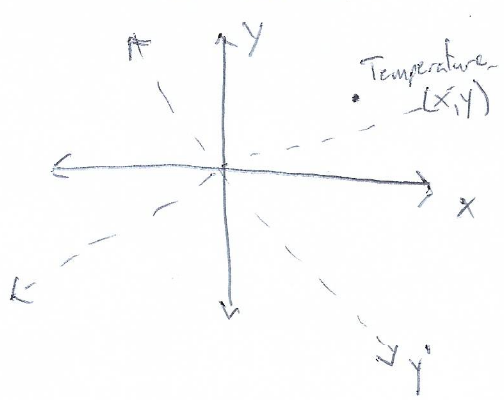
Vector Fields

Example: Electric Field $\vec{E}$

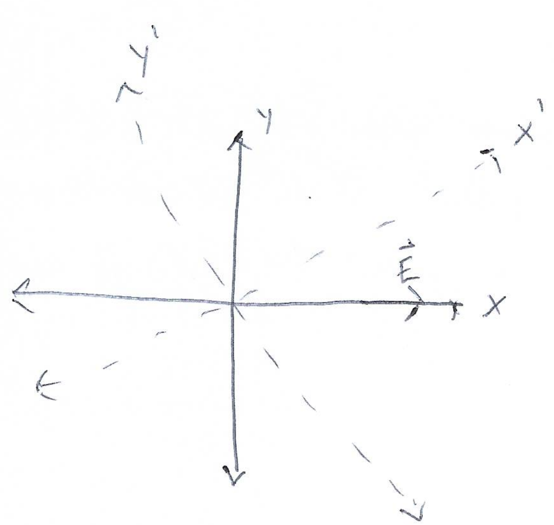
$\vec{E}(\vec{r},t)$

Tensor Fields $\overleftrightarrow{T}(\vec{r},t)$

Rotation of coordinates



Scalar



Vector

$\vec{E}(\vec{r}_0) = \hat{x} E_x$

$\vec{E}'(\vec{r}_0) = (\hat{x}'\cos\theta + \hat{y}'\sin\theta)E_x$

Maxwell's Equations: relating ~~instaneous~~ instantaneous rates of change of fields

Maxwell inuitioned equations then Einstein found relevance to SR

Source free

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Source

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{\partial\vec{E}}{\partial t}$$

relativistic correction

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B} = m\frac{d\vec{v}}{dt} = \gamma m\frac{d\vec{v}}{dt}$$

"Statics"

Fields are varying slowly

$$\vec{\nabla}\times\vec{E} \simeq \vec{0}, \quad \vec{\nabla}\times\vec{B} \simeq \mu_0\vec{J}$$

electrostatics, magnetostatics

Decoupled $\vec{E}$ and $\vec{B}$ in statics

$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ charges

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ currents

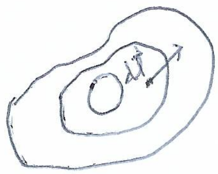
$\vec{\nabla} \equiv$ vector operator

(1) Acting on a scalar field $\vec{\nabla}\phi(\vec{r},t) \Rightarrow$ gradient

(2) Acting as dot product $\vec{\nabla}\cdot\vec{V}(\vec{r},t) \Rightarrow$ divergence

(3) Acting as a cross product $\vec{\nabla}\times\vec{V}(\vec{r},t) \Rightarrow$ curl

Consider scalar field $\phi(\vec{r}) = \phi(x,y,z)$



$d\vec{l}$ = infinitesimal displacement vector

Contour plot

$$d\phi = \phi(\vec{r} + d\vec{l}) - \phi(\vec{r})$$
$$d\vec{l} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$$

Multivariable Calculus

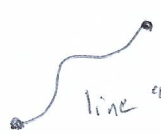
$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$
$$= \left(\frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z}\right)\cdot d\vec{l}$$
$$\equiv \vec{\nabla}\phi \cdot d\vec{l}$$

Cartesian coordinates

$$\boxed{\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$$

Fundamental Theorems: relate properties "inside" to properties at surface

1D

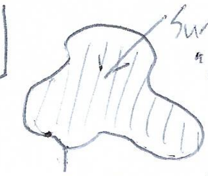


line "inside"

endpoints "surface"

$\nabla\phi$

2D

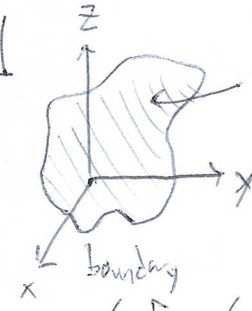


Surfaces S "inside"

Boundary contour, C "surface"

$\vec{\nabla}\times\vec{V}$

3D



Volume "inside"

boundary Surface S

1D Fundamental Theorem of calculus

$$\int_{\vec{r}_a}^{\vec{r}_b} d\phi$$

$$\int_{\vec{r}_a}^{\vec{r}_b} d\phi = \phi(\vec{r}_b) - \phi(\vec{r}_a)$$

$$\int_{\vec{r}_a}^{\vec{r}_b} \vec{\nabla}\phi \cdot d\vec{l} = \phi(\vec{r}_b) - \phi(\vec{r}_a)$$

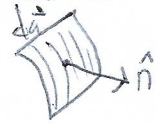
3D

Define $\oint_S \vec{V}(\vec{r}) \cdot d\vec{a} \equiv$ flux of $\vec{V}$ through S

Interpretation as fluid flow

$\rho \equiv$ mass density $\quad \vec{V}(\vec{r}) \equiv$ velocity field

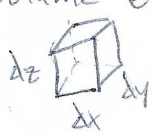
$\vec{J} = \rho \vec{V}(\vec{r})$ mass current

Element of area  $d\vec{a} = \hat{n}\, da$

$\vec{J} \cdot d\vec{a}$ = mass flow rate through $d\vec{a}$

$\oint_S \vec{J} \cdot d\vec{a}$ = total rate of mass flow through S

Consider a small volume element dV



One can show $\vec{\nabla} \cdot \vec{J}(\vec{r}) = \dfrac{\text{flux through surface}}{dV}$ = flux per unit volume = flux density

Now $\boxed{\oint \vec{J} \cdot d\vec{a} = \int dV\, \vec{\nabla} \cdot \vec{J}(\vec{r})}$

3D Fundamental Theorem

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Continuing math review

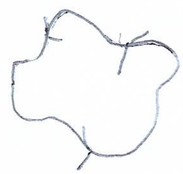
1D 2D 3D

$$\int_{\vec{r}_0}^{\vec{r}_1} d\vec{l}\cdot\vec{\nabla}\phi = \phi(\vec{r}_1) - \phi(\vec{r}_0)$$

$$\oint_V \vec{\nabla}\cdot\vec{J}\, d^3r = \oint_S \vec{J}\cdot d\vec{a}$$

$$\int_S (\vec{\nabla}\times\vec{J})\cdot d\vec{a} = \oint_C \vec{J}\cdot d\vec{l}$$

$\vec{\nabla}\times\vec{J} \equiv$ "Curl"



$\vec{\nabla}\times\vec{J}$ is the circulation density

$$\oint_C \vec{J}(\vec{r})\cdot d\vec{l}$$

- Tiny element of surface "one can show" $\oint_C d\vec{l}\cdot\vec{J} = \vec{\nabla}\times J\cdot d\vec{a}$

Illustrate with example

Divergence: Consider spherically symmetric field

$\vec{V}(\vec{r}) = V(r)\hat{r}$ Calculate $\oint_S \vec{V}(\vec{r})\cdot d\vec{a}$ sphere of radius R

assume $V(r) = A\, r^n$

$$\vec{\nabla}\cdot\vec{V}(\vec{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}(Ar^{n+2}) = \frac{1}{r^2}(n+2)(Ar^{n+1})$$
$$= A(n+2)\, r^{n-1}$$

$$\left.\begin{array}{l} V(R)\cdot 4\pi R^2 \\ AR^n\, 4\pi R^2 \end{array}\right\} 4\pi A R^{n+2}$$

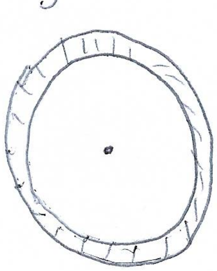
$n = -2$ is a problem

Calculate $\int_V \vec{\nabla}\cdot\vec{v}\,d^3r = \int_R d^3r \frac{1}{r^2} A(n+2)\, r^{n+1}$

$$= 4\pi A(n+2)\int_0^R dr\, \frac{r^3}{r^2}\, r^{n+1} = 4\pi A(n+2)\frac{R^{n+2}}{n+2} = 4\pi A R^{n+2}$$

Integral over sphere centered on origin is nonzero, for $n=-2 \rightarrow 4\pi A$

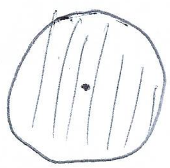
Integrate over volume not containing the origin



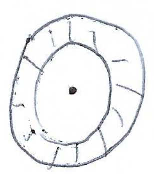
$$d\Phi = 4\pi\left[(R+dR)^2\, v(R+dR) - R^2 v(R)\right]$$
$$= 4\pi R\, v(R)\Big[\underbrace{\frac{d\ln v(R)}{d\ln R}}_{n} + 2\Big]dR$$

Now flux vanishes for $n=-2$

$n=-2$



Flux $= 4\pi A$



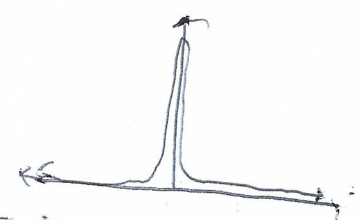
Flux $=0$

This means that

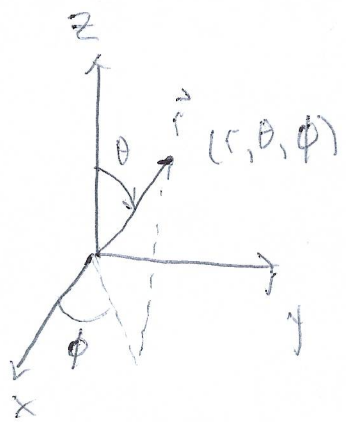
$$\lim_{R\to 0}\int \vec{\nabla}\cdot \frac{A}{r^2}\hat{r} = 4\pi A$$

This is the property of 3D Dirac delta function $\delta(\vec{r})$

$$\int_V d^3r\, \delta(\vec{r}) = 1$$



$\delta(\vec{r})$ is 3D version of this



These units vectors are functions of $\vec{r}$!

As opposed to $\hat{x}, \hat{y}, \hat{z}$

Simple example is $\vec{v}(\vec{r}) = v(r)\hat{r}$ of why you would use curvilinear coordinates

In cartesian coordinates $\vec{v}(\vec{r}) = v(\sqrt{x^2+y^2+z^2})\,\dfrac{(x\hat{x}+y\hat{y}+z\hat{z})}{(x^2+y^2+z^2)^{1/2}}$

Return to E and M!

Statics

$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{B} \cong \mu_0\vec{J} \qquad \vec{\nabla}\times\vec{E} \cong \vec{0}$

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

Electrostatics $\Rightarrow \vec{J} \to 0$

Three approaches

1. Superposition of point charges
2. For certain very symmetric charge distributions $\rho(\vec{r})$ "Gauss' Law"
3. in general introduce scalar potential
   statics $\vec{E}(\vec{r}) = -\nabla V(\vec{r})$
   $\nabla^2 V(\vec{r}) = -\rho/\epsilon_0$ Poisson's equation

① Electric field of point charges

Apply to "point charge"

Charge density of electron at origin $\rho(\vec{r}) = -e\delta(\vec{r})$

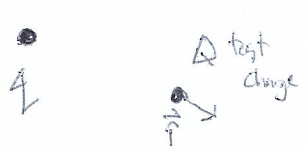
or charge q $\rho(\vec{r}) = q\delta(\vec{r})$

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{q\delta(\vec{r})}{\epsilon_0}$$

Recall if $\vec{v}(\vec{r}) = \frac{A\hat{r}}{r^2} \iff \vec{\nabla}\cdot\vec{v}(\vec{r}) = 4\pi A\,\delta(\vec{r})$

Mapping $4\pi A \leftrightarrow q/\epsilon_0$ or $A = \frac{q}{4\pi\epsilon_0}$

therefore $\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}$



Force on test charge Q

$$\vec{F} = Q\cdot\frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

Approximation in static case

$\boxed{\text{Force acts along vector between charges and varies as } \frac{1}{r^2} \text{ Coulomb's Law}}$

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Review last time

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \rho(\vec{r})/\epsilon_0$$

point charge at origin

$\rho(\vec{r}) = q\,\delta(\vec{r})$ therefore $\vec{E}(\vec{r}) = \frac{q}{\epsilon_0} \frac{\hat{r}}{r^2}$

$$\boxed{\text{Lorentz Force Law } \vec{F} = Q\vec{E}}$$

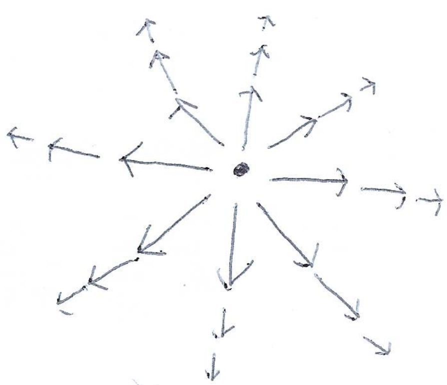
$\cdot\, Q$

q

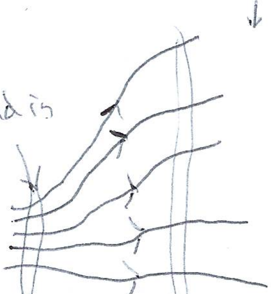
Coulomb's Law

<u>Field Lines</u>

Visualize vector field



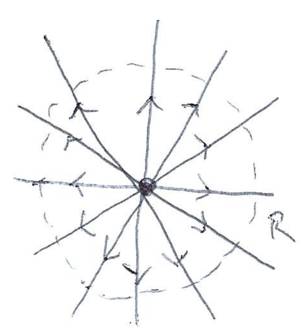
Electric field is large



Electric field is small

Field line for point charge

choose N to emerge from charge Q



# of field lines crossing there is N

area of sphere is $4\pi R^2$

Areal density of field lines

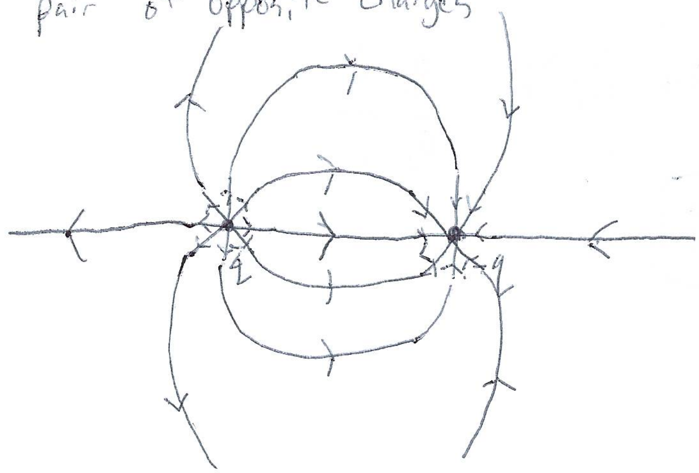
is $\frac{N}{4\pi R^2} \propto E(R)$

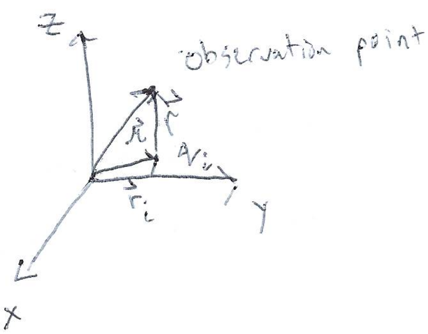
Rules for drawing field lines

1) Lines end on charges, or at ∞

2) Lines don't cross

Sketch pair of opposite charges





Continuous distributions of charge $\rho(\vec{r})$:

$\vec{r}_i \to \vec{r}\,'$

Then $\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}_i$, or $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

Point charge at $\vec{r}_i$ with charge q_i

$$\vec{E}(\vec{r}) = \frac{q_i}{\epsilon_0}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{q_i(\vec{r}-\vec{r}_i)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}_i|^3}$$

For many charges

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum q_i \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\rho(\vec{r}) = \langle \rho(\vec{r}) \rangle_{\substack{\text{time} \\ \text{space}}}$$

$$\rho(\vec{r}) = \left\langle \sum_i q_i \delta(\vec{r} - \vec{r}_i(t)) \right\rangle$$

$$\rho(\vec{r}) = \frac{1}{\text{Volume}} \int_{\text{Volume}} \rho(\vec{r})\, dV$$

$$= \frac{1}{\text{volume}} \int_{\text{volume}} dV \sum_i q_i \delta(\vec{r} - \vec{r}_i)$$

if $\vec{r}_i$ is inside volume, you get q

if $\vec{r}_i$ is outside volume, you get 0

$$\rho(\vec{r}) = \frac{Q_{inside}}{\text{Volume}}$$

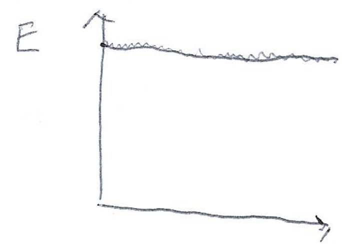
Continuous charge distribution

Positions of charges may fluctuate

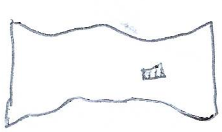
dV Avg density remains close to dN/dV

fluctuations small

measure field far away



Surface charge

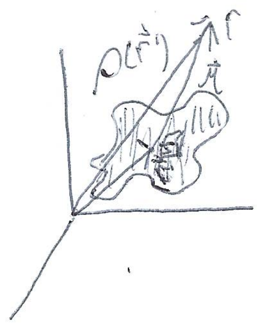


$$\sigma(\vec{r}) = \frac{Q_{inside}}{\text{area}}$$

Line charge

$$\lambda(\vec{r}) = \frac{Q_{inside}}{\text{length}}$$

Adapt $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i q_i \frac{\hat{\mathscr{r}}_i}{\mathscr{r}_i^2}$ for continuous charge distribution



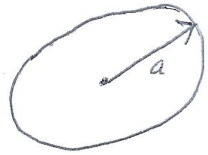
Now $\sum_i \Rightarrow \int d^3r$

$$\sum_i q_i \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \Rightarrow \int d^3r'\,\rho(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

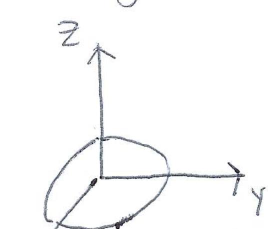
Remember $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

Homework

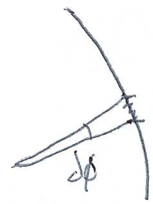
Consider electric field of ring of charge



charge per unit length = λ



$dq = \lambda\, dl$



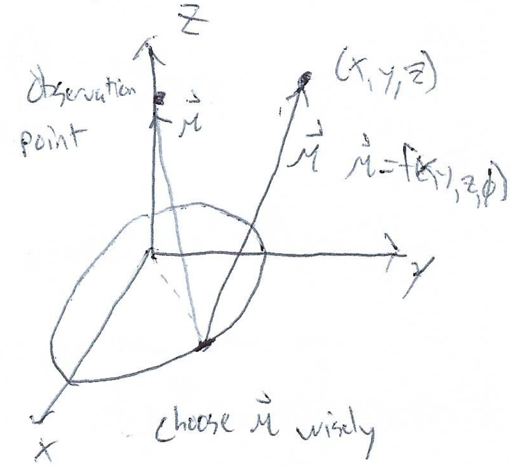
$dq = \lambda dl = \lambda a\, d\phi$

In principle $\rho(\vec{r}) \propto \delta(r-a)\,\delta(\theta - \pi/2)$

Mathematician way, not physicist

3D integral collapses to 1D integral

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} dq\,\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} a\lambda\, d\phi\,\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$



choose $\vec{\mathscr{r}}$ wisely

$\mathscr{r}^2$ is just pythagoras

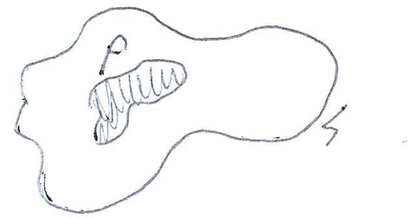
consider the symmetry

Gauss' Law

Fundamental Theorem vector field $\vec{v}(\vec{r})$

$$\int_V d^3r\, \vec{\nabla}\cdot\vec{v}(\vec{r}) = \oint_S \vec{v}(\vec{r})\cdot d\vec{a}$$

Recall $\vec{\nabla}\cdot\vec{E} = \rho(\vec{r})/\epsilon_0$



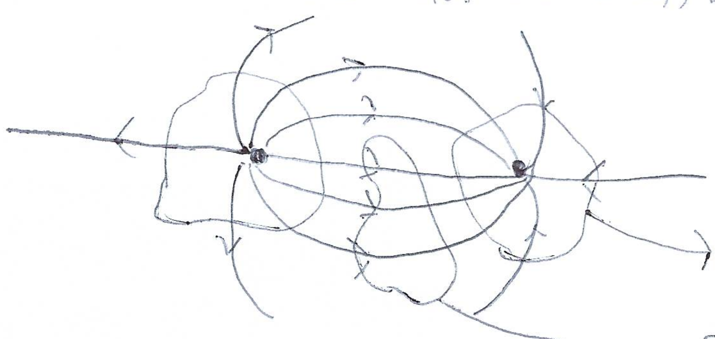
$$\int_V \vec{\nabla}\cdot\vec{E}(\vec{r})\, d^3r = \oint_S \vec{E}(\vec{r})\cdot d\vec{a}$$

Plug in $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$

$$\underbrace{\frac{1}{\epsilon_0}\int d^3r\, \rho(\vec{r})}_{\text{Charge inside}} = \underbrace{\oint_S \vec{E}(\vec{r})\cdot d\vec{a}}_{\text{Flux}}$$

Flux through the closed surface equals Q_{inside}/ϵ_0

Connection between field lines and Gauss' Law



Flux $\propto$ number of field lines passing through surface

Flux $\propto N$

Flux is 0

Use Gauss' Law to find $\vec{E}$

1) Spherical symmetry $\rho(r)$
2) Cylindrical symmetry $\rho(s)$
3) "Planar" symmetry $\rho(z)$

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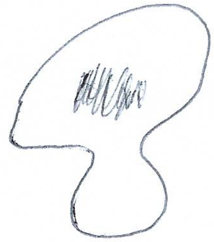
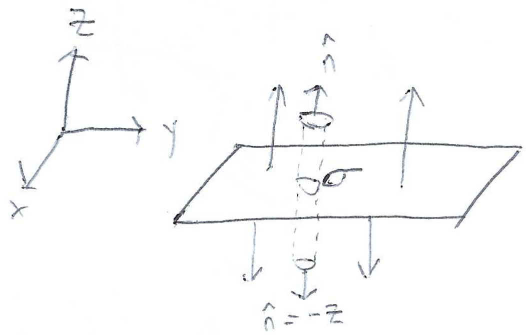
Gauss' Law $\quad \text{Flux} = \dfrac{Q_{in}}{\epsilon_0}$



Planar sheet of charge
uniform charge distribution



$\vec{E}$ is parallel to $\hat{z}$
due to symmetry
$E(z) = -E(z) = -E(-z)$

$$\text{Flux}\left[\vec{E}(z)\cdot\hat{n}(z) + \vec{E}(z)\cdot\hat{n}(-z)\right]A$$

$$\cancel{E(z) + E(-z)}$$

$$\left[\vec{E}(z)\cdot\hat{z} + \vec{E}(-z)\cdot(-\hat{z})\right]A$$

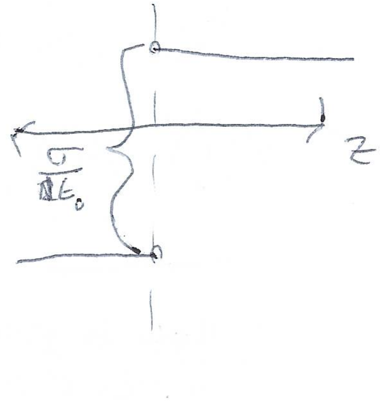
$$\left[\vec{E}(z) - \vec{E}(-z)\right]A$$

$$2\vec{E}(z) = A$$

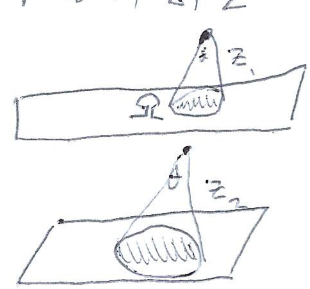
$$2E(z)A = \frac{\sigma A}{\epsilon_0}$$

$$E(z) = \frac{\sigma}{2\epsilon_0}$$

independent of z



Physical picture for E
independent of z



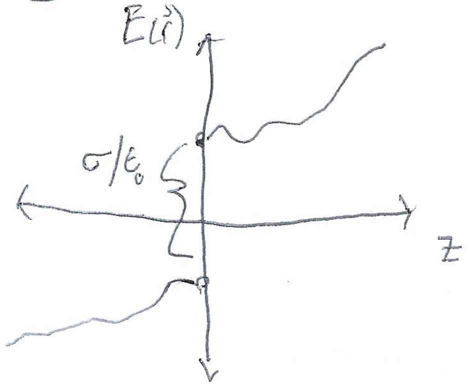
$Q = \sigma A_\Omega$
$= \sigma \Omega z^2$

Define E_Ω to be electric field
from charges in Ω

$E_\Omega \propto \dfrac{Q_\Omega z^2}{z^2}$ Amount of charge in solid angle

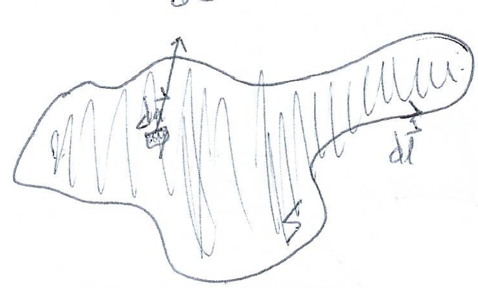
Curved Surface with $\sigma(\vec{r})$

As we approach surface result approaches uniform sheet of charge

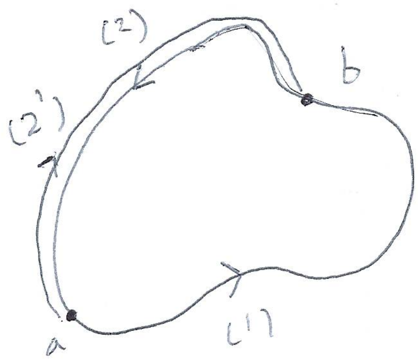


Concept of electrostatic Potential

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$$



2D contour



Fundamental Theorem

$$\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = \oint_C \vec{E}\cdot d\vec{l} \Rightarrow 0$$

$$\oint_C \vec{E}\cdot d\vec{l} = \int_{(1)} \vec{E}\cdot d\vec{l} + \int_{(2)} \vec{E}\cdot d\vec{l}$$

$$\int_{(1)} \vec{E}\cdot d\vec{l} = -\int_{(2)} \vec{E}\cdot d\vec{l}$$

$$\int_{(2')} \vec{E}\cdot d\vec{l} = -\int_{(2)} \vec{E}\cdot d\vec{l}$$

$$\int_{(1)} \vec{E}\cdot d\vec{l} = \int_{(2')} \vec{E}\cdot d\vec{l}$$

If $\vec{\nabla}\times\vec{E}(\vec{r}) = 0$

then the path integral between two points is independent of path

This means we can associate a scalar field with $\vec{E}(\vec{r})$ defined such that

$$-\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}(\vec{r})\cdot d\vec{l} = \underbrace{V(\vec{r}_b) - V(\vec{r}_a)}_{\text{Electrostatic potential}}$$

Recall 1D Fundamental Theorem

$$\int_{\vec{r}_a}^{\vec{r}_b} \vec{\nabla} V(\vec{r}) \cdot d\vec{l} = V(\vec{r}_b - \vec{r}_a) = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}(\vec{r}) \cdot d\vec{l}$$

$$\vec{\nabla} V(\vec{r}) = -E(\vec{r})$$

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

$$V(\vec{r}) + C \rightarrow V'(\vec{r})$$

Gauge transformation

$$\vec{\nabla} V(\vec{r}) = \vec{\nabla} V'(\vec{r})$$

In practice, choose $V=0$ at some point

choose $V(\vec{r}=\infty)=0$

find $V(\vec{r})$

Poisson's Equation

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

∇^2 scalar differential operator

$$\cancel{\vec{\nabla}(\vec{\nabla} \cdot \vec{E}} \quad \vec{E} = -\vec{\nabla} V$$

Cartesian coordinates

$$\vec{\nabla} \cdot \left\{-\vec{\nabla} V(\vec{r})\right\} = \rho/\epsilon_0$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\boxed{\nabla^2 V(\vec{r}) = -\rho/\epsilon_0}$$

In reality we don't look for ρ

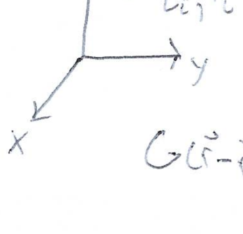
Superposition

$$\vec{\nabla} V_{pt}(\vec{r}) = -\frac{q}{\epsilon_0} \delta(\vec{r})$$

Math solve for Green function

$$\nabla^2 G(\vec{r}) = -\delta(\vec{r})$$

$$\text{So} \quad V_{pt}(\vec{r}) = \frac{q}{\epsilon_0} G(\vec{r})$$



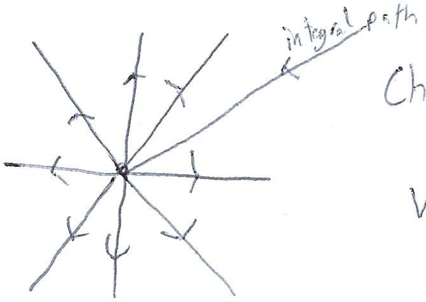
In general $V(\vec{r}) = \frac{1}{\epsilon_0} \sum_i q_i G(\vec{r} - \vec{r}_i)$

$G(\vec{r} - \vec{r}_i)$

Continuous charge distribution

$$V(\vec{r}) = \frac{1}{\epsilon_0}\int d^3r' \, \rho(\vec{r}\,') \, G(\vec{r}-\vec{r}\,')$$

Potential of point charge



Choose $V(r=\infty)=0$

$$V(\vec{r}_b) - V(\vec{r}_a) = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}\cdot d\vec{l}$$

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E}\cdot d\vec{l} \rightarrow -\int_{\infty}^{r} \frac{q}{4\pi\epsilon_0 r^2}\,dr = \frac{q}{4\pi\epsilon_0 r}$$

$$G(\vec{r}) = \frac{1}{4\pi r}$$

$$-\nabla V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r^2}\hat{r} = \vec{E}(\vec{r})$$

We solved Poisson Equation

Discrete

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\vec{r}-\vec{r}_i|}$$

Continuous $\rho(\vec{r})$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

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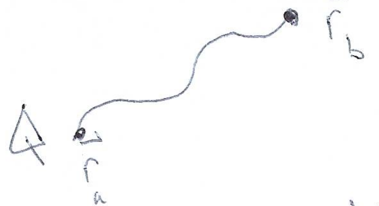
General solution to Poisson's Equation

$$\nabla^2 V(\vec{r}) = -\rho(\vec{r})/\epsilon_0$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

Potential and relationship to work and energy

Assume $\vec{E}(\vec{r})$



How much work?

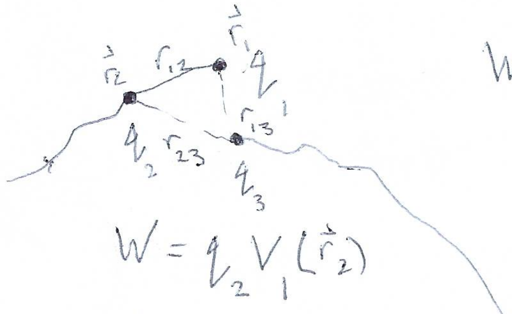
$$\vec{F} = -Q\vec{E}(\vec{r})$$

$$W = -Q\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}(\vec{r})\cdot d\vec{l}$$

$W = Q(V(\vec{r}_b) - V(\vec{r}_a))$ independent of path

potential difference is work per unit charge to move from $a \to b$

Work done to assemble a collection of point charges



$$W_2 = \frac{q_2 q_1}{4\pi\epsilon_0 r_{12}}$$

$$W = q_2 V_1(\vec{r}_2)$$

where $V_1 = \dfrac{q_1}{4\pi\epsilon_0 r_{12}}$

$$W_3 = W_2 + \frac{q_3 q_2}{4\pi\epsilon_0 r_{23}} + \frac{q_3 q_1}{4\pi\epsilon_0 r_{13}}$$

$$W_4 = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}}\right]$$

For n charges

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i \sum_{j>i}^{n} \frac{q_j}{r_{ij}}$$

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i \sum_{j>i}\frac{q_j}{r_{ij}} = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}\frac{q_i q_j}{r_{ij}} = \frac{1}{8\pi\epsilon_0}\sum q_i \sum_{i\neq j}\frac{q_j}{r_{ij}}$$

$$= \frac{1}{8\pi\epsilon_0}\sum_i q_i \sum_{i\neq j}\frac{q_j}{r_{ij}} = \frac{1}{2}\sum_i q_i \underbrace{\sum_{j\neq i}\frac{1}{4\pi\epsilon_0}\frac{q_j}{r_{ij}}}_{\text{Potential due to all charges not including ith charge}}$$

$$\boxed{W = \frac{1}{2}\sum_i q_i V(\vec{r}_i)}$$

Continuous charge distribution

$$\sum_i q_i \to \int d^3r' \rho(\vec{r}')$$

$$W = \frac{1}{2}\int d^3r\, \rho(\vec{r}) V(\vec{r})$$

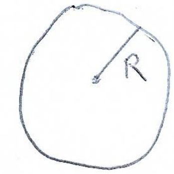
Rewrite this in a cool way using integration by parts

Consider $\cancel{\vec{\nabla}\cdot E\, V(\vec{r})\vec{E}(\vec{r})}$ $\vec{\nabla}\cdot[V(\vec{r})\vec{E}(\vec{r})]$

$$= V(\vec{r})\vec{\nabla}\cdot\vec{E}(\vec{r}) + \vec{E}(\vec{r})\cdot\vec{\nabla}V(\vec{r})$$

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \rho(\vec{r})/\epsilon_0$$
$$\vec{\nabla}V(\vec{r}) = -\vec{E}(\vec{r})$$

$$\int d^3r \left[\vec{\nabla}\cdot[V(\vec{r})\vec{E}(\vec{r})] = \frac{V(\vec{r})\rho(\vec{r})}{\epsilon_0} - E^2(\vec{r})\right]$$



$$\int d^3r \left[\vec{\nabla}\cdot V(\vec{r})\vec{E}(\vec{r})\right] = \frac{2W}{\epsilon_0} - \int d^3r\, E^2(\vec{r})$$

$\Downarrow$ Fundamental theorem

$$\oint_S d\vec{a}\cdot[V(\vec{r})\vec{E}(\vec{r})] \xrightarrow[\text{limit that } S\to\infty]{} 0$$

Argument Consider $R \to \infty$

the area of $S \propto R^2$

$V \propto 1/R$

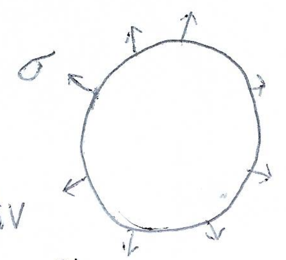
$E \propto 1/R^2$

Product (Area)(V)(E) $\propto 1/R \to 0$

$W \equiv$ Energy stored

$$\boxed{W = \frac{\epsilon_0}{2}\int d^3r\, E^2(\vec{r})}$$

Spherical sheet of charge Calculate stored energy



$dv = 4\pi r^2 dr$

$$\vec{E}(\vec{r}) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & \text{outside} \\ 0 & \text{inside} \end{cases} \Rightarrow \vec{E}\cdot\vec{E} = \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2$$

$$W = \frac{\epsilon_0}{2}\int_R^\infty d^3r \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 = \frac{\epsilon_0}{2}\int_R^\infty 4\pi r^2 dr \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2$$

$$W\,[\text{or } \mathcal{E}] = \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0}\left(-\frac{1}{r}\right)\Big|_R^\infty = \frac{Q^2}{8\pi\epsilon_0 R}$$

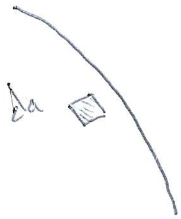
Wants to expand

$\cancel{\vec{F}(\vec{r})}$ $F(R)$ = Force $= -\frac{d\mathcal{E}}{dR}$

$F(R) = \frac{Q^2}{8\pi\epsilon_0 R^2}$; Surface tension $= F/A$

Notice that $Q = A\sigma$ so $F/A = \frac{A^2\sigma^2}{8\pi\epsilon_0 R^2 A} = \frac{\sigma^2}{2\epsilon_0}$ Surface tension

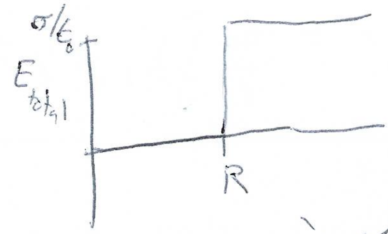
Look at this problem from the perspective of forces

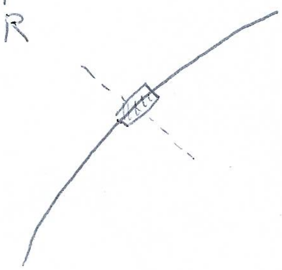


$$\vec{F} = \sigma \, da \, \vec{E}_{other}$$

$$\vec{E}_{total} = \vec{E}_{patch} + \vec{E}_{other}$$

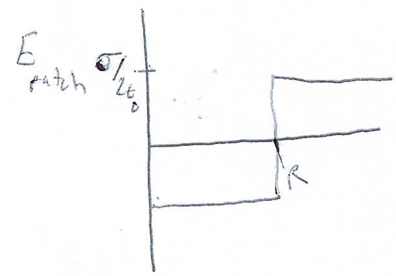
$$\vec{E}_{other} = \vec{E}_{total} - \vec{E}_{patch}$$





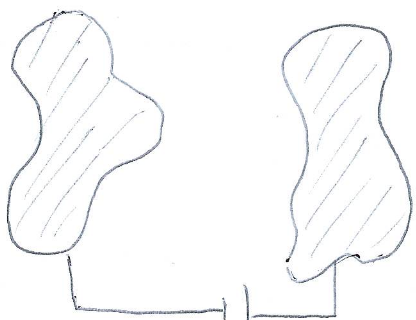
$$\vec{E}_{total} - \vec{E}_{patch} = \frac{\sigma}{2\epsilon_0} = \vec{E}_{other}$$

$\vec{E}_{other}$ is continuous!



$$F = \sigma \, da \frac{\sigma}{2\epsilon_0} \Rightarrow \frac{F}{da} = \frac{\sigma^2}{2\epsilon_0}$$

Conductors



V

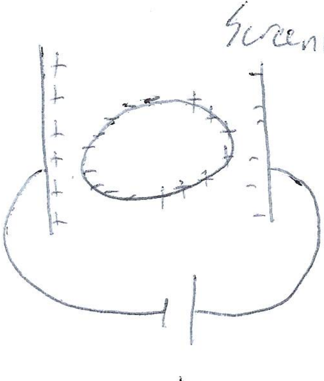
What is a conductor?

Basically σ = conductivity

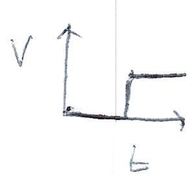
$\sigma \neq 0 \quad \sigma = [\Omega^{-1} L^{-1}]$

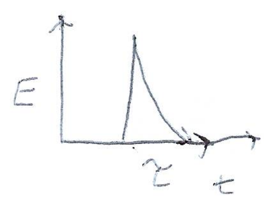
$J = \sigma E$

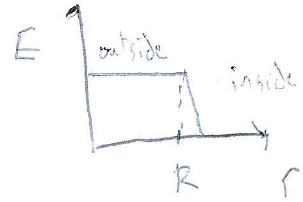
Screening $E = 0$ inside



V







Conductor has a certain length and time scale associated with it

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Conductor

$\sigma \neq 0$

$\sigma \to$ length scales } for screening
time scales }

Properties of ideal conductor

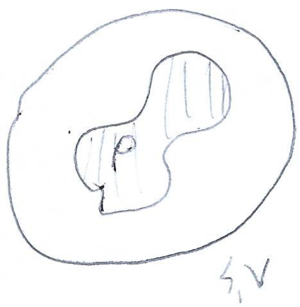
1) $\vec{E} = \vec{0}$ inside

2) $\vec{E}_{\text{outside}}$ is $\parallel$ to $\hat{n}$

3) conductor is an equipotential

How to understand $d\vec{E} = 0$

introduce conservation of charge



$$Q_{enc}(t) = \int_V d^3r \, \rho(\vec{r}, t)$$

Flux of charge through S

$$\text{Flux} = \oint_S \vec{J}(\vec{r}, t) \cdot d\vec{a}$$

$$\frac{d Q_{enc}(t)}{dt} = -\oint_S \vec{J}(\vec{r}, t) \cdot d\vec{a}$$

Convert to local form

Fund. Thrm.

$$\oint_S \vec{J} \cdot d\vec{a} = \int_V d^3r \, \vec{\nabla} \cdot \vec{J}$$

$$\underbrace{\int d^3r \, \frac{\partial}{\partial t} \rho(\vec{r}, t)}_{\frac{dQ}{dt}} = -\int d^3r \, \vec{\nabla} \cdot \vec{J}(\vec{r}, t)$$

Continuity eq

$$\frac{\partial}{\partial t} \rho(\vec{r}, t) = -\vec{\nabla} \cdot \vec{J}(\vec{r}, t)$$

Conservation of charge

$\vec{J}(\vec{r},t) \neq \sigma \vec{E}(\vec{r},t)$

This is already implicit in Maxwell Equations

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\left[\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right]$$

$$\frac{\partial}{\partial t}\left[\vec{\nabla}\cdot\vec{E}\right] = \frac{\partial\rho}{\epsilon_0\partial t} \qquad 0 = \mu\vec{\nabla}\cdot\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left[\vec{\nabla}\cdot\vec{E}\right]$$

$$\cancel{-\mu_0\vec{\nabla}\cdot\vec{J}} \qquad -c^2\mu_0\vec{\nabla}\cdot\vec{J} = \frac{1}{\epsilon_0}\frac{\partial\rho}{\partial t}$$

$$-\vec{\nabla}\cdot\vec{J} = \frac{\partial\rho}{\partial t} \qquad c^2 = \frac{1}{\mu_0\epsilon_0}$$

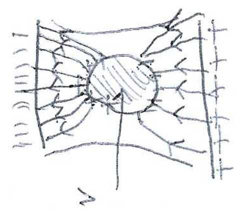
Static Regime

$\vec{\nabla}\cdot\vec{J} = 0$

but in a conductor $\vec{J} = \sigma\vec{E}$

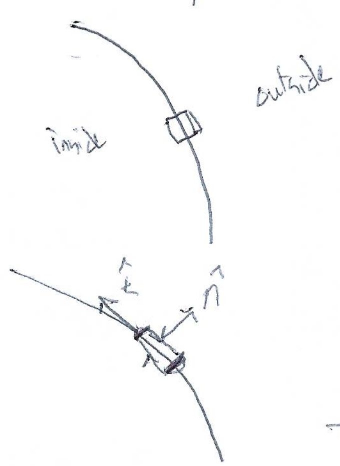
$$\left.\begin{array}{l}\therefore\ \vec{\nabla}\cdot\vec{E} = 0 \\ \text{also } \vec{\nabla}\times\vec{E} = 0\end{array}\right\} \vec{E}_{inside} = 0$$

Physical Terms



$\vec{E}=0$ inside

Boundary Conditions



Gauss Law

σ is charge density at surface

$$\frac{\sigma da}{\epsilon_0} = (\vec{E}_{outside} - \vec{E}_{inside})\cdot\hat{n}\,da$$

$$\vec{E}_{outside}\cdot\hat{n} = \frac{\sigma}{\epsilon_0}$$

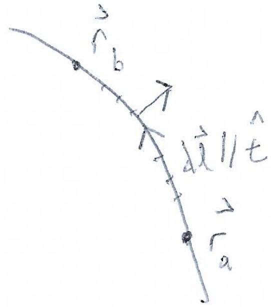
Tangential component

$$\oint_c \vec{E}\cdot d\vec{l} = \int d\vec{a}\cdot(\vec{\nabla}\times\vec{E}) = 0$$

$$(\vec{E}_{in} - \vec{E}_{out})\cdot\hat{t} = 0$$

$$\vec{E}_{out}\cdot\hat{t} = 0$$

Boundary Conditions

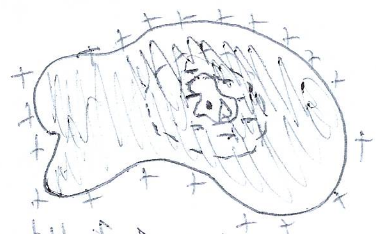


$$V(\vec{r}_b) - V(\vec{r}_a) = \int_{\vec{r}_a}^{\vec{r}_b} \vec{E} \cdot d\vec{l}$$

but $\vec{E}$ is perpendicular to $d\vec{l}$

$$V(\vec{r}_b) = V(\vec{r}_a)$$

Interesting Properties of multiply connected conductors

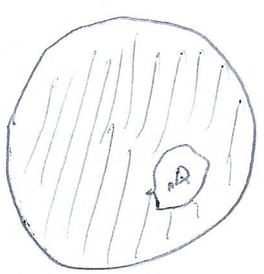


$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \equiv 0$$

can we tell if Q is there?

if Q is positive, compensating surface charge on cavity wall

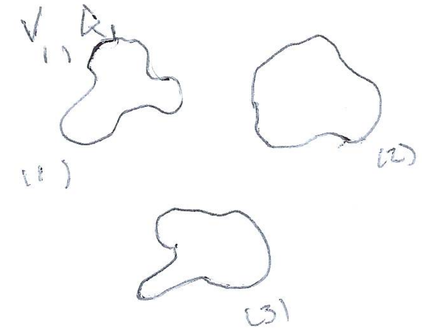
Positive surface charge at outer surface with total charge Q



Outside doesn't know cavity is oddly shaped or not centered

Capacitance matrix

array of conductors



Uniqueness of $V(\vec{r})$

$$V(\vec{r}) = V(Q_1, Q_2, Q_3, \ldots)$$

linearity $V_1 = P_{11} Q_1 + P_{12} Q_{12} + P_{13} Q_{13} + \ldots$

$$V_i = \sum_j P_{ij} Q_j$$

P_{ij} = coefficients of the potential

or $\vec{V} = \overleftrightarrow{P} \vec{Q}$

$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ \vdots \end{bmatrix}$$

Multiply by $\overleftrightarrow{P}^{-1}$

$$(\overleftrightarrow{P})^{-1} \vec{V} = \vec{Q}$$

$$Q_i = \sum_j C_{ij} V_j$$

$$(\overleftrightarrow{P})^{-1} \equiv \overleftrightarrow{C}$$

capacitance matrix

Stored energy $\mathcal{E}$

$$\mathcal{E} = \frac{1}{2} \int d^3 r \rho(\vec{r}) V(\vec{r})$$

Consider conductor 1

$$\mathcal{E}_1 = \frac{1}{2} \int_{(1)} d^3 r \rho V = \frac{V_1}{2} \int d^3 r \rho = \frac{Q V_1}{2}$$

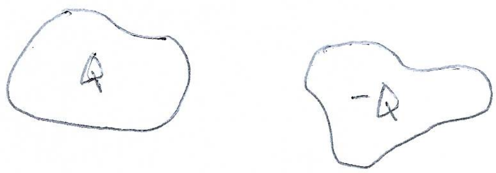
$$\mathcal{E}_1 = \frac{V_1}{2} \left(C_{11} V_1 + C_{12} V_2 + C_{13} V_3 + \ldots \right)$$

$$\mathcal{E} = \frac{1}{2} \sum_{ij} C_{ij} V_i V_j$$

C_{ij} is symmetric

$$C_{ij} = C_{ji}$$

Capacitance system of 2 conductors



$$V_1 = P_{11} Q_1 + P_{12} Q_2$$
$$= (P_{11} - P_{12}) Q$$
$$V_2 = \cancel{P_{22}} (P_{21} - P_{22}) Q$$
$$= (P_{12} - P_{22}) Q$$

Potential difference

$$V_2 - V_1 = (2P_{12} - (P_{22} + P_{11}))Q = V$$

$$C \equiv \frac{Q}{V} = \frac{1}{\cancel{2P_{12} - (}} = \frac{1}{(P_{11} + P_{22}) - 2P_{12}}$$

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$$\nabla^2 V(\vec{r}) = 0$$

Aside

$$\nabla^2 V = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$

$$Z(z) = e^{ikz}$$

$$V = e^{ikz} e^{-i\omega t}$$

$$\nabla^2 V = -\frac{\omega^2}{c^2} V$$

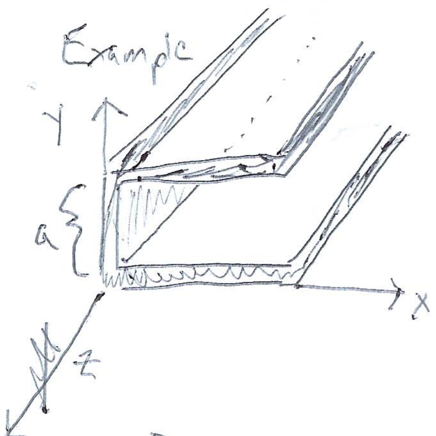
Laplace is special

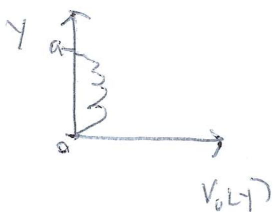
$\omega = 0$

Conclusion $C > 0$ exponential solutions

$C < 0$ sinusoidal solutions

Example





Boundary conditions

1) Two grounded metal plates

2) $V_0(y) = 0$ at $y = 0, a$

3) $V \to 0$ as $x \to \infty$

Translational symmetry along z

$Z(z) = \text{constant}$ (only for Laplace Equation)

Therefore solutions have the form

$$V(x,y) = X(x)\,Y(y)$$

$$C_x + C_y = 0$$

C_x should be positive due to boundary. likewise C_y should be negative

From boundary (3) $V \to 0$ as $x \to \infty$

$$X(x) = Ae^{-k_x x} \quad \text{where } k = \pm\sqrt{C_x}$$

$$C_y = -C_x$$

$$Y(y) = B\sin k_y y + C\cos k_y y$$

$$k_x = k_y \equiv k$$

Solution that satisfies bc #3

$$V(x,y) = (B\sin ky + C\cos ky)(e^{-kx}) \quad \text{True for all } k$$

Because $V(x,0) = \cancel{V_0}\ 0$

$$V(x,0) = \cancel{C\cos k}\ C\cos ky\, e^{-kx} = 0$$

$$C = 0$$

$$V(x,y) = B \sin(ky) e^{-kx}$$

Now $V(x,a) = 0$

$$V(x,a) = B\sin(ka)e^{-kx} = 0$$

$$\sin(ka) = 0$$

therefore we have $k_n a = n\pi$

$$V(x,y) = B_n \sin(k_n y) e^{-k_n x}$$

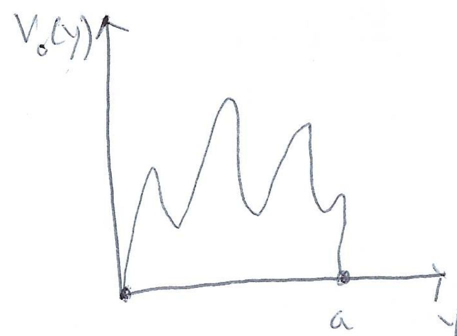
General Solution

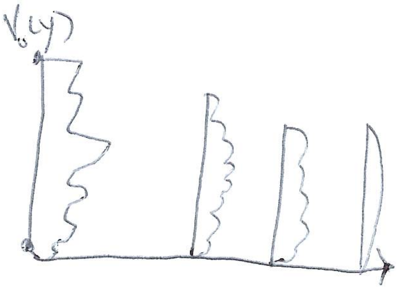
$$V(x,y) = \sum_n B_n \sin(k_n y) e^{-k_n x}$$

$$= \sum_n B_n \sin(k_n y) e^{-k_n x}$$

$$V(0,y) = V(y)$$

$$V(y) = \sum_n B_n \sin\frac{n\pi y}{a}$$ Fourier Series





Orthogonality

$$\int_0^a dy \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right) = \frac{a}{2}\delta_{nn'}$$

$$\delta_{nn'} = \begin{cases} 1 & \text{if } n = n' \\ 0 & \text{if } n \neq n' \end{cases}$$

$$\sin\frac{n'\pi}{a}y\left[V_0(y) = \sum_n B_n \sin\frac{n\pi y}{a}\right]$$

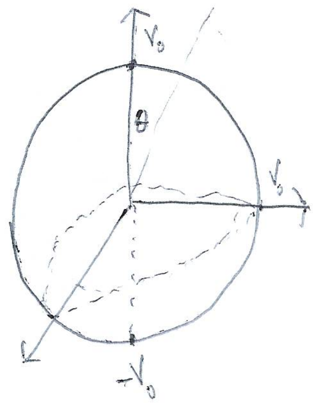
$$\int_0^a dy\, V_0(y) \sin\frac{n'\pi y}{a} = \sum_n B_n \int dy \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)$$

$$\int_0^a dy\, V_0(y)\sin\left(\frac{n'\pi y}{a}\right) = B_{n'}\frac{a}{2}$$

$$V(x,y) = \sum_n B_n \sin\left(\frac{n\pi y}{a}\right) e^{-\frac{n\pi x}{a}}$$

Finally $\boxed{B_n = \frac{2}{a}\int_0^a dy\, V_0(y) \sin\left(\frac{n\pi y}{a}\right)}$

Solve $\nabla^2 V(\vec{r})$ in polar coordinates



specify V on a sphere $V(r,\theta,\phi)\Big|_{r=R} = V_0\cos\theta$

of radius R

$V \to 0$ as $r \to \infty$

V depends only on latitude

Separation of variables

$$V(\vec{r}) = R(r)\,\Theta(\theta)\,\cancel{\Phi(\phi)}$$ no dependence

write out ∇^2 in polar coordinates

$$\Rightarrow \frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = C$$

$$\Rightarrow \frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -C$$

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Laplace in polar coordinates

$$V(r,\theta) = R(r)\,\Theta(\theta)$$

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \text{constant}$$

$$\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \text{constant}$$

Write constant as $l(l+1)$

Try $R(r) \propto r^j$

$$\frac{1}{r^j}\frac{d}{dr}\left(r^2 j r^{j-1}\right) = l(l+1)$$

$$j(j+1)r^j = r^j\, l(l+1)$$

if $j = l$ this is satisfied

$j = -(l+1)$

General solution to radial equation is

$$R(r) = A_l r^l + \cancel{B_l r^l} + \frac{B_l}{r^{l+1}} = A_l r^l + \frac{B_l}{r^{l+1}}$$

Polar equation ~~is~~ is simplified by $x = \cos\theta$

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx}$$

$$\frac{d}{dx}\left((1-x^2)\frac{d\Theta}{dx}\right) + l(l+1)\Theta(x) = 0$$

Trial solution $P(x) = x^a \sum_j a_j x^j$

For $x=1$ (North pole), $P(1)$ diverges unless $l=0$ or positive integer, in which case the series terminates

Legendre Polynomials are solutions $P_l(x)$

Some of the polynomials

Even $l \Rightarrow$ even powers

Odd $l \Rightarrow$ odd powers

$P_0(x) = 1$

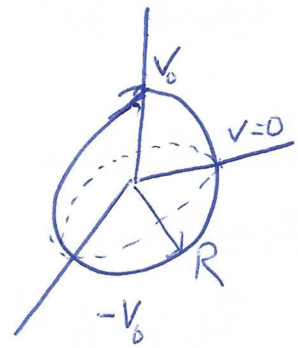
$P_1(x) = x = \cos\theta$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$\vdots$

General solution $V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$



"Starter problem"

(1) Boundary condition $V(R,\theta) = V_0 \cos\theta$

(2) V is finite inside (no charges)

(3) $V \to 0$ as $r \to \infty$

Inside sphere $r < R$

$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ since $\frac{B_l}{r^{l+1}}$ would blow up at $r=0$

$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0 \cos\theta$ Solve for A_l

The $P_l(\cos\theta)$ are $\left\{\begin{matrix}\text{complete} \\ \text{orthogonal}\end{matrix}\right\}$ functions on interval $-1 \le x \le 1$

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \begin{cases} 0 & \text{if } l \neq l' \\ \frac{2}{2l+1} & \text{if } l = l' \end{cases}$$

$\frac{2}{2l+1}$

$$P_{\ell'}(V(R,\theta)) = \left[\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = V_0 \cos\theta \right] P_{\ell'}$$

$$\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) \qquad \text{for boundary condition } V(R,x)$$

$$\int_{-1}^{1} dx\, P_{\ell'}(x) V(R,x) = \int_{-1}^{1} \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(x) \cdot P_{\ell'}(x)\, dx$$

$$= \int_{-1}^{1} dx\, P_{\ell'}(x) V(R,x) = A_{\ell'} R^{\ell'} \cdot \frac{2}{\ell'+1} \qquad \text{drop prime}$$

$$\boxed{\cancel{A_\ell = \left(\frac{\ell+1}{2}\right) \int_{-1}^{1} dx\, V(R,x) P_\ell(x)}} \qquad \boxed{A_\ell = \left(\frac{\ell+1}{2}\right)\left(\frac{1}{R^\ell}\right) \int_{-1}^{1} dx\, V(R,x) P_\ell(x)}$$

For our problem

$$V(R,x) = V_0 P_1(x)$$

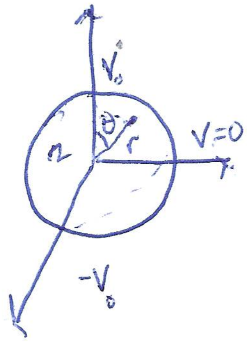
Only One non-vanishing term, $A_1 = V_0/R$

Solution for $r < R$ is just $\boxed{V(r,\theta) = \frac{V_0}{R} r \cos\theta}$

<u>Outside sphere</u> $V \to 0$, $r \to \infty$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad \text{Same steps...}$$

$$\boxed{V(r,\theta) = \frac{V_0 R^2}{r^2} \cos\theta}$$



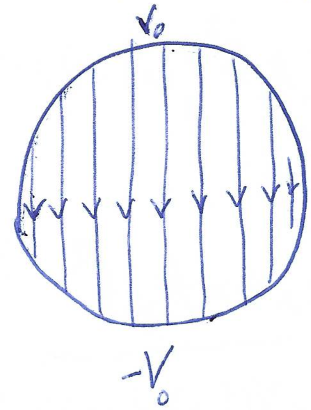
Look at $r<R$

Determine electric field $\qquad r\cos\theta = z$

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r}) \qquad V_{inside}(\vec{r}) = V_0 \frac{z}{R}$$

$$\vec{\nabla} V(\vec{r}) = \hat{z}\frac{\partial}{\partial z} V(z)$$

$$\boxed{\vec{E} = \frac{-V_0}{R}\hat{z}}$$

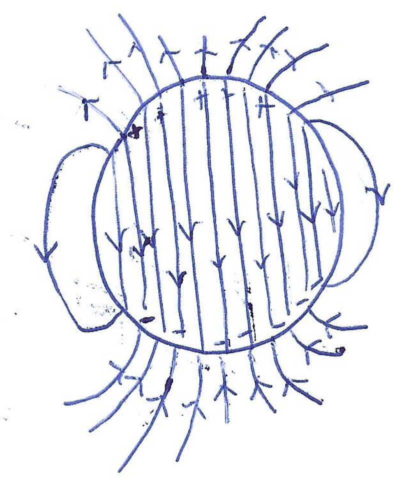


For $r>R$

$$V(r,\theta) = V_0 \frac{R^2}{r^2}\cos\theta \qquad \vec{E}(\vec{r}) = -\vec{\nabla} V(\hat{r},\theta)$$

$$\vec{E}(\vec{r}) = -\hat{r}\frac{\partial}{\partial r}V(r,\theta) - \hat{\theta}\frac{1}{r}\frac{\partial V(r,\theta)}{\partial\theta} = \frac{V_0 R^2}{r^3}\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right)$$

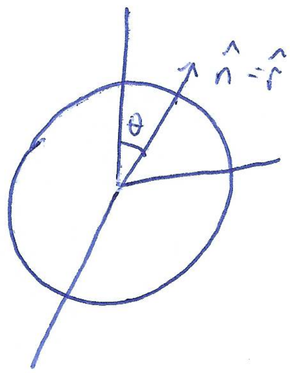
$$\vec{E}_{inside}(R,\theta) = \frac{-V_0}{R}\hat{z}, \quad \vec{E}_{outside}(R,\theta) = \frac{V_0}{R}\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right)$$



$\vec{E}$ is discontinuous at $r=R$

$\Rightarrow$ surface charge $\sigma(\theta)$

Gauss' Law $\Rightarrow \vec{E}_{out} \cdot \hat{n} - \vec{E}_{in} \cdot \hat{n} = \dfrac{\sigma}{\epsilon_0}$



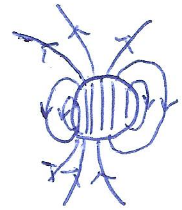
$$\vec{E}_{in} \cdot \hat{n} = -\frac{V_0}{R} \hat{z} \cdot \hat{r} = \frac{-V_0}{R} \cos\theta$$

$$\vec{E}_{out} \cdot \hat{n} = \frac{2V_0}{R} \cos\theta$$

$$\frac{\sigma}{\epsilon_0} = \frac{2V_0}{R}\cos\theta + \frac{V_0}{R}\cos\theta = \frac{3V_0}{R}\cos\theta$$

$$\sigma = \frac{3\epsilon_0 V_0 \cos\theta}{R}$$

Zoom out



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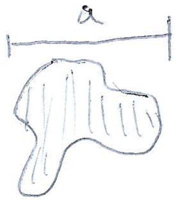
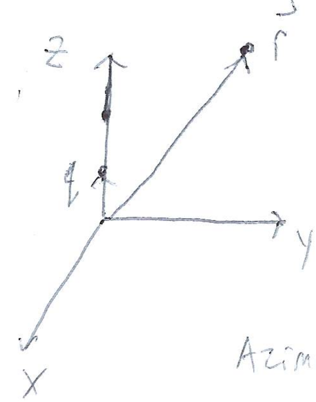
Mutipole Expansion



Localised $\rho(\vec{r})$

Develop a theory of the field for $\frac{a}{r} << 1$



1) Fields at large distances

2) Interactions with Fields

3) Interactions between molecules

Azimuthal symmetry

Charge is a position $z'\hat{z}$

Boundary condition $V \to 0$; $r \to \infty$

Solution has this form

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{B_l[\rho]}{r^{l+1}} P_l(\cos\theta)$$

But this just a point charge

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - z'\hat{z}|} = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{B_l[\rho]}{r^{l+1}} P_l(\cos\theta)$$

Works for all $\vec{r}$, including $\vec{r} = z\hat{z}$

$|\vec{r} - z'\hat{z}| \to z - z'$; $P_l(1) = 1$

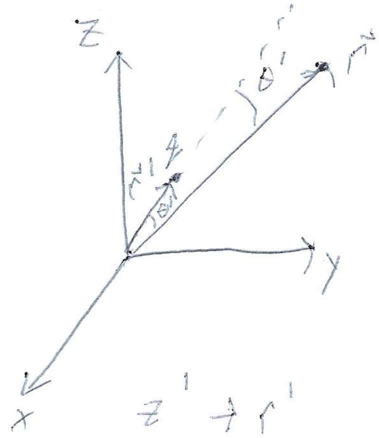
$$\frac{q}{z - z'} = \sum_{l=0}^{\infty} \frac{B_l}{z^{l+1}}$$

$$\frac{q}{z-z'} = \sum_{l=0}^{\infty} \frac{B_l}{z^{l+1}} \qquad \frac{1}{z-z'} = \frac{1}{z}\left(\frac{1}{1-z'/z}\right) = \frac{1}{z}\left[1+\frac{z'}{z}+\left(\frac{z'}{z}\right)^2+\cdots\right]$$

$$\frac{1}{z}\sum_{l=0}^{\infty} \frac{B_l}{z^l}, \quad \cancel{B_l = 1} \qquad B_l = z'^{\,l}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{q}{1}\sum_l \frac{z'^{\,l}}{r^{l+1}} P_l(\cos\theta)$$

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\, q\sum_l \frac{z'^{\,l}}{r^{l+1}} P_l(\cos\theta)}$$



$z' \to r'$

$\cos\theta \to \cos\theta' \to \hat{r}\cdot\hat{r}'$

Suppose point charge is not on z axis

Integrate $q\delta^3(\vec{r}-\vec{r}') \to \rho(\vec{r}')$

General multipole expansion for arbitrary $\rho(\vec{r}')$

$$\boxed{V(\vec{r}) = V(r,\theta) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int d^3r'\,\rho(\vec{r}')\, r'^{\,l}\, P_l(\overbrace{\cos\theta'}^{\hat{r}\cdot\hat{r}'})}$$

Amplitude weighted

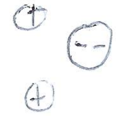
As $r \to \infty$, V converges to the lowest non-zero/non-vanishing term in the expansion

1. Monopole term $l=0$

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{r}\right)\underbrace{\int d^3r'\,\rho(\vec{r}')}_{\text{total charge}} = \frac{Q}{4\pi\epsilon_0 r}$$

$Q \equiv$ Monopole moment

② Dipole moment $\ell=1$ ex H_2O



$$V(r,\theta)=\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int d^3r'\rho(\vec{r}')\,r'\,\hat{r}'\cdot\hat{r}$$

$\hat{n} \equiv \hat{r}'$

$r'\hat{r}'=\vec{r}'$

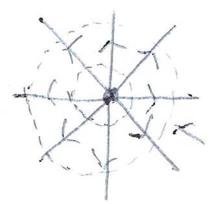
$$V(\vec{r})=\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot\underbrace{\int d^3r'\rho(\vec{r}')\,\vec{r}'}_{\text{dipole moment (vector!)} \equiv \vec{p}}$$

$$\boxed{V(\vec{r})=\frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}}$$

Electric fields

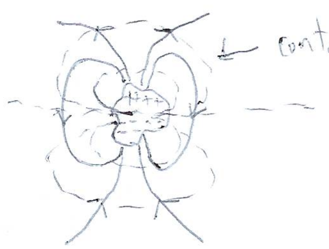
~~$\rho(\vec{r})$~~

$\ell=0$



contours of constant potential

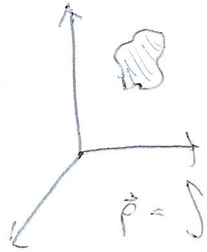
$\ell=1$



← contours of constant potential

Comments on Dipole moment

1)



Shift origin by $\vec{r}_0$

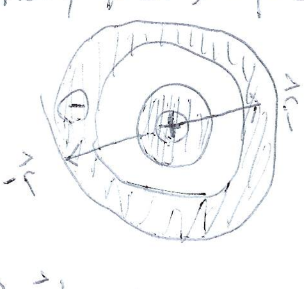
$$\vec{p}=\int d^3r'\rho(\vec{r}')\,\vec{r}'$$

shift origin $\int d^3r'\rho(\vec{r}')(\vec{r}'-\vec{r}_0)=\int d^3r'\rho(\vec{r}')-\overbrace{\int d^3r'\rho(\vec{r}')}^{\text{vanishes if } Q=0}\vec{r}_0$

① if $Q = 0$ then $\vec{p}$ is independent of choice of origin

② $\vec{p} \equiv 0$ if the localized charge has inversion symmetry

Inversion symmetry $\rho(\vec{r}) = \rho(-\vec{r})$



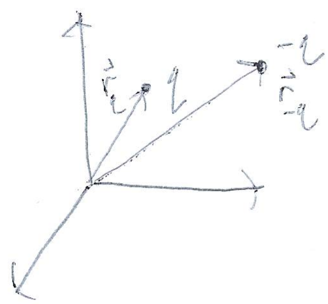
$\rho(\vec{r}) \neq \rho(-\vec{r})$

$$\int d^3r' \rho(\vec{r}')\, \vec{r}' \to 0$$

(even) (odd)

Exam 2 weeks
Thursday
week March 4th

Dipole moment of opposite point charges

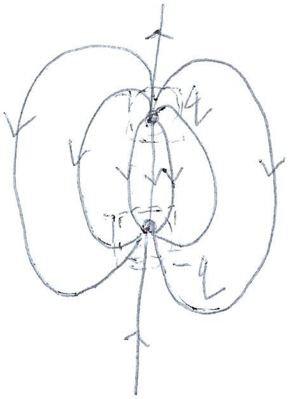


$$\vec{p} = \int d^3r'\, \vec{r}' \left[q\,\delta(\vec{r}' - \vec{r}_q) - q\,\delta(\vec{r}' - \vec{r}_{-q}) \right]$$

$$= q(\vec{r}_q - \vec{r}_{-q}) = q\vec{d} \quad \text{where } \vec{d} \equiv \vec{r}_q - \vec{r}_{-q}$$

$\boxed{\vec{p} = q\vec{d}}$ independent of choice of origin

Physical dipole

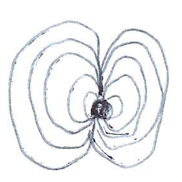


Asymptotically

$$V(r,\theta) \xrightarrow[r\to\infty]{} \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

Perfect ~~Dip~~ dipole

let $q \to \infty$, $d \to 0$ such that $qd \to$ constant p



$\frac{p\cos\theta}{4\pi\epsilon_0 r^2}$

everywhere

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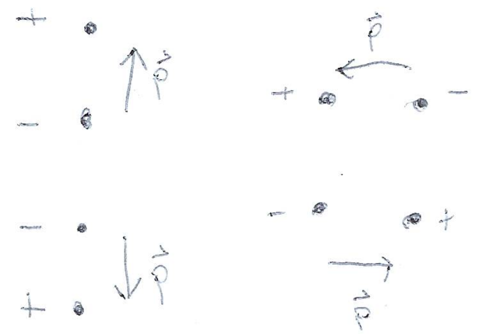
Multipole Expansion

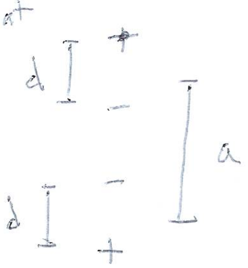
$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\sum_l \frac{1}{r^{l+1}}\int d^3r'\,\rho(\vec{r}')\,r'^l\,P_l(\hat{r}'\cdot\hat{r})$$

Dipole term $l=1$

Examples with lowest nonzero term is $l=2$



Look at



$\hat{r}' = \hat{z}$, $\hat{r}\cdot\hat{r}' = \cos\theta$

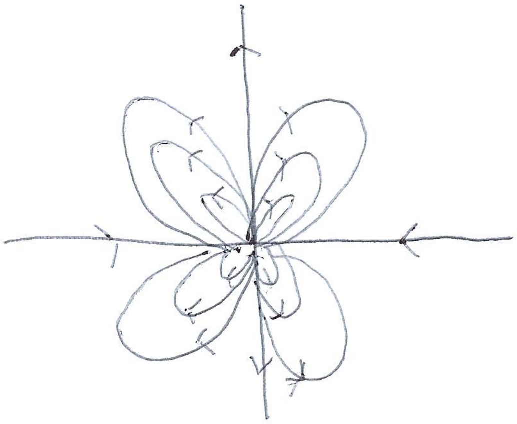
$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\,\frac{1}{2}(3\cos^2\theta - 1)\int d^3r'\,\rho(\vec{r}')\,r'^2$$

$\rho(\vec{r})$ is a sum of delta functions

$$\int d^3r'\,\rho(\vec{r}')\,r'^2 = 4pa \qquad \cancel{p \text{ is}} \quad p = qd$$

$$V(r,\theta) = \frac{pa}{2\pi\epsilon_0 r^3}(3\cos^2\theta - 1) \qquad \vec{E} = -\vec{\nabla}V$$

$$= -\hat{r}\frac{\partial V}{\partial r} - \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta}$$



Field of 2 antiparallel dipoles along $\hat{z}$ axis

$\ell = 2$

$$V(r,\theta) = \frac{1}{2\pi\epsilon_0 r^3}\int d^3r'\, r'^2 \rho(\vec{r}\,')\, \frac{1}{2}\left(3 \cancel{\cos\theta} (\hat{r}'\cdot\hat{r})^2 - 1\right)$$

Express in cartesian coordinates

$$(\hat{r}\cdot\hat{r}')^2 = \frac{1}{r^2 r'^2}\left(x^2x'^2 + y^2y'^2 + z^2z'^2 + 2xyx'y' + 2yzy'z' + 2zxz'x'\right)$$

write $1 = \dfrac{x^2+y^2+z^2}{r^2}$ consider term proportional to x^2

$$\frac{1}{2}\left(3\frac{x^2x'^2}{r^2r'^2} - \frac{x^2}{r^2}\right) = \frac{1}{2}\frac{x^2}{r^2}\left(3\frac{x'^2}{r'^2} - 1\right)$$

Also terms proportional to xy

$$2\,\frac{1}{2}\,3\,\frac{xy}{r^2}\left(\frac{x'y'}{r'^2}\right)$$

This shows that we can write V in the following way

$$V(r,\theta) = \frac{1}{2r^3}\sum_{i,j}\hat{r}_i\hat{r}_j Q_{ij}$$

Example $\hat{r}_1 = \frac{x}{r}$, $\hat{r}_2 = \frac{y}{r}$, $\hat{r}_3 = \frac{z}{r}$

Q_{ij} = elements of quadrupole tensor

$$Q_{ij} = \int d^3r'\left(3r'_i r'_j - r'^2\delta_{ij}\right)\rho(\vec{r}\,')$$

Multipole Expansion

$V(r,\theta) \propto$ Scalar + vector + quadrupole

[charge] [vector] [tensor]

$2\ell+1$ (1) s (3) p (5) d

Q_{ij} is symmetric, traceless tensor

independent elements 5

$+\ \cdot$

$-\ \cdot$

$-\ \cdot$

$+\ \cdot$

$$V(r,\theta) = \frac{1}{2r^3}\sum_{ij}\hat{r}_i\hat{r}_j Q_{ij}$$

$$Q_{zz} = \int d^3r'\,(3z'^2 - z'^2)\rho(\vec{r}') = 2\int d^3r'\,z'^2\rho(\vec{r}')$$

$$Q_{xx} = -\int d^3r'\,z'^2\rho(\vec{r}') = -\frac{1}{2}Q_{zz}$$

$$Q_{yy} = Q_{xx} = -\frac{1}{2}Q_{zz}$$

all other terms are zero

$$\overleftrightarrow{Q} = \begin{bmatrix} -Q_{zz}/2 & 0 & 0 \\ 0 & -Q_{zz}/2 & 0 \\ 0 & 0 & Q_{zz} \end{bmatrix}$$

$$V = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\left(\frac{z^2}{r^2}Q_{zz} - \frac{x^2}{r^2}\frac{Q_{zz}}{2} - \frac{y^2}{r^2}\frac{Q_{zz}}{2}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}Q_{zz}\left(\frac{z^2 - (x^2+y^2)/2}{r^2}\right)$$

$$\frac{z^2}{r^2} - \frac{x^2+y^2+z^2}{2r^2} + \frac{z^2}{2r^2} = \frac{3z^2}{2r^2} - \frac{1}{2} = \frac{3}{2}\cos^2\theta - \frac{1}{2} = \frac{1}{2}(3\cos^2\theta - 1)$$

$$V(r,\theta) = \frac{1}{2\pi\epsilon_0 r^3} \frac{Q_{zz}}{2} (3\cos^2\theta - 1)$$

or

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0 r^3} Q_{zz} P_2(\cos\theta)$$

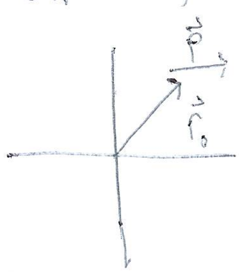
Force and torque on dipole

Start with energy

$$\mathcal{E} = \int d^3r\, \rho(\vec{r})\, V(\vec{r})$$

This ~~clear~~ is clearly zero if V is constant

Expand $V(\vec{r})$ at about $\vec{r}_0$, which is where the dipole is



$$V(\vec{r}-\vec{r}_0) \simeq V(\vec{r}_0) + (\vec{r}+\vec{r}_0)\cdot\vec{\nabla}V\big|_{\vec{r}_0}$$

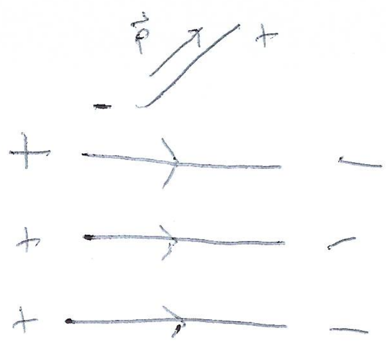
$$\mathcal{E} = \int d^3r\,\rho(\vec{r})\left(V(\vec{r}_0) + (\vec{r}+\vec{r}_0)\cdot\vec{\nabla}V(\vec{r}_0)\right)$$

$$= V(\vec{r}_0)\int d^3r\,\rho(\vec{r}) + \vec{\nabla}V(\vec{r}_0)\cdot\int d^3r\,(\vec{r}-\vec{r}_0)\,\rho(\vec{r})$$

Assuming 0 = charge

$$\mathcal{E} = \underbrace{\vec{\nabla}V(\vec{r}_0)}_{-\vec{E}}\cdot\underbrace{\int d^3r\,\vec{r}\rho(\vec{r})}_{\vec{p}}$$

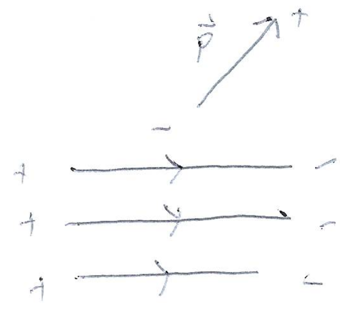
$$\mathcal{E} = -\vec{p}\cdot\vec{E}(\vec{r}_0)$$



Linear force

$$\vec{F} = -\vec{\nabla}\,\mathcal{E}(\vec{r}) = \vec{\nabla}(\vec{p}\cdot\vec{E}(\vec{r}))$$

$\vec{F} = 0$ for uniform field



$$\vec{\nabla}(\vec{p}\cdot\vec{E}(\vec{r})) = (\vec{p}\cdot\vec{\nabla})\vec{E}(\vec{r})$$

$$F_x = \left(p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z}\right) E_x$$

$\vec{F} = 0$ for uniform $\vec{E}$

$$\vec{N} = \vec{r}\times\vec{F} = \int d^3r\, \vec{r}\times \rho(\vec{r})\vec{E}(\vec{r})$$

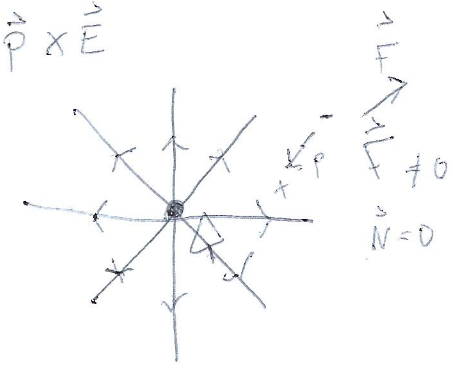
However there is a torque

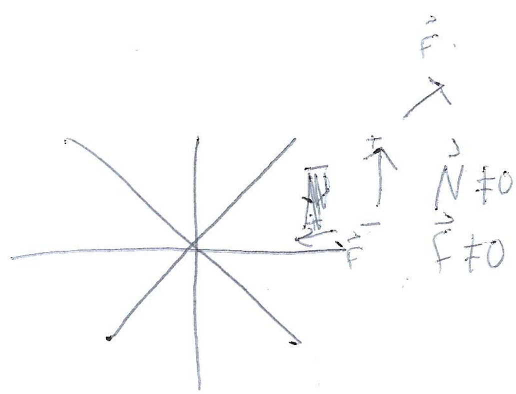
constant $\vec{E}$

$$\vec{N} = \left[\int d^3r\, \vec{r}\rho(\vec{r})\right]\times\vec{E}$$

$$\vec{N} = \vec{p}\times\vec{E}$$

Examples





Midterm

3 problems

Jackson Griffiths wrong about Polarization

Zangwill modern theory of polarization

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Electric Fields in Matter

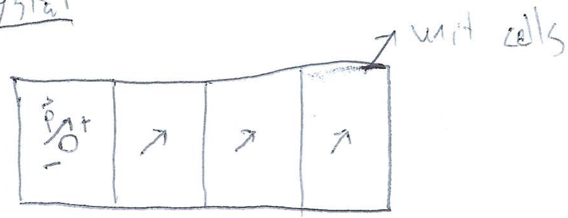
"Modern Theory of Polarization"

replaces textbook version "Lorentz theory"

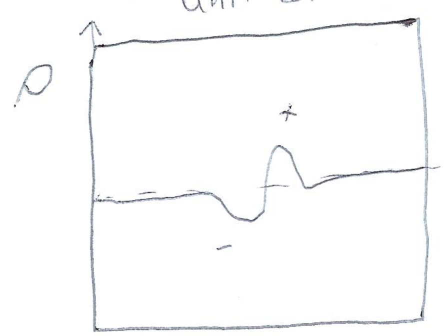
Polarization $= \vec{P}$

in Lorentz theory defines $\vec{P}$ as dipole moment per unit volume

Crystal



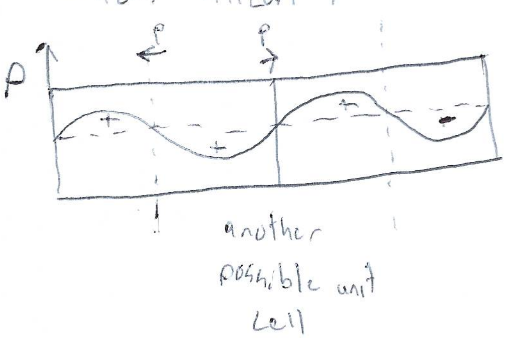
$\vec{P} = \frac{\vec{p}}{V}$; V = volume of unit cell



Calculate dipole moment

$$\vec{p} = \int_V d^3r\, \vec{r} \rho(\vec{r})$$

Consider Silicon, GaAs



clear that $\vec{p}$ depends on choice of unit cell

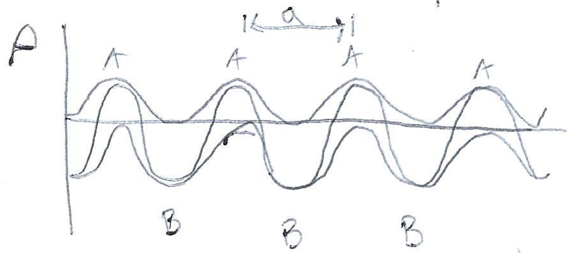
Polarization not well defined

Difference in Polarization is

Polarization: Dielectrics and Ferro-electric

(Si) ($BaTiO_3$)

Dielectric materials: simple model



$\rho = \rho_A + \rho_B = \rho_0 \cos\frac{2\pi x}{a}$

Calculate dipole moment per unit cell

$$\vec{p} = \int_0^a \frac{\rho_0}{a}\, dx\, x \cos\left(\frac{2\pi x}{a}\right)\hat{x}$$

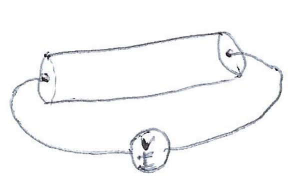
But what about $\vec{p} = \frac{\rho_0}{a}\int_{x_0}^{a+x_0} dx\, x \cos\left(\frac{2\pi x}{a}\right)\hat{x}$

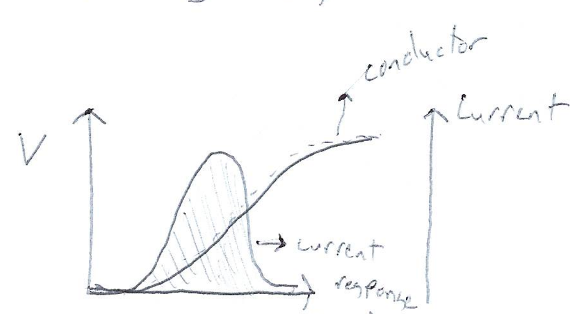
$$\cancel{\vec{p} =}\ \boxed{\vec{p}(x_0) = 2\pi \rho_0 a \sin\left(\frac{2\pi x_0}{a}\right)}$$

Talk about $\delta \vec{P}_j$ change in polarization

Polarize a dielectric by applying an $\vec{E}$ field

Ramp voltage slowly





$$\boxed{\int dt\, I(t) = \Delta}$$

"Modern" definition of Polarization

$$\vec{P} \neq \vec{p}/V_{ol}$$

Really $\frac{\partial \vec{P}}{\partial t} = \vec{J}_b$, $\vec{J}_b$ = bound current

Integrate

$$\int_0^\infty dt \, \frac{\partial \vec{P}}{\partial t} = \int_0^\infty dt \, \vec{J}_b$$

$$\delta \vec{P} = \int_0^\infty dt \, \vec{J}_b$$

$\hat{n}$ is unit normal

$$\delta \vec{P} \cdot \hat{n} = \int_0^\infty dt \, \vec{J}_b \cdot \hat{n} = \frac{\text{Charge}}{\text{Area}}$$

Starting Point is

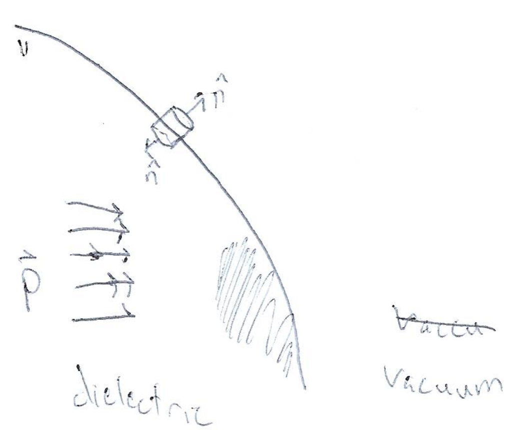
$$\frac{\partial \vec{P}}{\partial t} \equiv \vec{J}_b$$

Conservation of charge $\vec{\nabla} \cdot \vec{J}_b = -\frac{\partial \rho_b}{\partial t}$ ρ_b = bound charge density

$$\frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{P} = -\frac{\partial}{\partial t} \rho_b$$

$$\vec{\nabla} \cdot \vec{P} = -\rho_b$$

Consider surface



Gauss' Law for pillbox

$$\int_V d^3r \, \vec{\nabla} \cdot \vec{P} = \int_S \vec{P} \cdot \hat{n} \, da$$

$$\int_V d^3r \, \vec{\nabla} \cdot \vec{P} = -\int_V d^3r \, \rho_b = -\delta Q_b$$

$$\int \vec{P} \cdot \hat{n} \, da = -\vec{P} \cancel{da} \cdot \hat{n} \, da$$

$$\cancel{\vec{P} da} = -P \cdot \hat{n} \, da = -\delta Q_b$$

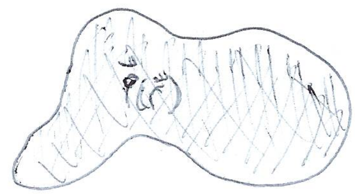
$$\vec{P} \cdot \hat{n} = \frac{\delta Q_b}{da} = \sigma_b$$

σ_b = bound area charge density

Summarizing $\vec{P}$ is related to charge density introduced by breaking of inversion symmetry

Dielectrics – symmetry breaking induced by $\vec{E}$ field

$$\boxed{\begin{array}{ll} \text{Bulk bound charge.} & \vec{\nabla}\cdot\vec{P} = -\rho_b \\ \text{Surface bound charge} & \vec{P}\cdot\hat{n} = \sigma_b \end{array}}$$

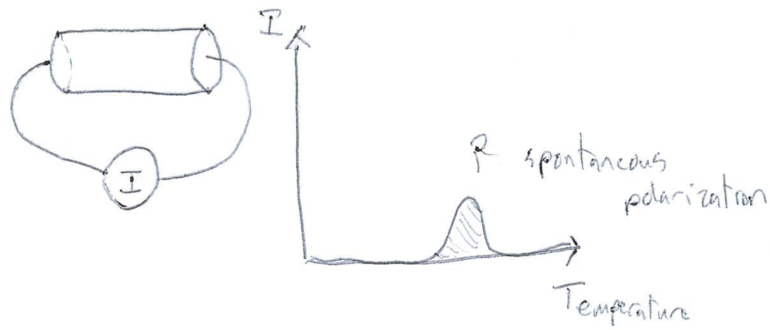


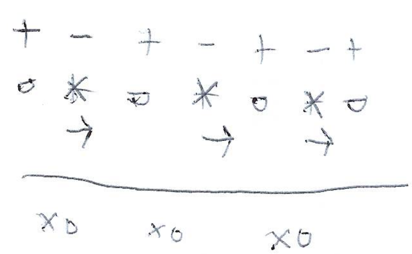
calculate total charge

$$Q = \int_V d^3r\,\rho_b(\vec{r}) + \oint_S da\,\sigma_b(\vec{r})$$

$$= -\int_V d^3r\,\vec{\nabla}\cdot\vec{P}(\vec{r}) + \oint_S \vec{P}\cdot\hat{n}\,da \equiv 0 \text{ by Fundamental Theorem}$$

Ferroelectrics: spontaneously break inversion symmetry upon cooling





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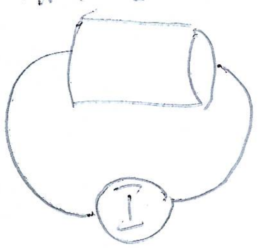
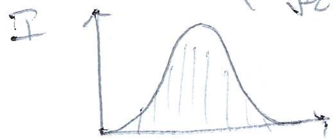
Reminder

Apply E_z T



$Q = \int I dt$



E_z T

$$\vec{J}_b = \frac{\partial \vec{P}}{\partial t} \Rightarrow \vec{\nabla}\cdot\vec{P} = -\rho_b \Rightarrow \vec{P}\cdot\hat{n} = \sigma_b$$

1) Ferroelectric, given $\vec{P}(\vec{r})$

2) Dielectric, apply $\vec{E}$, find $\vec{P}(\vec{r})$

Introduce auxilliary field $\vec{D}$

Make it up to help solve problems

$\rho = \rho_f + \rho_b$; $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$

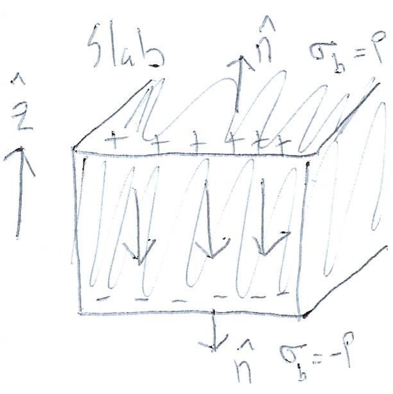
$\epsilon_0 \vec{\nabla}\cdot\vec{E} = \rho_f - \vec{\nabla}\cdot\vec{P}$

$$\rho_f = \epsilon_0 \vec{\nabla}\cdot\vec{E} + \vec{\nabla}\cdot\vec{P} = \vec{\nabla}\cdot(\epsilon_0\vec{E} + \vec{P}) = \boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$$

Beware $\vec{\nabla}\times\vec{D}$ is not necessarily zero

Some basic examples of polarized media

Slab



$\vec{P}\uparrow$ uniform

$\vec{\nabla}\cdot\vec{P} = 0$

$\rho_b = 0 = \rho_b$

What is the $\vec{E}$ field inside?

$\vec{P}\cdot\hat{n} = \pm\sigma_b$

$$\vec{E} = -\frac{\sigma_b}{\epsilon_0}\hat{z} = \frac{-P}{\epsilon_0}\hat{z} = \frac{-\vec{P}}{\epsilon_0}$$

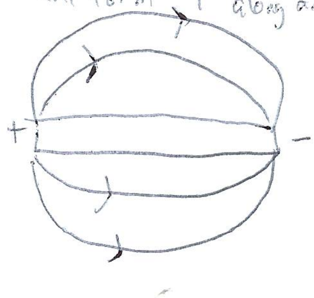
$\vec{E} = \frac{-\vec{P}}{\epsilon_0}$ notice: if assume $\vec{D} = 0$ then $\vec{E} = \frac{-\vec{P}}{\epsilon_0}$

Polarized needle

as we stretch it long and thin, it looks like 2 point charges of opposite charge

$\vec{P}$

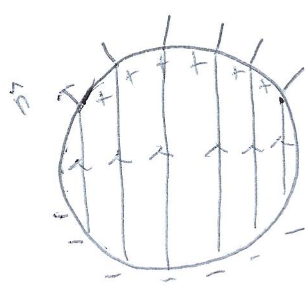
uniform $\vec{P}$ along axis



$\vec{E}$ field inside is not uniform

$\vec{D} \neq 0$ in this case

Uniformly Polarized sphere



field lines of $\vec{P}$

$\vec{\nabla} \cdot \vec{P} = 0 \Rightarrow \rho_b = 0$

$\vec{P} \cdot \hat{n} = P\cos\theta = \sigma_b$

$\sigma_b = P\cos\theta$

This is a problem we've solved!

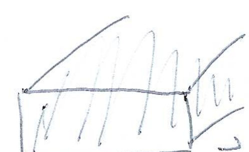
Recall boundary condition

$V(R,\theta) = V_0 \cos\theta$

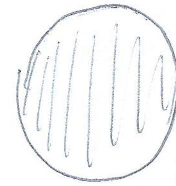
Solution $\vec{E}_{inside} = \frac{-V_0}{R}\hat{z}$, $\sigma = \frac{3\epsilon_0 V_0}{R}\cos\theta$

$\frac{3\epsilon_0 V_0}{R} \Longleftrightarrow P$ $\quad \frac{V_0}{R} \Longleftrightarrow \frac{P}{3\epsilon_0}$

Answer $\vec{E}_{\text{inside dielectric}} = \frac{-\vec{P}}{3\epsilon_0}$

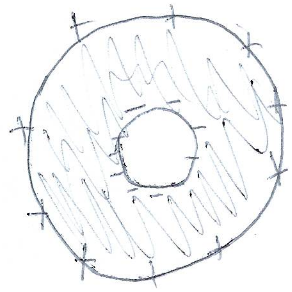


$\vec{E} = \frac{-\vec{P}}{\epsilon_0}$



$\vec{E} = \frac{-\vec{P}}{3\epsilon_0}$

4.15 in Griffiths

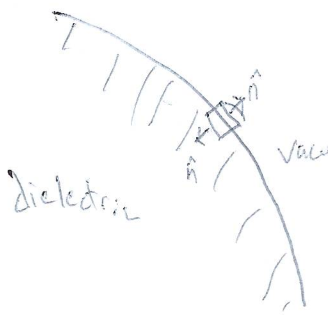


$\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$

$\vec{\nabla}\vec{P} \neq 0$

$\rho_b \neq 0 \quad \vec{P}\cdot\hat{n} \neq 0$

Good thing about $\vec{D}$



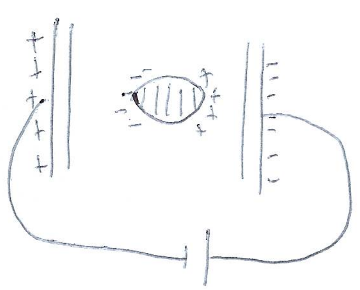
Here $\rho_f = 0$

$\vec{\nabla}\cdot\vec{D} = 0$

$\left(\vec{D}_{\text{just inside}} - \vec{D}_{\text{just outside}}\right)\cdot\hat{n} = 0$

Continuity of normal component of $\vec{D}$

In general



Find $\vec{E}$ This leads to a "self consistancy Problem"

$\vec{E} = \vec{E}_{\text{applied}} + \vec{E}_{\text{induced by Polarization}}$

Constitutive relation

What is $\vec{P}$ for given $\vec{E}$.

Most general case

$$P_i = \epsilon_0 \sum_j \chi_{ij} E_j + \epsilon_0 \sum_{jk} \chi^{(2)} E_j E_k$$

This defines linear susceptibility tensor χ_{ij}

first order responce "linear"

second order response

Linear response, isotropic

$$\overleftrightarrow{\chi} = \chi \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Reduces to } \vec{P} = \epsilon_0 \chi \vec{E}$$

Multiplicity of definitions

Recall $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$$= \epsilon_0 \vec{E} + \epsilon_0 \chi \vec{E} = \underbrace{\epsilon_0 (1+\chi)}_{\epsilon_0} \vec{E}$$

$$\epsilon = \epsilon_0 (1+\chi) \Rightarrow \vec{D} = \epsilon \vec{E}$$

Another dimensionless parameter $K = \frac{\epsilon}{\epsilon_0} = 1+\chi$

Example Problems

$$\rho_f = q\,\delta(\vec{r})$$

ρ_{ind} = induced polarization charge $\equiv \rho_b$

$$\rho = \rho_f + \rho_{ind}$$

∞ dielectric medium Assume $\vec{P} = \epsilon_0 \chi \vec{E}$

$$\vec{\nabla} \cdot \vec{P} = \epsilon_0 \chi \vec{\nabla} \cdot \vec{E}$$

$$-\rho_{ind} = \epsilon_0 \chi \left(\frac{\rho}{\epsilon_0}\right) = \chi \rho$$

$$\rho = \rho_f + \rho_{ind} = \rho_f - \chi \rho$$

$$\rho = \frac{\rho_f}{1+\chi} = \frac{\rho_f}{K} \qquad \rho = \frac{q\,\delta(\vec{r})}{K}$$

Screening

$$q \to \frac{q}{K}$$

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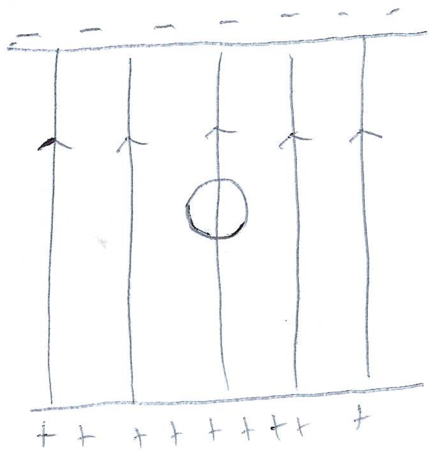
Dielectric sphere uniform field

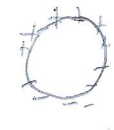


plates really far away

Boundary condition $\vec{E}(\vec{r}) \to E_0 \hat{z}$ far from sphere

Total field $\vec{E}(\vec{r}) = E_0 \hat{z} + \vec{E}_p(\vec{r})$

$\vec{E}_p$ is electric field from bound charge



$\vec{E}_{pin}$ ~~electric field~~

$\vec{E}_{pin} = \vec{E}_p$ inside sphere, $\vec{E}_{in}$ = total field inside

$\vec{E}_{in}(\vec{r}) = E_0 \hat{z} + E_{pin}(\vec{r})$

Dielectric matter $\vec{P}(\vec{r}) = \epsilon_0 \chi \vec{E}_{in}(\vec{r})$

~~Guess~~ Because uniform for $\vec{P}$ inside a sphere produces uniform $\vec{E}$

We can guess uniform $\vec{P}$

Then $\vec{E}_{pin} = \frac{-1}{3\epsilon_0}\vec{P} = \frac{-1}{3\epsilon_0}\epsilon_0 \chi \vec{E}_{in} \Rightarrow \boxed{\vec{E}_{pin} = -\frac{\chi}{3}\vec{E}_{in}}$

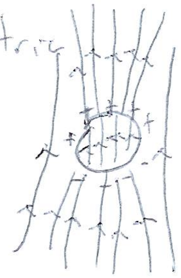
$\vec{E}_{in} = E_0 \hat{z} - \frac{\chi}{3}\vec{E}_{in}$

$$\boxed{\vec{E}_{in} = \frac{E_0 \hat{z}}{1+\chi/3}}$$

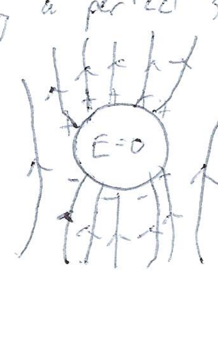
Partial screening

Dielectric

$E_{in} = \frac{E_0}{1+\chi/3}$



Conductor $\Rightarrow$ a "perfect screening"

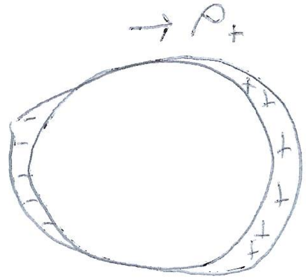


Energy stored in dielectric media

Recall for vacuum energy stored in field

in $\mathcal{U} = \frac{\epsilon_0}{2}\int d^3r\, \vec{E}^2(\vec{r})$

How much work is done to polarize a dielectric?



ρ_+ is $\oplus$ charge density

ρ_+ moves under influence of some field $\vec{E}$

$\vec{J}$ is current density

$\vec{J} = \rho_+ \vec{v}$, $\vec{v}$ is velocity

The rate at which work is done

$$\frac{\partial W}{\partial t} = \vec{F}\cdot\vec{v}, \quad \vec{F} = \rho_+ \vec{E}$$

$$\frac{dW}{dt} = \rho_+ \vec{E}\cdot\vec{v} = \vec{J}\cdot\vec{E}$$

Recall $\vec{J} = \frac{\partial \vec{P}}{\partial t}$ $\qquad \frac{dW}{dt} = \vec{E}\cdot\frac{d\vec{P}}{dt} \Rightarrow \boxed{\delta W = \vec{E}\cdot\delta\vec{P}}$

The total change in energy density is δW plus energy stored in the field

∂u = total change in energy density

$$\boxed{\partial u = \partial u_{\text{field}} + \delta W}$$

$$dU_{\text{field}} = d\left(\frac{\epsilon_0}{2}E^2\right) = \epsilon_0 \vec{E} \cdot \delta \vec{E}$$

$$dU = \epsilon_0 \vec{E} \cdot \delta \vec{E} + \vec{E} \cdot \delta \vec{P} = \vec{E} \cdot (\epsilon_0 \delta \vec{E} + \delta \vec{P})$$

Recall $\vec{D} = \epsilon_0 \vec{E} + \vec{P} \Rightarrow \boxed{dU = \vec{E} \cdot \delta \vec{D}}$

We should integrate over all space

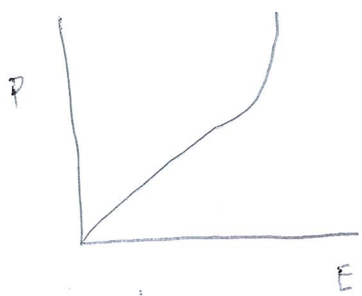
$$dU = \int d^3r \, \vec{E}(\vec{r}) \cdot \delta \vec{D}(\vec{r})$$

$$dW = \int d^3r \, \vec{E}(\vec{r}) \cdot \delta \vec{P}(\vec{r})$$

Now polarize to some final state $P(\vec{r})$

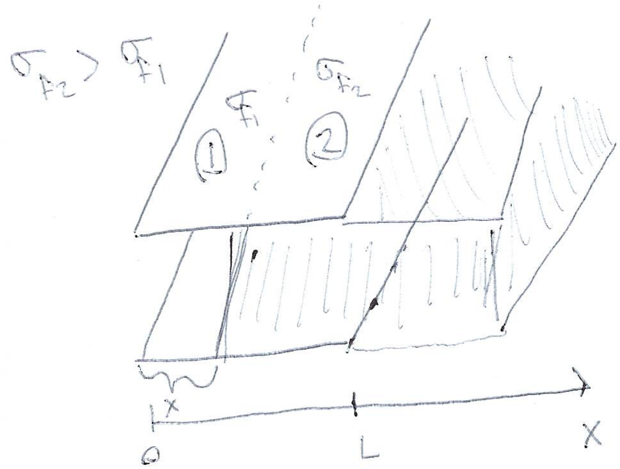
$$U = \int d^3r \int_0^D \vec{E} \cdot \delta \vec{D} \qquad W = \int d^3r \int_0^P \vec{E} \cdot \delta \vec{P}$$

Typical dielectric



Assume linear regime $\vec{P} = \epsilon_0 \chi \vec{E}$; $\vec{D} = \epsilon \vec{E}$

Total energy in system
$$\boxed{\begin{aligned} U &= \frac{\epsilon}{2} \int d^3r \, E^2(\vec{r}) \\ W &= \left(\frac{\epsilon - \epsilon_0}{2}\right) \int d^3r \, \vec{E}^2(\vec{r}) \end{aligned}}$$



dielectric partially inbetween parallel plates

Find the Force on the dielectric

Assume isolated system with free charges Q, $-Q$ on plates

Find $u(x)$

Note E is the same in 1 and 2

1 $E = \frac{\sigma_{F_1}}{\epsilon_0} \Rightarrow \sigma_{F_1} = \epsilon_0 E$; σ_{F_1} is free charge/area on plate

2 $E = \frac{\sigma_{F_2} + \sigma_b}{\epsilon_0} = \frac{\sigma_{F_2} - P}{\epsilon_0} = \frac{\sigma_{F_2} - \epsilon_0 \chi E}{\epsilon_0}$ $\quad E = \frac{\sigma_{F_2}}{\epsilon}$ or $\sigma_{F_2} = \epsilon E$

Now relate to total charge Q

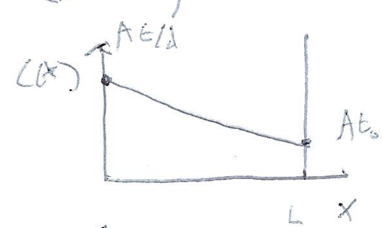
$$Q = \sigma_{F_1} A\left(\frac{x}{L}\right) + \sigma_{F_2} A\left(\frac{L-x}{L}\right) = \text{constant}$$

Express in terms first of E ~~then write~~

$$Q = \frac{EA}{L}\left(\epsilon_0 x + \epsilon(L-x)\right)$$ or in terms of Voltage $V = Ed$

$$Q = V \frac{A}{Ld}\left(\epsilon_0 x + \epsilon(L-x)\right)$$ Recall $Q = C(x)V$

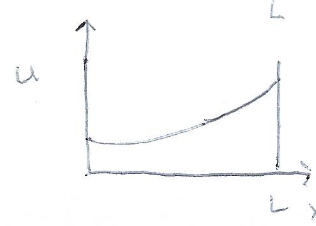
$C(x)$ is the capacitance $C(x) = \frac{A}{Ld}\left(\epsilon_0 x + \epsilon(L-x)\right)$



calculate stored energy $u = \frac{\epsilon}{2}\int E^2 d^3r$

Express u in terms of Q : $u = \frac{1}{2}\frac{Q^2}{C(x)}$

$$F = -\frac{du}{dx} = -\frac{1}{2}\frac{Q^2}{C^2(x)}\frac{A}{Ld}(\epsilon - \epsilon_0)$$



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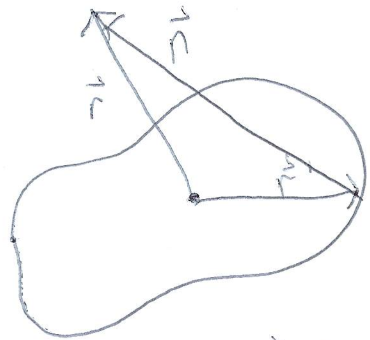
Biot - Savart



$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint d\vec{l} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Statics $\vec{\nabla} \cdot \vec{J} = 0$

Magnetostatics

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu \vec{J} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

Statics $\vec{\nabla} \times \vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla} V$ $\qquad$ $V(\vec{r})$ is scalar potential

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$ $\qquad$ $\vec{A}(\vec{r})$ is vector potential

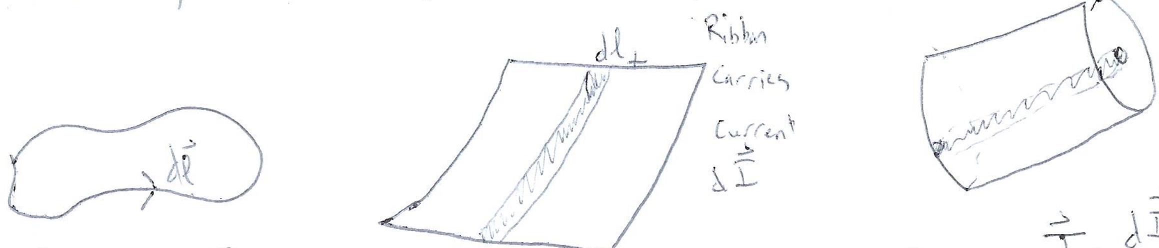
Vector identity $\nabla \cdot (\vec{\nabla} \times V) = 0$

Charge matter interactions

velocity dependent force

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Preliminary current in different dimensions



1D wire I

2D $\vec{K} \equiv \frac{dI}{dl_\perp}$

3D $\vec{J} = \frac{d\vec{I}}{da_\perp}$

$\vec{B} = \vec{\nabla}\times\vec{A}$ plug into Maxwell equation

$$\vec{\nabla}\times\vec{B} = \mu\vec{J}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} \qquad \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$$

Choose $\vec{A}$ such that $\vec{\nabla}\cdot\vec{A} = 0$

Gauge transformation

Because $\vec{\nabla}\times\vec{\nabla}\chi(\vec{r}) = 0$ we are free to make the transformation

$$\vec{A} \to \vec{A}' \text{ where } \vec{A}' = \vec{A} + \vec{\nabla}\chi(\vec{r})$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{\nabla}\chi = \vec{\nabla}\times\vec{A}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}'$$

Suppose initially $\vec{\nabla}\cdot\vec{A} = \Lambda(\vec{r})$

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$

$$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\chi = \Lambda(\vec{r}) + \nabla^2\chi = 0$$

~~R~~ therefore we want to choose $\nabla^2\chi = -\Lambda(\vec{r})$

You can always find such a $\chi(\vec{r})$ Coulomb Gauge

You can always set $\vec{\nabla}\cdot\vec{A} \equiv 0$ [Coulomb gauge]

then $\nabla^2\vec{A} = -\mu_0\vec{J}$

This is really 3 equations

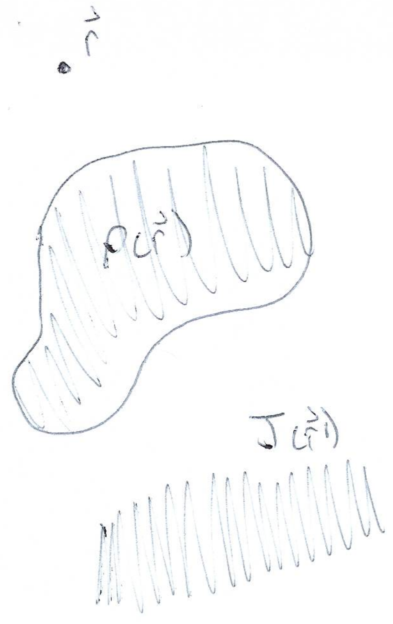
$$\nabla^2 A_x = -\mu_0 J_x$$

$$\nabla^2 A_y = -\mu_0 J_y$$

$$\nabla^2 A_z = -\mu_0 J_z$$

Three copies of Poisson's Equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d^3r'$$

$$A_x(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r \frac{J_x(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Summarize in a vector equation

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \; \vec{\nabla}\times\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

∇ is derivative with respect to $\vec{r}$

Vector chain rules

$$\frac{\vec{\nabla}\times\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} = -\vec{J}(\vec{r}')\times\left(\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

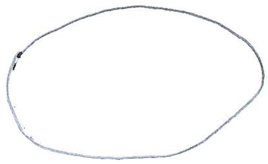
$$\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = \frac{-(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$\vec{\nabla}$ (V of pt charge) = Electric Field of point charge

Finally: $$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\vec{J}(\vec{r}')\times\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

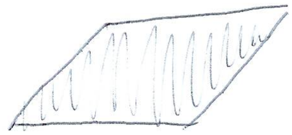
Biot – Savart

Example of current distributions



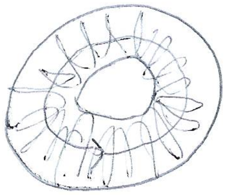
loop

∞ straight wire



∞ plane of current

Solenoid



toroid

doughnut

Consider contrast with electrostatics

λ

loop of charge

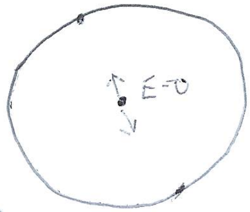
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \oint \lambda \, dl' \frac{\hat{\mu}}{\mu^2}$$

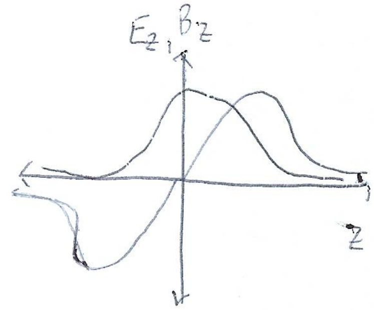
I

Loop of current

$$\vec{B} = \frac{\mu_0}{4\pi} \oint \frac{I \, d\vec{l} \times \hat{\mu}}{\mu^2}$$

Looking down on loop of charge





Solve for $\vec{E}$ on axis

$$\vec{E} = \hat{z} \frac{Q}{4\pi\epsilon_0} \frac{z}{(R^2+z^2)^{3/2}}$$

$d\vec{B}$ ~~comes from~~ from $I d\vec{l}'$

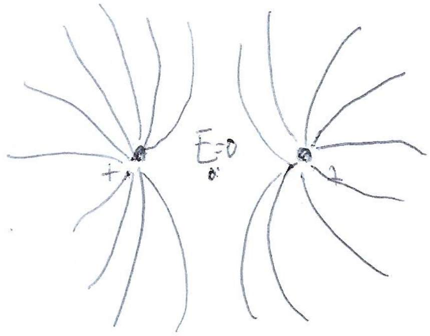
$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l}' \times \hat{\mu}}{\mu^2} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l}' \times (z\hat{z} - \vec{r}')}{\mu^3}$$

$d\vec{l}' \times z\hat{z}$ vanishes by symmetry when you integrate over the loop

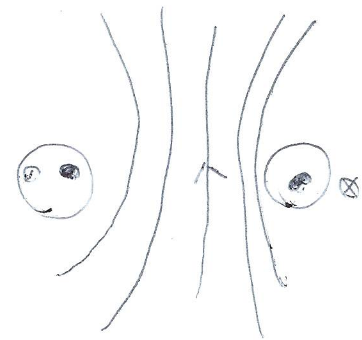
$d\vec{l}' \times (-\vec{r}')$ doesn't vanish and points in z direction

$$\vec{B}_z(z) = \frac{\mu_0 I}{4\pi} \hat{z} \left(\frac{1}{(z^2+R^2)^{3/2}}\right) \oint dl' R = \frac{\mu_0 I}{2} \hat{z} \left(\frac{R^2}{(z^2+R^2)^{3/2}}\right)$$

Loop of charge



Loop of current



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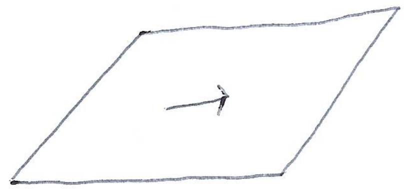
Line and plane of current

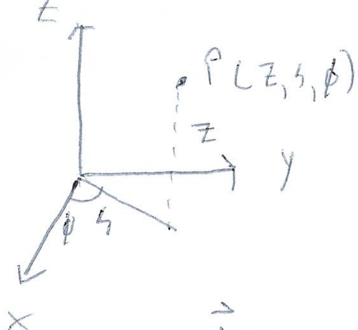
I



$\vec{K}$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}')}{\mathscr{r}} = \frac{\mu_0}{4\pi}\vec{J}\int d^3r' \frac{1}{\mathscr{r}}$$

In this case: $\vec{A}$, $\vec{J}$ are parallel for line current



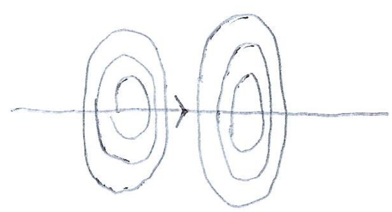
So for $\vec{A}(z,s,\phi) = A_z(z,s,\phi)$

Translational symmetry $\Rightarrow$ A_z is independent of z

Rotational symmetry $\Rightarrow$ A_z is independent of ϕ

$$\vec{A}(s) = A_z(s)\,\hat{z}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\partial A_z(s)}{\partial s}\,\hat{\phi}$$



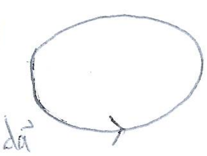
Ampere's Law

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\int \vec{J}\cdot d\vec{a} = \mu_0 I_{enc}$$

arbitrary field $V(\vec{r})$

$$\oint \vec{V}\cdot d\vec{l} = \int(\vec{\nabla}\times\vec{V})\cdot d\vec{a}$$

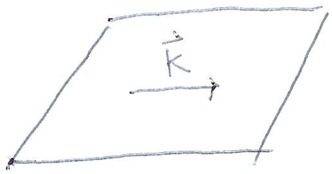


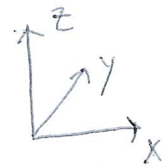
Take loop of radius s

$$B_\phi(s)\,2\pi s = \mu_0 I$$

$$B_\phi(s) = \frac{\mu_0 I}{2\pi s}$$

Plane of Current





$\vec{K} = K\hat{x}$ $\qquad$ $\vec{A}(\vec{r}) = A_x(x, y, z)\hat{x}$

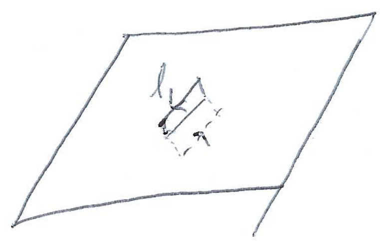
$\vec{A}(\vec{r}) = A_x(z)\hat{x}$

$\vec{B}(z) = \vec{\nabla}\times\vec{A}_x(z)\hat{x}$ $\qquad$ $\vec{B}(z) = -\hat{y}\,\frac{\partial A_x(z)}{\partial z}$, show that $A_x(z) = A_x(-z)$

Biot-Savart $\quad A_x(z) = \frac{\mu_0}{4\pi}K\int d^2r'\,\frac{1}{|\vec{r}-\vec{r}'|}$ $\qquad \frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(x-x')\hat{x} + (y-y')\hat{y} + (z\ \cancel{\ })\hat{z}}$

But $B(z) \propto \frac{\partial A_x(z)}{\partial z}$ $\qquad$ even function of z

This means that $B(z) = -B(-z)$



$\oint \vec{B}\cdot d\vec{l} = \left(-B_y(z) + B_y(-z)\right)l$

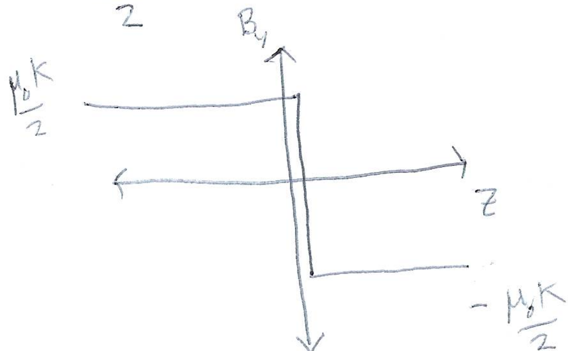
$I_{enc} = Kl$ $\qquad B_y(-z) - B_y(z) = -\mu_0 K$

$-2B_y(z) = \mu_0 K$

$B_y(z) = \frac{-\mu_0 K}{2}$ $\qquad$ B is independent of z

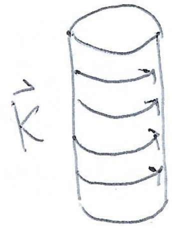
B is discontinuous

edge on



Jump in B of $\mu_0 K$

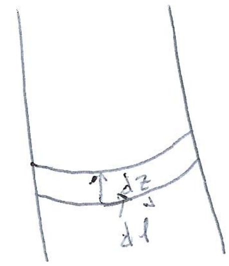
Consider solenoid



plane of charge wrapped into a cylinder

Biot-Savart

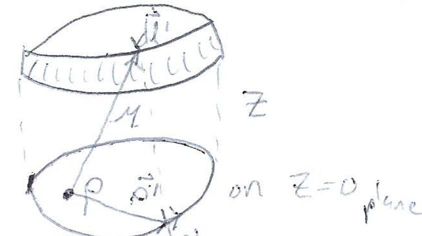
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^2r' \, \frac{\vec{K}(\vec{r}\,') \times \vec{\mathscr{r}}}{\mathscr{r}^3}$$



$d^2r' = dz'\,dl'$

$\cancel{\vec{\mathscr{r}} = |\vec{r} - \vec{r}\,'|}$ $\quad \mathscr{r} = \vec{r} - \vec{r}\,'$

$$= -(\vec{\rho}\,' + z\hat{z})$$



$$\vec{B}(P) = \frac{\mu_0 K}{4\pi}\int_{-\infty}^{\infty} dz \oint \frac{d\vec{l}\,' \times \vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$d\vec{l}\,' \times \vec{\mathscr{r}} = -d\vec{l}\,' \times (\vec{\rho}\,' + z\hat{z})$$

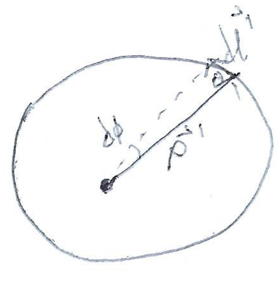
$$\vec{B}(P) = -\frac{\mu_0 K}{4\pi}\int_{-\infty}^{\infty} dz \left(\oint \frac{d\vec{l}\,' \times \vec{\rho}\,'}{\mathscr{r}^3} + \oint \frac{d\vec{l}\,' \times z\hat{z}}{\mathscr{r}^3}\right)$$

$$\vec{B}(P) = -\frac{\mu_0 K}{4\pi}\left(\oint d\vec{l}\,' \times \vec{\rho}\,' \int \frac{dz}{\mathscr{r}^3} + \cancel{\oint \frac{d\vec{l}\,' \times \hat{z}}{\mathscr{r}^3}\int dz \frac{z}{\mathscr{r}^3}}\right)$$

0 by symmetry

$$\vec{B}(P) = -\frac{\mu_0 K}{2\pi}\oint \frac{d\vec{l}\,' \times \vec{\rho}\,'}{\rho'^2}$$

$$\int_{-\infty}^{\infty} \frac{dz}{\mathscr{r}^3} = \int_{-\infty}^{\infty} \frac{dz}{(\rho'^2 + z^2)^{3/2}} = \frac{2}{\rho'^2}$$



Note $\vec{B} = B(\vec{r})\hat{z}$ $\qquad |d\vec{l}\,' \times \vec{\rho}\,'| = -\rho' dl' \sin\theta$

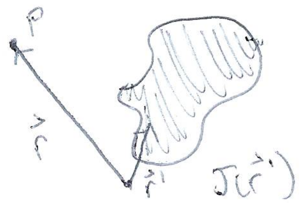
Law of sines $\dfrac{A}{\sin\theta_A} = \dfrac{B}{\sin\theta_B} = \dfrac{C}{\sin\theta_C}$

$$\frac{dl'}{d\phi} = \frac{\rho'}{\sin\theta} \Rightarrow \sin\theta = \frac{\rho' d\phi}{dl'} \qquad |d\vec{l}\,' \times \vec{\rho}\,'| = -\rho'^2 d\phi$$

$$|\vec{B}(P)| = \frac{\mu_0 K}{2\pi}\oint \frac{\rho'^2 d\phi}{\rho'^2} = \mu_0 K$$

$\vec{B} = \mu_0 K \hat{z}$ <u>inside</u> $\qquad$ What happens outside?

Magnetic multipole expansion



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}')}{\mathscr{r}}$$

Expand $\frac{1}{\mathscr{r}} = \frac{1}{|\vec{r}-\vec{r}'|}$ in powers of $\left(\frac{r'}{r}\right)$

Since this integral is annoying to solve

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

$$= \frac{1}{r} + \frac{1}{r}\left(\frac{r'}{r}\right)\cos\theta$$

$$= \frac{1}{r} + \frac{1}{r^2}(\hat{r}\cdot\vec{r}') + \cdots$$

monopole　　dipole　　...

Monopole

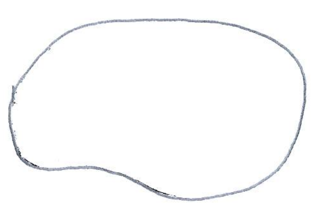
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi r}\int d^3r' \vec{J}(\vec{r}')$$

Consider $\vec{\nabla}\cdot x\vec{J}(\vec{r}) = x\vec{\nabla}\cdot\vec{J} + \vec{J}\cdot\vec{\nabla}x$　　$\vec{\nabla}x = \hat{x}\frac{\partial x}{\partial x} = \hat{x}$

$$= \vec{J}\cdot\hat{x} = J_x$$

$$A_{mono,x} = \frac{\mu_0}{4\pi r}\int d^3r' J_x(\vec{r}') = \frac{\mu_0}{4\pi}\int d^3r' \vec{\nabla}\cdot x\vec{J}(\vec{r}') = \frac{\mu_0}{4\pi}\oint d\vec{a}\cdot\left(x\vec{J}(\vec{r})\right) = \vec{0}$$

$$\vec{A}_{mono}(\vec{r}) = \vec{0}$$



$$\int d^3r' \vec{J}(\vec{r}') \Rightarrow \oint I\, d\vec{l} = 0$$

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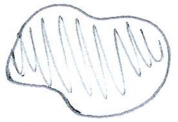
Multiple Expansion of $\vec{A}$

$\vec{V}$ 

$$\vec{A}_{dip}(\vec{r}) = \cancel{\frac{\mu_0}{4\pi}} \frac{\mu_0}{2\pi r^3}\int d^3r'\, \vec{J}(\vec{r}\,')\, \vec{r}\cdot\vec{r}\,'$$

$$\vec{A}_{dip\,x}(\vec{r}) = \cancel{\frac{\mu_0}{4\pi r^3}\int} \frac{\mu_0}{4\pi r^3}\left[x\int d^3r'\, J_x x' + y\int d^3r'\, J_y y' + z\int d^3r'\, J_z z'\right]$$

This the form of matrix multiplication

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \left[\overleftrightarrow{T}\right]\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad T_{ij} = \int d^3r'\, J_i(\vec{r}\,')\, r'_j$$

Consider $\vec{\nabla}\cdot(xy\vec{J}) = x\vec{\nabla}\cdot(y\vec{J}) + y\,\vec{\nabla}\cdot(x\vec{J})$

$$\vec{\nabla}\cdot(xy\vec{J}) = xJ_y + yJ_x$$

Consider $(\vec{r}\times\vec{J})_z = xJ_y - yJ_x$

$$\vec{\nabla}\cdot(xy\vec{J}) - (\vec{r}\times\vec{J})_z = 2yJ_x$$

$$T_{xy} = \frac{1}{2}\int d^3r'\left(\underbrace{\vec{\nabla}\cdot(xy\vec{J})}_{\text{goes to zero}} - (\vec{r}\times\vec{J})_z\right)$$

$$T_{xy} = -\frac{1}{2}\int d^3r\,(\vec{r}\times\vec{J})_z$$

General Form $\quad T_{ij} = \frac{1}{2}\epsilon_{ikj}\int d^3r\,(\vec{r}\times\vec{J})_k$

$$\epsilon_{ikj} = \begin{cases} -1 & \text{for even permutations} \\ +1 & \text{for odd permutations} \\ 0 & \text{if 2 indices are the same} \end{cases}$$

Structure of T_{ij}

$$\begin{bmatrix} 0 & T_{xy} & T_{xz} \\ -T_{xy} & 0 & T_{yz} \\ -T_{xz} & -T_{yz} & 0 \end{bmatrix}$$

Only 3 independent elements

T_{xy}, T_{xz}, T_{yz} are called the components of the magnetic dipole moment

$$m_k = \epsilon_{ikj} T_{ij} = \frac{1}{2}\int d^3r \left(\vec{r} \times \vec{J}(\vec{r})\right)_k$$

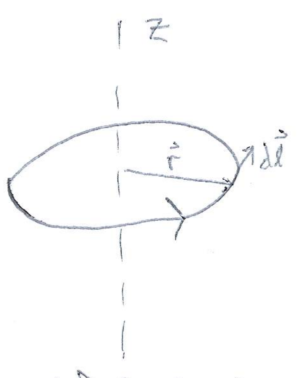
$$m_x = -T_{yz}, \quad m_y = +T_{xz}, \quad m_z = -T_{xy}$$

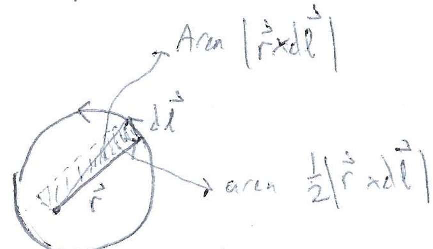
$$A_x(\vec{r}) = \frac{\mu_0}{4\pi r^3} \sum T_{xj} r_j = \frac{\mu_0}{4\pi r^3}\left(T_{xy} y + T_{xz} z\right) = \frac{\mu_0}{4\pi r^3}\left(-m_z y + m_y z\right)$$

$$= \frac{\mu_0}{4\pi r^3}\left(\vec{m} \times \vec{r}\right)_x$$

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left(\vec{m} \times \vec{r}\right)$$

Consider planar current loop (xy plane)





Area of Loop is $\frac{1}{2}\oint |\vec{r} \times d\vec{l}|$

$$\vec{m} = \frac{1}{2}\int d^3r \; \vec{r} \times \vec{J}(\vec{r})$$

$$= \frac{I}{2}\oint \vec{r} \times d\vec{l} \parallel \hat{z}$$

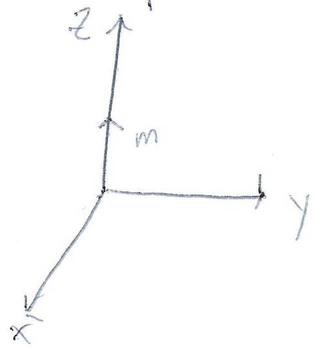
Therefore $\vec{m} = I \cdot \text{area}$

$$= I \cdot A\,\hat{z}$$

$$\vec{B}_{dip}(\vec{r}) = \vec{\nabla} \times \vec{A}_{dip}(\vec{r})$$

$$\vec{B}_{dip} \propto \vec{\nabla} \times \frac{(\vec{m} \times \vec{r})}{r^3}$$

Assume $\vec{m}$ is parallel to z



$$\frac{\vec{m} \times \vec{r}}{r^3} = \frac{m}{r^2} \sin\theta \, \hat{\phi}$$

$$\vec{\nabla} \times \left(\frac{m}{r^2} \sin\theta \, \hat{\phi} \right)$$

$$= m \frac{1}{r \sin\theta} \left(\frac{m}{r} \frac{\partial}{r^2 \partial\theta} \sin^2\theta \right) \hat{r} - m \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{m}{r} \sin\theta \right) \right) \hat{\theta}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3} \left(2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta} \right)$$

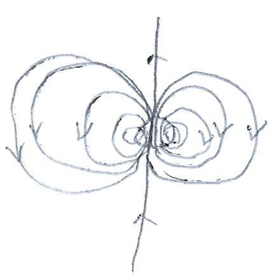
In general case $\vec{m} = (\vec{m} \cdot \hat{r})\hat{r} + (\vec{m} \cdot \hat{\theta})\hat{\theta} = m\cos\theta \, \hat{r} - m\sin\theta \, \hat{\theta}$

Consider $3(\vec{m} \cdot \hat{r})\hat{r} = 3m\cos\theta \, \hat{r}$

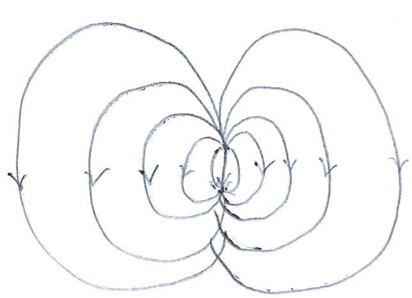
$3(\hat{m} \cdot \hat{r})\hat{r} - \vec{m} = 2\cos\hat{\theta} \, \hat{r} + m\sin\theta \, \hat{\theta}$

$$\vec{B}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3} \left(3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m} \right)$$

Magnetic field



Electric dipole



Same at large distances

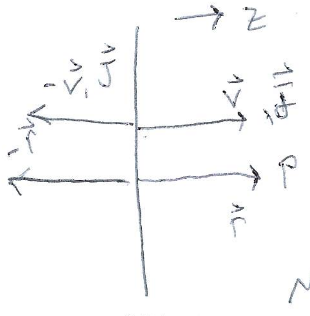
Why do we call $\vec{m}$ an ~~actual v~~ axial vector?

Properties under rotation

Proper rotation $R_{\hat{n}}(\theta)$

Improper rotations, mirror reflections or inversions

Consider improper rotation $M_z \equiv$ mirror reflection about xy plane



$$M_z\{\vec{r}\} = x\hat{x} + y\hat{y} - z\hat{z}$$

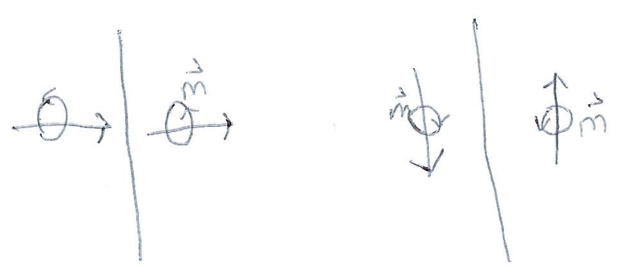
vectors transform like $\vec{r}$

examples are $\vec{v}$ and $\vec{J}$

$$\vec{m} \propto \vec{r} \times \vec{J}$$

$$\vec{m} \propto \hat{x}(yJ_z - zJ_y) + \hat{y}(zJ_x - xJ_z) + \hat{z}(xJ_y - yJ_x)$$

case a) case b)



What is this good for?

Suppose that a medium has mirror symmetry M_z?

$$M_z\{\vec{m}\} = \vec{m}$$

If $\vec{m}$ is parallel to $\hat{z}$ the $M_z\{\vec{m}\} = -\vec{m}$

Therefore $m_x = 0$

m_z can be nonzero

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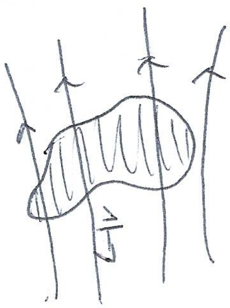
Force and Torque



$$\vec{F} = \int d^3r\, \rho \vec{v} \times \vec{B} = \int d^3r\, \vec{J} \times \vec{B}$$

Expand $\vec{B}(\vec{r})$

$$B_i(\vec{r}) = B_i(0) + \vec{r} \cdot \vec{\nabla} B_i(\vec{r})\Big|_{r=0} + \ldots$$

Consider F_x Look at lowest order

$$F_x = \int d^3r \left(J_y B_z(0) - J_z B_y(0) \right) = 0$$

Next order term

$$F_x = \int d^3r \left(J_y (\vec{r} \cdot \vec{\nabla} B_z) - J_z (\vec{r} \cdot \vec{\nabla} B_y) \right)$$

Look at first term

$$\vec{\nabla} B_z\Big|_0 \cdot \int d^3r\, J_y \vec{r} = \vec{\nabla} B_z\Big|_0 \cdot (\hat{x} m_z - \hat{z} m_x)$$

2nd term

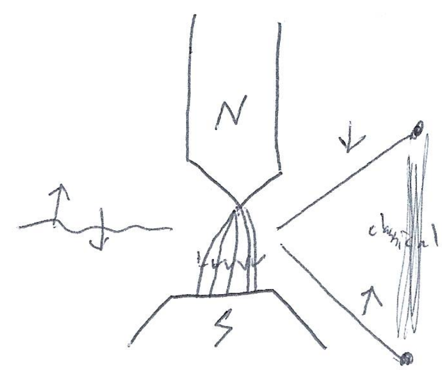
$$\vec{\nabla} B_y \cdot \int d^3r\, J_z \vec{r} = \vec{\nabla} B_y\Big|_0 \cdot (-\hat{x} m_y + \hat{y} m_z)$$

$$F_x = m_z \frac{\partial B_z}{\partial x} + m_y \frac{\partial B_y}{\partial y} - \left(m_x \frac{\partial B_z}{\partial z} + m_x \frac{\partial B_y}{\partial y} \right)$$

But $\vec{\nabla} \cdot \vec{B} = 0$, $m_x \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) = 0$

$$F_x = m_z \frac{\partial B_z}{\partial x} + m_y \frac{\partial B_y}{\partial x} + m_x \frac{\partial B_x}{\partial x} = \frac{\partial}{\partial x}(\vec{m} \cdot \vec{B})$$

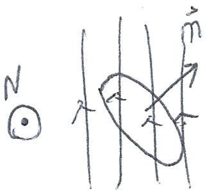
$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$$



irl
2 discrete points

Stern-Gerlach experiment

Torque



$$\vec{N} = \int d^3r\, \vec{r}\times\vec{F} = \int d^3r\, \vec{r}\times(\vec{J}\times\vec{B})$$

$$\vec{N} = \int d^3r \left(\vec{J}(\vec{B}\cdot\vec{r}) - \vec{B}(\vec{J}\cdot\vec{r})\right)$$

Assume $\vec{B}$ is uniform $\longrightarrow 0 \quad T_{ii} = 0$

$$\vec{N} = \int d^3r\, \vec{J}(\vec{B}\cdot\vec{r}) - \vec{B}\int d^3r\,(\vec{J}\cdot\vec{r})$$

$$= B_x \int d^3r\, \vec{J}\cancel{x} + B_y \int d^3r\, \vec{J} y + B_z \int d^3r\, \vec{J} z$$

$$N_x = -B_y \overset{\to J_x y}{m_z} + B_z m_y = (\vec{m}\times\vec{B})_x$$

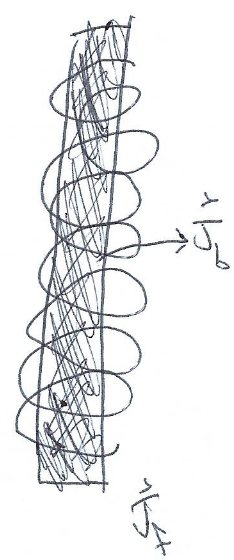
$$\boxed{\vec{N} = \vec{m}\times\vec{B}}$$

Magnetism in Matter only scratch surface biggest part of condensed matter physics

information storage/manipulation

Associated classically with bound currents $\vec{J}_b$

Recall $\rho = \rho_b + \rho_f$ Here $\vec{J} = \vec{J}_b + \vec{J}_f$



Polarization Dielectrics vs Ferroelectrics

Magnetization paramagnets vs ~~ferroelectric~~ ferromagnets

Define Magnetization

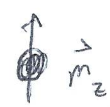
Recall $\frac{\partial \vec{P}}{\partial t} = \vec{J}_b$

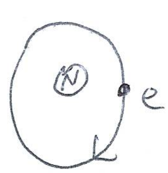
Now $\vec{\nabla}\times\vec{M}(\vec{r}) = \vec{J}_b$

For most cases

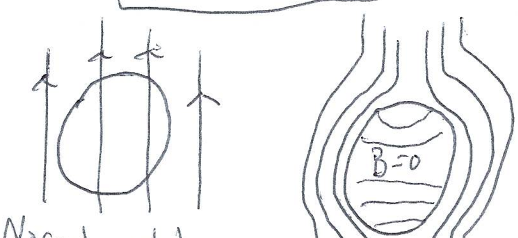
$\vec{M}$ as magnetic dipole density

$\vec{M}(\vec{r}) = \frac{1}{\Delta V} \int_{\Delta V} d^3r \, \vec{m}(\vec{r})$ Breaks down in case of super conductor

1) electron spin 

2) orbital motion 

Super conductor



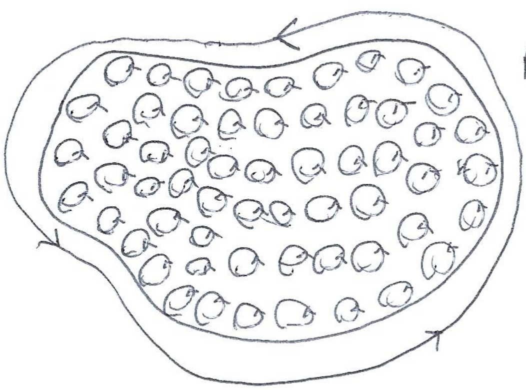
Normal metal

Super conductor

Magnetization

macroscopic currents on surface

Classical picture



many current loops

if $\vec{M}$ is uniform

$\vec{J}_b = \vec{0}$ inside and ~~$\vec{J}$~~ nonzero surface current

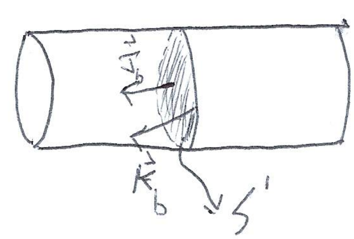
Defining property of loop ~~current~~ (bound) currents

Current through any interior surface must be zero

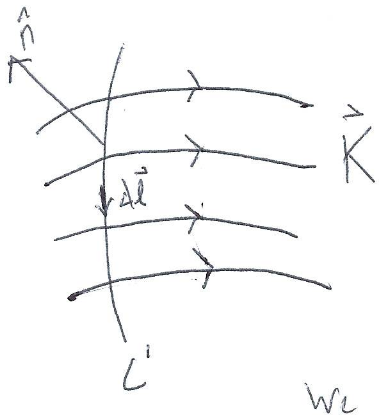
$\vec{J}_b$ bulk current density

$\vec{K}_b$ surface current density

$I_{S'}$ = bulk current $\quad I_{S'} = \int_{S'} d\vec{a} \cdot \vec{J}_b(\vec{r})$



View of surface



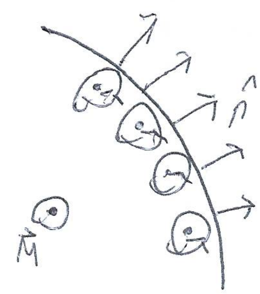
Current flowing through $d\vec{l}$

$d\vec{l} \cdot (\vec{K} \times \hat{n})$

$$I_{C'} = \oint d\vec{l} \cdot (\vec{K}_b \times \hat{n})$$

We require

$$I_{S'} + I_{C'} = 0$$



This is true if $\vec{K}_b = \vec{M}(\vec{r}_s) \times \hat{n}$

$$I_{S'} = \int d\vec{a} \cdot \vec{J}_b(\vec{r}) = \int d\vec{a} \cdot (\vec{\nabla} \times \vec{M}(\vec{r}))$$

$$I_{S'} = \oint_{C'} d\vec{l} \cdot \vec{M}(\vec{r}_s)$$

$$I_{C'} = \oint_{C'} d\vec{l} \cdot \left(\vec{M}(\vec{r}_s) \times \hat{n}\right) \times \hat{n}$$

$$= \oint d\vec{l} \cdot \left(\underbrace{\hat{n}(\vec{M}(\vec{r}_s) \cdot \hat{n})}_{0 \ \because \ d\vec{l} \cdot \hat{n} = 0} - \vec{M}(\vec{r}_s)(\hat{n} \cdot \hat{n}) \right)$$

$$I_{C'} = -\oint d\vec{l} \cdot \vec{M}(\vec{r}_s)$$

$$= -I_{S'}$$

Analogy $\boxed{\vec{P} \quad \rho_b = -\vec{\nabla} \cdot \vec{P} \quad \sigma_b = \vec{P} \cdot \hat{n}}$

$\vec{M}$ $\quad \vec{J}_b = \vec{\nabla} \times \vec{M}(\vec{r})$

$\vec{K}_b = \vec{M}(\vec{r}_s) \times \hat{n}$

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Electrodynamics

Maxwell's Equations

$$\vec{\nabla} \times \vec{E} = \frac{-\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

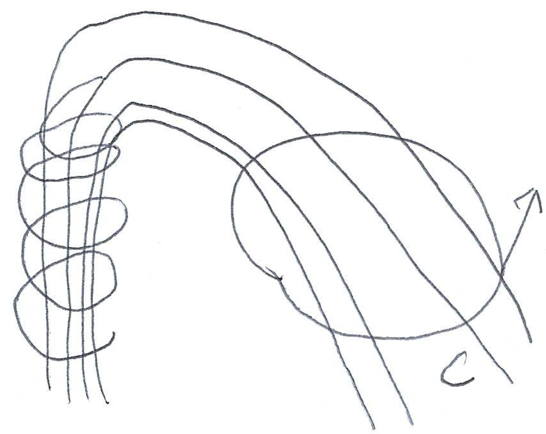
$\vec{\nabla} \times \vec{E} = 0$ (??) in conflict with $q(\vec{E} + \vec{v} \times \vec{B}) = \vec{F}$

and relativity

Galilean Invariance

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ in conflict with charge conservation

Consider



Spatially varying B field

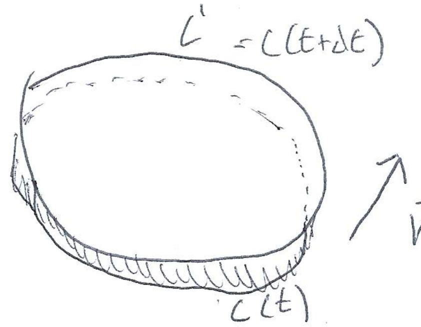
$$\oint \vec{f} \cdot d\vec{l} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

Loop moves with velocity $\vec{v}$

$\vec{f} = \vec{F}/q$ force per unit charge

$\vec{f} = \vec{v} \times \vec{B}$; consider $\oint \vec{f} \cdot d\vec{l}$, work done on q in circuit around C

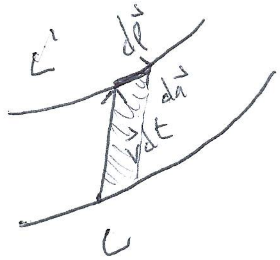
$\oint \vec{f} \cdot d\vec{l}$ = electromotive force = $\mathcal{E}$



C bounds surface S

C' bounds surface S' = S(t+dt)

We can choose S(t+dt) − S(t) to be the "ribbon" of surface connecting C to C'



$$d\vec a = \vec v dt \times d\vec l$$

Magnetic flux through S $\quad \Phi_S = \int_S \vec B \cdot d\vec a$

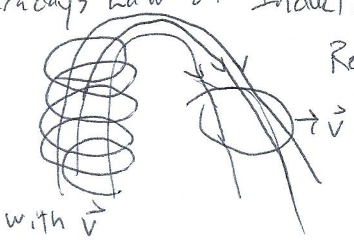
$$d\Phi = \int_{S'} \vec B \cdot d\vec a - \int_S \vec B \cdot d\vec a = \int_{\Delta S} \vec B \cdot d\vec a$$

$$d\Phi = \oint_C \vec B \cdot (\vec v dt \times d\vec l) \qquad \frac{d\Phi}{dt} = \oint \vec B \cdot (\vec v \times d\vec l)$$

$$\frac{d\Phi}{dt} = -\oint_C (\vec v \times \vec B) \cdot d\vec l = -\mathcal{E} \qquad \boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$

Faraday's Law of Induction

2 ways of looking at this experiment



1) $\mathcal{E} = \oint (\vec v \times \vec B) \cdot d\vec l$

$\boxed{2)\ \mathcal{E} = -\frac{d\Phi}{dt}}$

Universal Physical Law

Reference frame F' moving with $\vec v$
charges are not moving

(1) would be wrong

$\mathcal{E} = \frac{-d\Phi}{dt}$ implies something surprising

In frame f' we must have $\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = \frac{-d\Phi}{dt}$

$$\oint_c \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_s \vec{B} \cdot d\vec{a}$$

$$\parallel$$

$$\int_s (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = -\frac{d}{dt}\int_s \vec{B} \cdot d\vec{a}$$

Implies Local relation

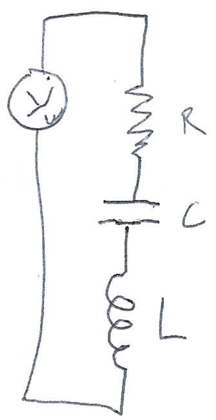
$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \equiv \mathcal{E} = -\frac{d\Phi}{dt}$$

Quasistatic phenomena

drop that term

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

Circuit Theory

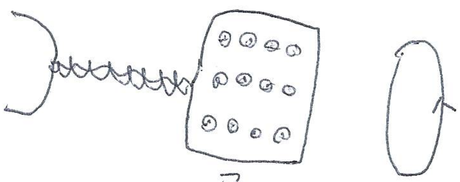


Validity of quasistatic approximation

$\omega < \frac{c}{L}$ L is typical dimension of device or circuit

Faraday ⇒ motors and generators

Generator



B

Flip Loop measurement of $\vec{B}$

$$\Phi = BA\cos\theta$$

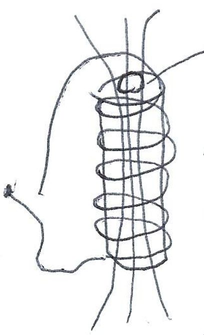
$$\frac{d\Phi}{dt} = -BA\sin\theta\frac{d\theta}{dt} = -\omega BA\sin\theta = -\mathcal{E}$$

Loop has resistance R

$$I(t) = \frac{\mathcal{E}}{R} = \frac{\omega BA\sin\omega t}{R}$$

Flip measurement

$$Q = \int_0^{\pi} I(t)\,dt = \frac{BA}{R}\int_0^{\pi} dt\,\sin\theta\frac{d\theta}{dt} = \boxed{\frac{2BA}{R} = Q}$$



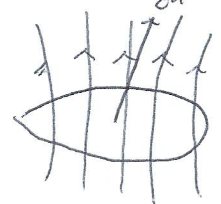
loop of wire

closing loop shoots loop up

$\uparrow \hat{z}$

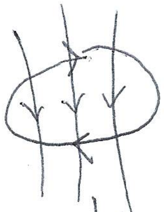
$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$d\vec{a}$



Φ increases when we close the switch

Induced current is in direction that opposes the change in flux "Lenz's Law"



$\downarrow \vec{m}$

induced current

magnetic moment points down

$$\vec{m} = -m\hat{z}$$

Recall $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$

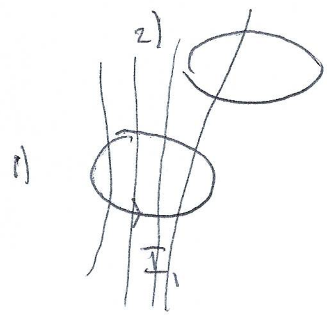
$$F_z = \frac{\partial}{\partial z} m_z B_z = m_z\frac{\partial B_z}{\partial z}$$

$m_z < 0$, $\frac{\partial B_z}{\partial z} < 0$

$F_z > 0$

Inductance

Consider the interaction between two loops of wire



Loops are of linked by "flux linkage"

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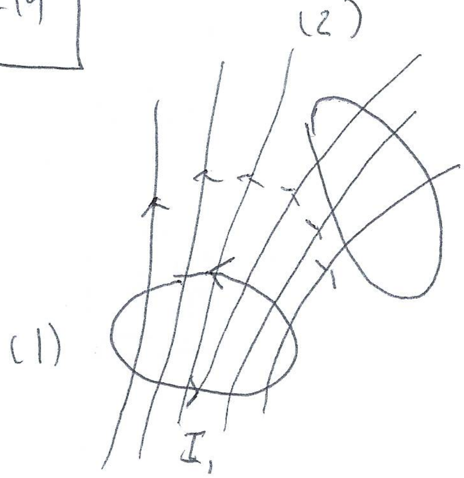
Inductance



Biot-Savart

$$\vec{B}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\vec{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

Identity about Φ

$$\Phi = \int \vec{B} \cdot d\vec{a} = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{a}$$
$$= \oint_C \vec{A} \cdot d\vec{l}$$

Flux through (2)

$$\Phi_2 = \int_{S_2} \vec{B}_1 \cdot d\vec{a}_2 = \oint_{C_2} \vec{A}_1 \cdot d\vec{l}_2 \; ; \; \vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{d\vec{l}_1}{\mathscr{r}}$$

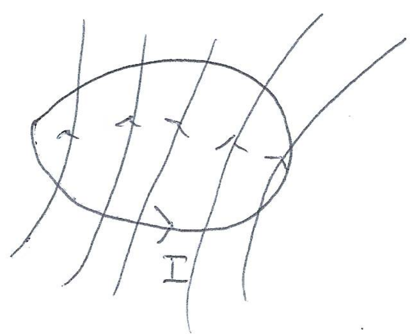
$$\boxed{\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}}}$$

Defines Mutual Inductance

$$\Phi_2 = M_{21} I_1$$

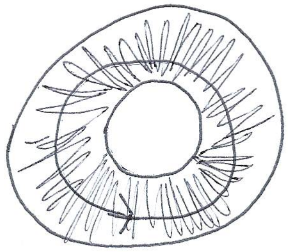
$$M_{21} = M_{12}$$

Self-inductance



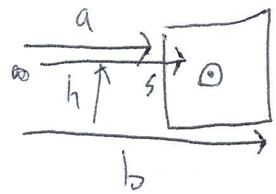
$$\Phi = L I$$

$L \equiv$ self inductance



Assume rectangular cross-section

$N = \#$ of loops



Toroid

Ampere's Law $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

$$2\pi s\, B(s) = \mu_0 I_{enc} = \mu_0 I N$$

$$B(s) = \frac{\mu_0 I N}{2\pi s}$$

Flux through single loop

$$\Phi = \frac{\mu_0 I N}{2\pi} \int_0^h dz \int_a^b ds\left(\frac{1}{s}\right) = \frac{\mu_0 I N h}{2\pi} \ln\left(\frac{b}{a}\right)$$

Total flux

$$\Phi = \frac{\mu_0 I N^2 h}{2\pi} \ln\left(\frac{b}{a}\right) \qquad L = \frac{\Phi}{I} = \frac{\mu_0 N^2 h \ln\left(\frac{b}{a}\right)}{2\pi}$$

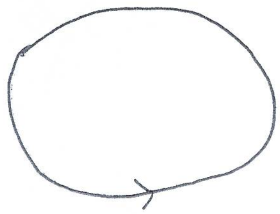
typical order of magnitude $L \sim \underbrace{\frac{(4\pi \times 10^{-7})}{2\pi}}_{\mu_0} \times \underbrace{(10^2)^2}_{N^2} \underbrace{(10^{-2})}_{h}$

$$L \sim 20\,\mu\text{farad}$$

Recall

$$U_{ele} = \frac{\epsilon_0}{2}\int d^3r\, E^2$$

Consider work required to generate B field



I

$$\mathcal{E} = -L\frac{dI}{dt} \qquad dW = -\mathcal{E}$$

~~dW~~ rate of doing work to change I

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}$$

$$dW = LI\,dI$$

Integrate from $I=0$ to I $\boxed{W = \frac{1}{2}LI^2}$

Recall

C

Express W in terms of $\vec{B}$

$$U = \frac{1}{2}CV^2 \qquad \Phi = LI = \oint \vec{A}\cdot d\vec{l}$$

$$W = \frac{1}{2}LII = \frac{1}{2}I\oint \vec{A}\cdot d\vec{l} = \frac{1}{2}\oint(\vec{A}\cdot \vec{I})dl$$

Generalizes to $W = \frac{1}{2}\int d^3r\, \vec{A}(\vec{r})\cdot\vec{J}(\vec{r})$

Quasistatics $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

$$W = \frac{1}{2\mu_0}\int d^3r\, \vec{A}(\vec{r})\cdot(\vec{\nabla}\times\vec{B})$$

Product Rule 6

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) = B^2 - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

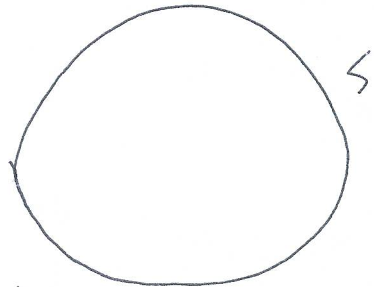
$$W = \frac{1}{2\mu_0}\int d^3r\, B^2 - \frac{1}{2\mu_0}\int d^3r\, \vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \frac{1}{2\mu_0}\int d^3r\, B^2 - \frac{1}{2\mu_0}\oint_S d\vec{a}\cdot(\vec{A}\times\vec{B})$$

Let S go to ∞ $\quad W = \frac{1}{2\mu_0}\int d^3r\, B^2$

$\vec{\nabla} \times \vec{E} \to -\frac{\partial \vec{B}}{\partial t}$

$$\boxed{\vec{\nabla} \times \vec{B} \cancel{=} \mu_0 \vec{J}}$$

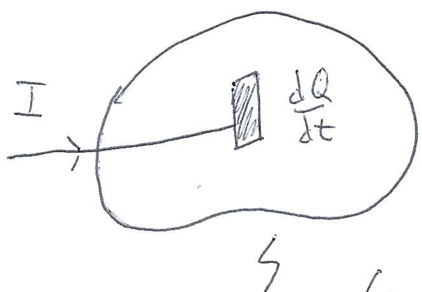
Incompatible with charge conservation



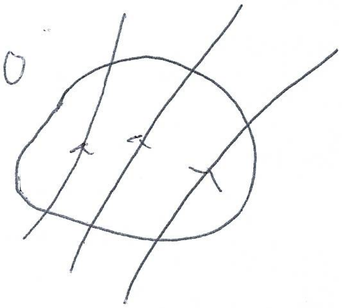
Consider $\vec{\nabla} \cdot (\qquad)$

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J}$ says $\vec{\nabla} \cdot \vec{J} = 0$

$$\int_V d^3r \, \vec{\nabla} \cdot \vec{J} = \oint_S \vec{J} \cdot d\vec{a}$$



$\int \vec{J} \cdot d\vec{a} \stackrel{?}{=} 0$



Conservation of charge

$$\boxed{\oint \vec{J} \cdot d\vec{a} = -\frac{d}{dt} Q_{enc}}$$

$\Downarrow$

$$\boxed{\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}}$$

$\vec{\nabla} \times \vec{B} \stackrel{?}{=} \mu_0 \vec{J} + (\text{something})$

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0 = \mu_0 \vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot (?)$

$\vec{\nabla} \cdot (?) = -\mu_0 \vec{\nabla} \cdot \vec{J} = \mu_0 \frac{\partial \rho}{\partial t}$

$$\boxed{(?) = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

$\vec{\nabla} \cdot \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right) = \mu_0 \frac{\partial \rho}{\partial t}$

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

Max well Equations

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E}=\frac{-\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Source Free

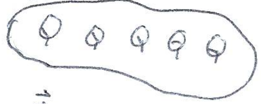
Maxwell's Equations in Matter

$$\rho=\rho_f+\rho_b$$

$$\vec{J}=\vec{J}_f+\vec{J}_{matter}$$

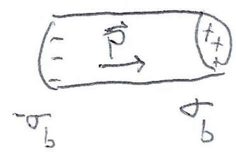
$$\vec{J}=\vec{J}_f+\vec{J}_b+\vec{J}_p$$

Magnetic material



$$\vec{J}_b=\vec{\nabla}\times\vec{M}$$

Polarized material



$$\vec{J}_p=\frac{\partial\vec{P}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho_f+\rho_b}{\epsilon_0} \qquad \rho_b=-\nabla\cdot\vec{P}$$

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho_f}{\epsilon_0}-\frac{\vec{\nabla}\cdot\vec{P}}{\epsilon_0} \qquad \vec{\nabla}\cdot(\epsilon_0\vec{E}+\vec{P})=\rho_f \qquad \boxed{\vec{\nabla}\cdot\vec{D}=\rho_f}$$

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}_f+\mu_0\vec{J}_b+\mu_0\vec{J}_p+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$=\mu_0\vec{J}_f+\mu_0\vec{\nabla}\times\vec{M}+\mu_0\frac{\partial\vec{P}}{\partial t}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\underbrace{(\vec{B}-\mu_0\vec{M})}_{\mu_0\vec{H}}=\mu_0\vec{J}_f+\mu_0\frac{\partial}{\partial t}\underbrace{(\epsilon_0\vec{E}+\vec{P})}_{\vec{D}}$$

$$\boxed{\mu_0\vec{\nabla}\times\vec{H}=\mu_0\vec{J}_f+\mu_0\frac{\partial\vec{D}}{\partial t}} \qquad \boxed{\vec{\nabla}\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t}}$$

Constitutive Relations

$$\vec{P}=P\{E,B\} \; , \; \vec{M}=M\{E,B\}$$

$\boxed{\text{Phys 110A 25 Apr 19}}$

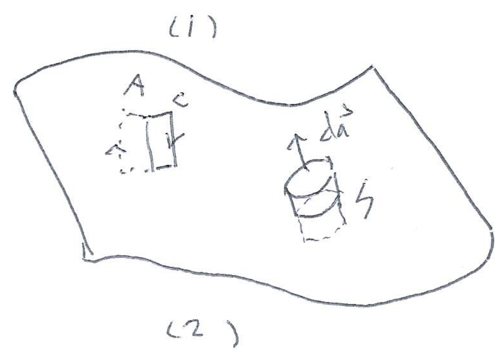
Maxwell Equations in Matter

$$\vec\nabla\cdot\vec B = 0 \qquad \vec\nabla\cdot\vec D = \rho_f \qquad \vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t} \qquad \vec\nabla\times\vec H = \vec J_f + \frac{\partial\vec D}{\partial t}$$

Constitutive Relations

$$\vec P = \vec P[\vec E, \vec B] \quad \vec M = \vec M[\vec E, \vec B]$$

Boundary conditions at interface



$$\vec\nabla\cdot\vec B = 0 \longleftrightarrow \oint_S \vec B\cdot d\vec a = 0$$

$$\vec\nabla\cdot\vec D = \rho_f \longleftrightarrow \oint \vec D\cdot d\vec a = Q_{free}$$

$$(\vec D_1 - \vec D_2)\cdot d\vec a = (D_{1\perp} - D_{2\perp})\,da = Q_{free}$$

$$\boxed{D_{1\perp} - D_{2\perp} = \sigma_f} \qquad \boxed{B_{1\perp} - B_{2\perp} = 0}$$

Integral form

$$\oint \vec E\cdot d\vec l = -\frac{d}{dt}\int_S \vec B\cdot d\vec a \qquad \oint \vec H\cdot d\vec l = I_{free,enc} + \frac{d}{dt}\int_S \vec D\cdot d\vec a$$

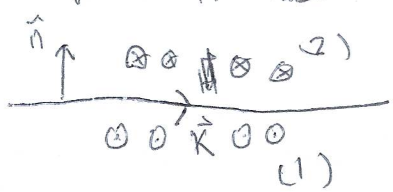
As area $\to 0$ $\quad \oint \vec E\cdot d\vec l = 0 \quad \oint \vec H\cdot d\vec l = I_{free,enc}$

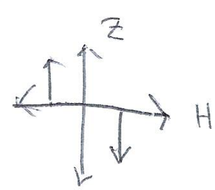
$\Downarrow$

Tangential $\vec E$ is continuous $\qquad I_{f,enc} \neq$ if there is a surface current $\vec K$

$$I_f = \vec K_f\cdot\hat n\times d\vec l \qquad \vec H_{1,\parallel} - \vec H_{2,\parallel} = \vec K\times\hat n$$

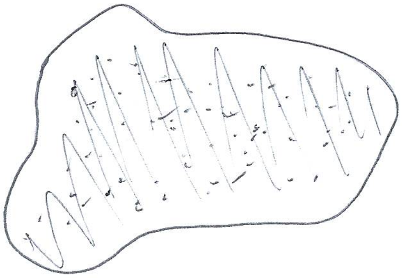
Example planar interface





Conservation of energy and momentum

Charged particles and E, B fields



$\mathcal{E}$ energy

$$\frac{d\mathcal{E}}{dt} = -\text{ Flux of } \mathcal{E}$$

$$\frac{d\vec{p}}{dt} = -\text{ Flux of } \vec{p}$$

Consider $\mathcal{E}$

$$\mathcal{E} = U_{em} + \mathcal{E}_{mech} \qquad \text{Equate } \frac{d\mathcal{E}_{mech}}{dt} = \frac{dW}{dt}$$

W is work done on the charge by the EM field

Consider single charge q

$$\vec{F} = q\vec{E} + q\,\vec{v}\times\vec{B}$$

$$dW = \vec{F}\cdot d\vec{l} \qquad \frac{dW}{dt} = \vec{F}\cdot\frac{d\vec{l}}{dt} = \vec{F}\cdot\vec{v} = q\,\vec{E}\cdot\vec{v}$$

For distribution of charges

$$\frac{dW}{dt} = \int d^3r\, \vec{J}\cdot\vec{E}$$

Eliminate $\vec{J}$ $\quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

Note to self

$$\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}E^2$$

$$\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\frac{dW}{dt} = \int d^3r\left(\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\cdot\vec{E} - \epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right)$$

Now

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot\underbrace{(\vec{\nabla}\times\vec{E})}_{-\frac{\partial\vec{B}}{\partial t}} - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$

$$\frac{dW}{dt} = -\frac{1}{\mu_0}\int d^3r\,\vec{\nabla}\cdot(\vec{E}\times\vec{B}) - \overbrace{\int d^3r\,\frac{\partial}{\partial t}\underbrace{\left(\frac{B^2}{\mu_0} + \epsilon_0E^2\right)}_{u\,\equiv\,\text{energy density stored in EM field}}}^{\frac{\partial}{\partial t}U_{EM}}$$

$$\cancel{\frac{d}{dt}\mathcal{E}_{mech} + \frac{d}{dt}U_{EM} = -\frac{1}{\mu_0}\int d^3r\, \vec{\nabla}\cdot(\vec{E}\times\vec{B})}$$

$$\boxed{\frac{d}{dt}\mathcal{E}_{mech} + \frac{d}{dt}U_{EM} = -\frac{1}{\mu_0}\int d^3r\, \vec{\nabla}\cdot(\vec{E}\times\vec{B})}$$

Time rate of change of total energy = − Flux of energy through S

Conservation of energy

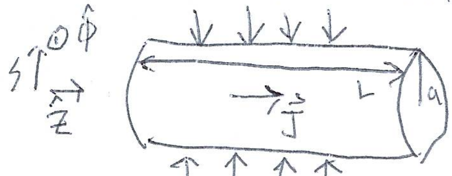
$$\int_V d^3r\, \vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \oint_S (\vec{E}\times\vec{B})\cdot d\vec{a}$$

This identifies $\frac{1}{\mu_0}\vec{E}\times\vec{B}$ as energy density current

This is Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

Example

Consider current flowing resistive wire



Ohm's Law $\vec{J} = \sigma\vec{E}$

steady-state

$$\frac{d\mathcal{E}}{dt} = -\int \vec{S}\cdot d\vec{a}$$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) \text{ , express } \vec{E} \text{ and } \vec{B} \text{ in terms of } \vec{J} \quad \vec{E} = \frac{\vec{J}}{\sigma} =$$

$$E_z = \frac{I}{\pi a^2\sigma}, \quad B_\phi = \frac{\mu_0 I}{2\pi a}$$

Ampere $\Rightarrow 2\pi a B_\phi = \mu_0 J \pi a^2$

$$B_\phi = \frac{\mu_0 J a}{2}$$

convert $J \to I$

$I = J\pi a^2$

$$S_s = -\frac{1}{\mu_0}\left(\frac{I}{\pi a^2\sigma}\right)\left(\frac{\mu_0 I}{2\pi a}\right) = \frac{-I^2}{2\pi^2 a^3\sigma}$$

Total rate that energy flows in

$$\oint_S \vec{S}\cdot d\vec{a} = \frac{-I^2}{2\pi^2 a^3\sigma}(2\pi a L) = \frac{-I^2 L}{\pi a^2\sigma}$$

R

Recall $R = \frac{\rho L}{A}$ $\quad\rho$ = resistivity

$$\oint_S \vec{S}\cdot d\vec{a} = I^2 R$$

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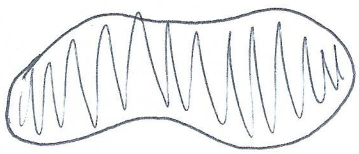
Energy Conservation



Charges and fields

$$\frac{dW}{dt} = \frac{d\mathcal{E}_{mech}}{dt} = \int d^3r\, \vec{J}\cdot\vec{E}$$

$$= -\int d^3r \frac{1}{\mu_0} \vec{\nabla}\cdot(\vec{E}\times\vec{B}) - \frac{du_{EM}}{dt}$$

$$\frac{d\mathcal{E}_{mech}}{dt} + \frac{du_{EM}}{dt} + \int d^3r\, \vec{\nabla}\cdot\vec{S} = 0$$

$$\underbrace{\frac{d}{dt}(\mathcal{E}_{mech} + U_{EM})}_{\text{rate of change of total energy}} + \underbrace{\oint_S \vec{S}\cdot d\vec{a}}_{\text{energy flux through surface}} = 0$$

Next conservation of momentum

For energy $\vec{S} = (\underbrace{\text{energy density}}_{\text{scalar}} \times \underbrace{\text{velocity}}_{\text{vector}})$

Consider momentum density current $= (\underbrace{\text{momentum density}}_{\text{vector}} \times \underbrace{\text{velocity}}_{\text{vector}})$

Example: $\overleftrightarrow{T}$, T_{xy} = flux of x-momentum moving in y direction

2nd rank tensor

$$\frac{d\vec{P}_{mech}}{dt} = \vec{F}_{mech} = \int d^3r\, \rho(\vec{E} + \vec{v}\times\vec{B}) = \int d^3r (\rho\vec{E} + \vec{J}\times\vec{B})$$

use $\cancel{\rho = \epsilon_0 \vec{\nabla}\times\vec{B}}$ $\rho = \epsilon_0 \vec{\nabla}\cdot\vec{E}$; $\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}$

$$\rho\vec{E} + \vec{J}\times\vec{B} = -\epsilon_0\mu_0\frac{\partial\vec{S}}{\partial t} + \epsilon_0\left((\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \frac{1}{2}\vec{\nabla}(E^2)\right) + \frac{1}{\mu_0}\left((\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}(B^2)\right)$$

Define Maxwell Stress Tensor

$$T_{ij} \equiv \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

$$\frac{d\vec{P}_{mech}}{dt} = -\int d^3r\, \epsilon_0\mu_0\frac{\partial\vec{S}}{\partial t} + \int d^3r (\vec{\nabla}\cdot\overleftrightarrow{T})$$

$$\underbrace{\frac{d\vec{P}_{mech}}{dt} + \underbrace{\int d^3r\, \epsilon_0\mu_0\frac{\partial\vec{S}}{\partial t}}_{\frac{\partial}{\partial t}\vec{P}_{EM}}}_{\text{rate of change of total momentum}} = \int d^3r(\vec{\nabla}\cdot\overleftrightarrow{T}) = \underbrace{\oint d\vec{a}\cdot\overleftrightarrow{T}}_{\text{momentum flux}}$$

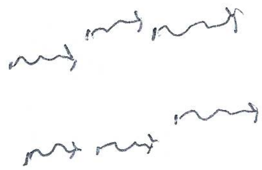
Summary

	Density	Current
Energy	$u = \frac{1}{2}(\epsilon_0 E^2 + B^2/\mu_0)$	$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$
Momentum	$\vec{g} = \epsilon_0\mu_0\vec{S}$	$-\overleftrightarrow{T}$

$$\vec{S} = \frac{1}{\epsilon_0\mu_0}\vec{g}$$

Consider a gas of photons

h = planck constant



velocity $= c$ $\quad \mathcal{E}_{photon} = hf \quad f$ = Frequency

$$p_{photon} = \frac{h}{\lambda}$$

eliminate h $\quad \mathcal{E}_{photon} = p_{ph} c \quad n_{photon} = \frac{\text{numbe of photons}}{\text{Volume}}$

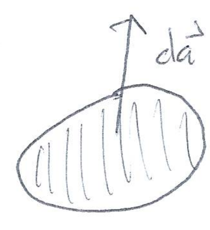
energy current $S = n_{photon} \mathcal{E}_{photon} c$, $g = n_{photon} p_{photon}$

$$S = \frac{g}{p_{photon}} \cdot \mathcal{E}_{photon} c = c^2 g \qquad c^2 = \frac{1}{\epsilon_0 \mu_0}$$

For steady state phenomena

$$\vec{F}_{mech} = \oint \overleftrightarrow{T} \cdot d\vec{a}$$

Consider force on small patch of surface



$d\vec{a} = \hat{z} da$

$$\delta \vec{F}_{mech} = \overleftrightarrow{T} \cdot d\vec{a}$$

$$= (\underbrace{T_{zx} \hat{x} + T_{zy} \hat{y}}_{\text{shear}} + \underbrace{T_{zz} \hat{z}}_{\text{pressure}}) \, da$$

Example $\dfrac{\delta \vec{F}_{mech,z}}{da} = \text{pressure}$

Example charged sphere

First: just charge on surface σ

$$\oint_{\text{north}} \overleftrightarrow{T} \cdot d\vec{a} \qquad d\vec{a} = \hat{r} R^2 \sin\theta \, d\theta \, d\phi = \hat{r}\, da$$

$$\delta F_z = \left(\overleftrightarrow{T} \cdot d\vec{a}\right)_z = T_{zx}\, \hat{x}\cdot\hat{r} + T_{zy}\, \hat{y}\cdot\hat{r} + T_{zz}\, \hat{z}\cdot\hat{r}$$

$$\hat{r} = \hat{z}\cos\theta + \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi$$

$$\delta F_z = \left(T_{zz}\cos\theta + T_{zy}\sin\theta\sin\phi + T_{zx}\cos\phi\sin\theta\right) da$$

$$T_{zx} = \epsilon_0 E_z E_x \;;\; T_{zy} = \epsilon_0 E_z E_y \;;\; T_{zz} = \epsilon_0 E_z^2 - \frac{\epsilon_0}{2} E^2$$

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{r} \qquad T_{zx} = \epsilon_0 \left(\frac{\sigma}{\epsilon_0}\right)^2 \cos\theta \sin\theta \cos\phi$$

$$T_{zy} = \epsilon_0 \left(\frac{\sigma}{\epsilon_0}\right)^2 \cos\theta \sin\theta \sin\phi$$

$$T_{zz} = \cancel{\epsilon_0\left(\frac{\sigma}{\epsilon_0}\right)^2}\; \epsilon_0 \left(\frac{\sigma}{\epsilon_0}\right)^2 \left(\cos^2\theta - \frac{1}{2}\right)$$

$$\delta F_z = \sum_i T_{zi}\, da_i = \epsilon_0 \left(\frac{\sigma}{\epsilon_0}\right)^2 R^2 \sin\theta\cos\theta \underbrace{\left(\underbrace{\underbrace{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi}_{\sin^2\theta} + \cos^2\theta}_{1} - \frac{1}{2}\right)}_{\frac{1}{2}} d\theta\, d\phi$$

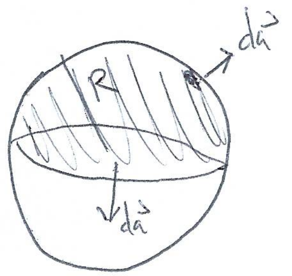
Total Force $F_z = \oint_{\text{North}} da\, \delta F_z$

$$F_z = \left(\frac{\epsilon_0}{2}\right)\left(\frac{\sigma}{\epsilon_0}\right)^2 2\pi R^2 \underbrace{\int_0^{\pi/2} d\theta \sin\theta\cos\theta}_{-\frac{\cos^2\theta}{2}\Big|_0^{\pi/2}}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q^2}{8R^2}$$

$$Q = 4\pi R^2 \sigma$$

Example charged sphere



Now spread charge uniformly

Include equatorial plane

$$d\vec{a} = -\hat{z}\,da = -\hat{z}\,r\,dr\,d\phi$$

$$\delta F_z = T_{zz}\,da \;; \quad \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}(\hat{x}\cos\phi + y\sin\phi)$$

$$T_{zz} = \epsilon_0 E_z^2 - \frac{1}{2}\epsilon_0 E^2 = -\frac{1}{2}\epsilon_0\left(\frac{Q(r)}{4\pi\epsilon_0 r^2}\right)^2 \;, \quad Q(r) = \frac{Qr^3}{R^3}$$

$$F_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 \int_0^{2\pi} d\phi \int_0^R r\,dr\,r^2$$

$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 2\pi\,\frac{1}{4}R^4$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q^2}{16} \qquad \text{Total is } \frac{Q^2}{4\pi\epsilon_0}\left(\frac{1}{8} + \frac{1}{16}\right)$$

110 A 30 Jan 19
Discussion

~~No~~ Room vacant before class

dr

David Dunsky

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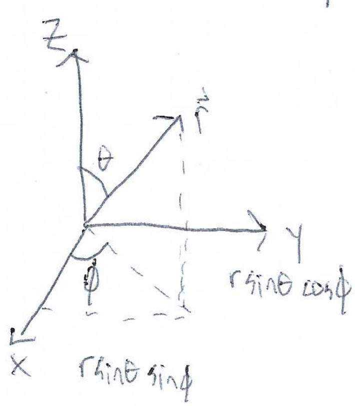
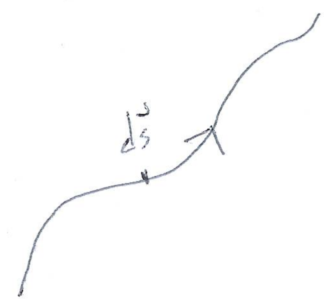
$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{\nabla}\cdot\vec{v} = \vec{\nabla}\cdot(v_x\hat{x} + v_y\hat{y} + v_z\hat{z}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{y}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$



$$\vec{r}\cdot\hat{x} = x = r\sin\theta\cos\phi$$
$$\vec{r}\cdot\hat{y} = y = r\sin\theta\sin\phi$$
$$\vec{r}\cdot\hat{z} = z = r\cos\theta$$



$$d\vec{s} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$d\vec{s}\cdot d\vec{s} = |d\vec{s}|^2 = dx^2 + dy^2 + dz^2$$

$$dx = d(r\sin\theta\cos\phi) = dr\sin\theta\cos\phi + r\cos\theta\cos\phi\,d\theta - r\sin\theta\sin\phi\,d\phi$$

$$dy = d(r\sin\theta\sin\phi) = dr\sin\theta\cos\phi + r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi$$

$$|d\vec{s}|^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

$$dz = d(r\cos\theta) = dr\cos\theta - r\sin\theta\,d\theta$$

$$dx^2 = dr^2\sin^2\theta\cos^2\phi + 2r\,dr\sin\theta\cos^2\phi\cos\theta\,d\theta - r\,dr\sin^2\theta\cos\phi\sin\phi\,d\phi + r^2\cos^2\theta\cos^2\phi - r^2\sin\theta\cos\theta\cos\phi\sin\phi\,d\phi$$

$$- r\,dr =$$

$$dx^2 = dr^2 + \cancel{r^2}d\theta^2$$

$$dy^2 = r^2 d\theta^2$$

$$dz^2 = r^2\sin^2\theta\,d\phi^2$$

$$d\vec{s} = (A\,dr\,\hat{r} + B\,d\theta\,\hat{\theta} + C\,d\phi\,\hat{\phi})\cdot(\quad) = A^2dr^2 + B^2d\theta^2 + C^2d\phi^2$$

$$A = 1,\quad B = r,\quad C = r\sin\theta$$

$$d\vec{s} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$df(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = (\hat{x}dx + \hat{y}dy + \hat{z}dz)\cdot\left(\hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}\right)$$

$$= (\vec{\nabla} f)\cdot(d\vec{s})$$

$$\underline{df} = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi = (\vec{\nabla} f)\cdot(dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi})$$

$$= dr(\nabla f)_r + r\,d\theta(\nabla f)_\theta + r\sin\theta\,d\phi(\nabla f)_\phi$$

$$(\nabla f)_r = \frac{\partial f}{\partial r} \qquad \vec{\nabla} f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$(\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$

$$(\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

1) The electric potential of a dipole is given by $V_{dip}(r,\theta) = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$

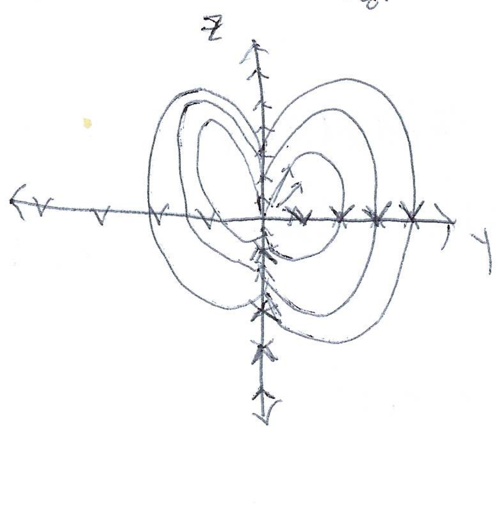
Find and sketch $\vec{E} = -\vec{\nabla} V$

$\vec{a}\cdot\vec{b} = ab\cos\theta$

$$\vec{\nabla} V = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right)$$

$$= \hat{r}\left(\frac{-2p\cos\theta}{4\pi\epsilon_0 r^2}\right) + \hat{\theta}\left(\frac{-p\sin\theta}{4\pi\epsilon_0 r^3 \cancel{\sin\theta}}\right) + \hat{\phi}(0) = \frac{-p\cos\theta}{2\pi\epsilon_0 r^2}\hat{r} + \frac{p\sin\theta}{4\pi\epsilon_0 r^3 \cancel{\sin\theta}}\hat{\theta}$$

$$-\vec{\nabla} V = +\frac{p\cos\theta}{2\pi\epsilon_0 r^3}\hat{r} + \frac{p\sin\theta}{4\pi\epsilon_0 r^3}\hat{\theta} = \vec{E}$$



Divergence in spherical coordinates

$$\vec{\nabla}\cdot\vec{v} = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}\right)$$

$$= \hat{r}\cdot\frac{\partial\vec{v}}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\vec{v}}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\vec{v}}{\partial\phi}$$

$$\frac{\partial\vec{v}}{\partial\theta} = \frac{\partial}{\partial\theta}\left(v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}\right) = \left(\frac{\partial v_r}{\partial\theta}\hat{r} + v_r\frac{\partial\hat{r}}{\partial\theta}\right) + \left(\frac{\partial v_\theta}{\partial\theta}\hat{\theta} + v_\theta\frac{\partial\hat{\theta}}{\partial\theta}\right) + \left(\frac{\partial v_\phi}{\partial\phi} + v_\phi\frac{\partial\hat{\phi}}{\partial\phi}\right)$$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

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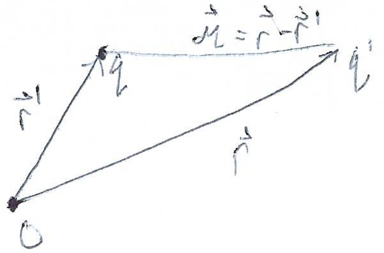
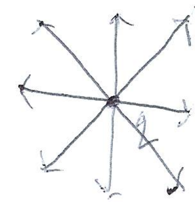
Discussion

Electric Fields and Gauss' Law



$$\frac{\vec{F}_q}{q_1} = \frac{q'q}{4\pi\epsilon_0 \mathscr{r}^2_{q'}}\hat{\mathscr{r}} = \vec{E}(\vec{r})$$



II Principle of Superposition

$$\vec{E}(\vec{r}) = \sum_i \frac{q_i}{4\pi\epsilon_0 \mathscr{r}_i^2}\hat{\mathscr{r}}_i \xrightarrow{\text{continuum limit}} \int_{\text{charges}} \frac{dq}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}}$$

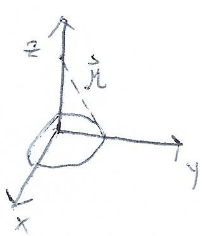
$$d\vec{E} = \frac{dq}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}}$$

$\cdot\, dq$

$\overline{ds}$ $\overline{da}$ $\overline{dV}$

$dq = \underset{\text{line}}{\lambda\, ds} = \underset{\text{area}}{\sigma\, dA} = \rho\, dV$

Find $\vec{E}$ a distance z above the axis of uniformly charged disc of radius R take limit $R >> z$ and $z >> R$



$$\int \frac{\sigma\, dA\, z\,\hat{z}}{4\pi\epsilon(z^2+r^2)\sqrt{z^2+r^2}} = \iint \frac{\sigma z r\, d\phi\, dr}{4\pi\epsilon_0(z^2+r^2)^{3/2}}\hat{z} \qquad \mathscr{r} = r^2 + z^2$$

$$= \int_0^R \int_0^{2\pi} \frac{\sigma z r}{4\pi\epsilon_0 (z^2+r^2)^{3/2}}\, d\phi\, dr\,\hat{z} = \int_0^R \frac{\sigma z r}{2\epsilon_0 (z^2+r^2)^{3/2}}\, dr\,\hat{z}$$

$u = z^2 + r^2$

$du = 2z\, dz + 2r\, dr$

$$= \frac{\sigma z}{2\epsilon_0}\int_0^R \frac{1}{2}\frac{1}{u^{3/2}}\, du\,\hat{z} = -\frac{\sigma z}{2\epsilon_0}\frac{1}{\sqrt{z^2+r^2}}\Bigg|_0^R \hat{z} = -\frac{\sigma z}{2\epsilon_0}\left(\frac{1}{\sqrt{z^2+R^2}} - \frac{1}{z}\right)\hat{z}$$

$$= -\frac{\sigma z}{2\epsilon_0}\left(\frac{z - \sqrt{z^2+R^2}}{z\sqrt{z^2+R^2}}\right)\hat{z} = -\frac{\sigma}{2\epsilon_0}\left(\frac{z - \sqrt{z^2+R^2}}{\sqrt{z^2+R^2}}\right)\hat{z}$$

For $R >> z$ $\quad \frac{\sigma}{2\epsilon_0}\hat{z}$

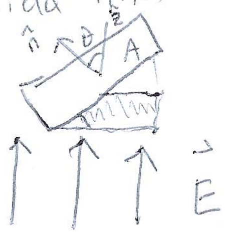
$\frac{R}{z} << 1$ $\quad$ For $\quad \frac{1}{\sqrt{z^2+R^2}} = \frac{1}{\sqrt{z^2\left(1+\frac{R^2}{z^2}\right)}} = \frac{1}{|z|}\left(1 + \left(\frac{R}{z}\right)^2\right)^{-1/2} = \frac{1}{|z|}\left(1 - \frac{1}{2}\left(\frac{R^2}{z^2}\right)\right)$

For $R << z$

$$\vec{E}(z) = \hat{z}\,\frac{\sigma}{4\epsilon_0}\left(\frac{R^2}{z^2}\right) = \frac{q}{4\pi\epsilon_0 z^2}$$

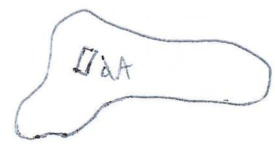
Gauss' Law

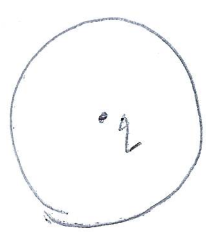
$$\# \text{ field lines} \propto A \propto \vec{E} \qquad \vec{E} = \int \frac{dq}{4\pi\epsilon_0 r^2}\,\hat{r}$$



$$\Phi = \#\text{ field lines} = EA\cos\theta = \vec{E}\cdot\vec{A}$$

$$\Phi = \int d\Phi = \int \vec{E}\cdot d\vec{A}$$





$$\Phi = \oint \vec{E}\cdot d\vec{A} = \int \frac{dq}{4\pi\epsilon_0 R^2}\hat{R}\cdot d\vec{A}$$

$$= \int \frac{dq}{(4\pi\epsilon_0 R^2)}\left(R^2 \sin\theta\, d\theta\, d\phi\right)$$

$$= \int \frac{q}{4\pi\epsilon_0 R^2}\sin\theta\, d\theta\, d\phi$$

$$\int \frac{2\pi q}{4\pi\epsilon_0 R^2}\sin\theta\, d\theta = \frac{q}{\epsilon_0}$$

$$\Phi = \oint \vec{E}\cdot d\vec{A} = \frac{q_{inside}}{\epsilon_0}$$

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{in}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_{volume} \rho\, dV$$

$$\int_{volume} (\vec{\nabla}\cdot\vec{E})\, dV = \int \frac{\rho}{\epsilon_0}\, dV \qquad \vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

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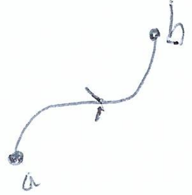
Discussion

Image Charges

$$V(b) - V(a) = -\int_a^b \vec{E}\cdot d\vec{s} \rightarrow \vec{E} = -\vec{\nabla} V$$



$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

$$\int_V \vec{\nabla}\cdot\vec{E}\, d\tau = \int_V \frac{\rho}{\epsilon_0} d\tau$$

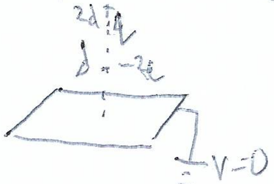
$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} = \vec{\nabla}\cdot(-\vec{\nabla}V) = -\nabla^2 V$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}} \leftarrow \text{Poisson's Equation}$$

unique

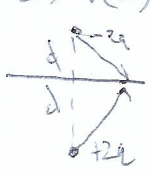
A solution to $\nabla^2 V$ and boundary conditions then it is <u>the</u> solution

Ex 1) 3.7 Find force on charge q if the xy plane is a conductor ~~with~~ a held at V=0

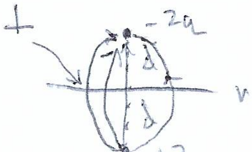


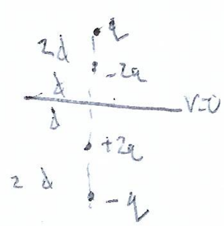
$$V(r) - V(\infty) = -\int_\infty^r \left(\frac{q\,\hat{r}}{4\pi\epsilon_0 r^2}\right)\cdot d\vec{s} = -\int_\infty^r \frac{q}{4\pi\epsilon_0 r^2} dr = \frac{q}{4\pi\epsilon_0 r}$$

5 16 9 14 4



$$V(s) = \frac{-2q}{4\pi\epsilon_0\sqrt{d^2+s^2}} + \frac{2q}{4\pi\epsilon_0\sqrt{d^2+s^2}} = 0$$





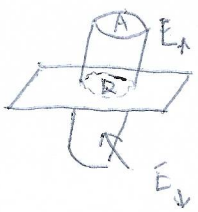
$$F_q = \frac{q}{4\pi\epsilon_0}\left(\frac{-2q}{d^2} + \frac{2q}{(3d)^2} - \frac{q}{(4d)^2}\right) = \frac{q^2}{4\pi\epsilon_0}\left(-\frac{1}{d^2} + \frac{2}{9d^2} - \frac{1}{16d^2}\right)$$

$$= \frac{q^2}{4\pi\epsilon_0}\left(\frac{-144+32-9}{144}\right) = \frac{q^2}{4\pi\epsilon_0}\left(\frac{-121}{144}\right)$$

$$\vec F_q = q\vec E = q\left[\frac{-2q}{4\pi\epsilon_0(2d)^2}\hat z + \frac{2q}{4\pi\epsilon_0(4d)^2}\hat z - \frac{q}{4\pi\epsilon_0(6d)^2}\hat z\right]$$

$$= \frac{\hat z\, q^2}{4\pi\epsilon_0 d^2}\left(-\frac{29}{72}\right)$$

b) What is the induced surface charge density σ



$$\oint \vec E \cdot d\vec A = E_{\uparrow\perp}A - E_{\downarrow\perp}A = \frac{Q_{in}}{\epsilon_0}$$

$$E_\uparrow - E_\downarrow = \frac{\sigma}{\epsilon_0} \qquad E_\downarrow = 0$$

$$E_\uparrow = \frac{\sigma}{\epsilon_0}$$

$$E_{\uparrow\perp} - E_{\downarrow\perp}\Big|_{\text{surface}} = \frac{\sigma}{\epsilon_0}\Big|_{\text{surface}}$$

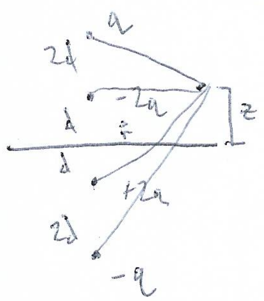
$$\vec E = -\vec\nabla V$$

$$\hat n \cdot \vec E = \hat n \cdot (-\vec\nabla V)$$

$$-\frac{\partial V}{\partial z}\Big|_{z=0} = \frac{\sigma}{\epsilon_0}$$

$$E_n = -\frac{\partial V}{\partial n}\Big|_{\text{surface}}$$

$\uparrow$ directional derivative

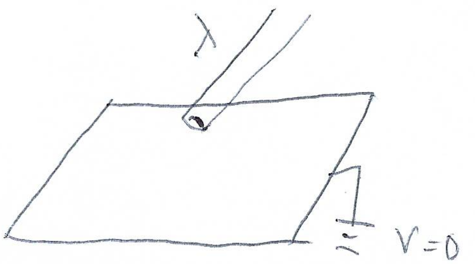


$$V(\vec r) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{s^2+(3d-z)^2}} - \frac{2q}{\sqrt{s^2+(d-z)^2}} + \frac{2q}{\sqrt{s^2+(d+z)^2}} - \frac{q}{\sqrt{s^2+(3d+z)^2}}\right]$$

$$\frac{\partial V}{\partial z} = \frac{q}{4\pi\epsilon_0}\left[\frac{3d-z}{(s^2+(3d-z)^2)^{3/2}} - \frac{2(d-z)}{(s^2+(d-z)^2)^{3/2}} \mp \frac{2(d+z)}{\left(\sqrt{s^2+(d+z)^2}\right)^3} + \frac{(3d+z)}{(s^2+(3d+z)^2)^{3/2}}\right]_{z=0}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{6d}{(s^2+9d^2)^{3/2}} - \frac{4d}{(s^2+d^2)^{3/2}}\right] \rightarrow \sigma(s) = -\epsilon_0\frac{\partial V}{\partial z}$$

$$\sigma(s) = \frac{-q}{4\pi}\left[\frac{6d}{(s^2+9d^2)^{3/2}} - \frac{4d}{(s^2+d^2)^{3/2}}\right]$$

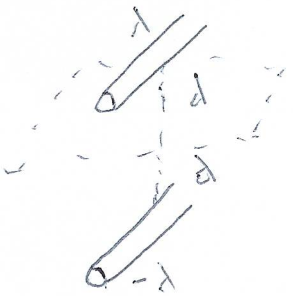


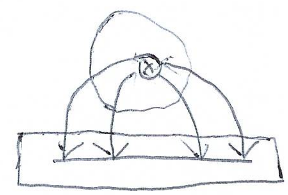
a) what is voltage everywhere

b) what is the electric field everywhere

c) what is the induced surface charge density on the conducting surface

d) what is the ~~induced surface~~ total induced charge on the surface





$$\Phi_w = \frac{\lambda l}{\epsilon_0} = \text{\# of field lines}$$

$$\Phi_p = \frac{q_{in}}{\epsilon_0} = \text{\# of field lines } \cancel{\text{from}}$$

$$\frac{q_{in}}{\epsilon_0} = -\frac{\lambda l}{\epsilon_0} \Rightarrow q = -\lambda l$$

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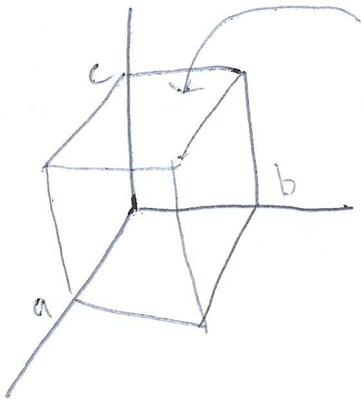
Discussion



$\Phi(x,y,0) = V(x,y)$

Separation of Variables

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

$\vec{E} = -\vec{\nabla}\Phi$

$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$

$\boxed{\nabla^2\Phi = 0}$ Laplace's equation

$\Phi(x,y,z) = X(x)\,Y(y)\,Z(z)$

$$\nabla^2\Phi = \frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi + \frac{\partial^2}{\partial z^2}\Phi = 0$$

$$\frac{d^2}{dx^2}XYZ + X\frac{d^2Y}{dy^2}Z + XY\frac{d^2Z}{dz^2} = 0$$

$$X''YZ + XY''Z + XYZ'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$\frac{X''}{X} = \alpha^2, \quad \frac{Y''}{Y} = \beta^2 = \frac{Z''}{Z} = \gamma^2, \quad \alpha^2 + \beta^2 + \gamma^2 = 0$$

If $\alpha^2 > 0$, $X(x) = Ae^{\alpha x} + Be^{-\alpha x}$

If $\alpha^2 < 0$, $X(x) = A\cos\alpha x + B\sin\alpha x$ $\rightarrow$ α^2 negative

β^2 negative

γ^2 positive

Redefine $\alpha^2 \to -\alpha^2$ $\qquad \gamma^2 = \alpha^2 + \beta^2$

$\beta^2 \to -\beta^2$

$$\frac{X''}{X} = -\alpha^2 , \quad \frac{Y''}{Y} = -\beta^2 , \quad \frac{Z''}{Z} = \gamma^2$$

$$\Phi_p(x,y,z) = (A\cos\alpha x + B\sin\alpha x)(C\cos\beta y + D\sin\beta y)(Ee^{\gamma z} + Fe^{-\gamma z})$$

$X(x) = A\cos\alpha x + B\sin\alpha x$

$Y(y) = C\cos\beta y + D\sin\beta y$

$Z(z) = Ee^{\gamma z} + Fe^{-\gamma z}$

$\Phi(0,y,z) = 0 \quad \Phi(a,y,z) = 0$

$\Phi(x,0,z) = 0 \quad \Phi(x,b,z) = 0$

$\Phi(x,y,0) = 0 \quad \Phi(x,y,c) = \Phi$

$\Phi_p(0,y,z) = A\,Y(y)Z(z) = 0 \to A = 0$

$\Phi_p(x,0,z) = C\,X(x)Z(z) = 0 \to C = 0$

$\Phi_p(x,y,0) = X(x)Y(y)(E+F) = 0 \to E = -F$

$$\Phi_p(x,y,z) = (B\sin\alpha x)(D\sin\beta y)(Ee^{\gamma z} - Ee^{-\gamma z})$$
$$= A\sin\alpha x\sin\beta y(e^{\gamma z} - e^{-\gamma z})$$

$\Phi_p(a,y,z) = A\sin\alpha a\sin\beta y(e^{\gamma z} - e^{-\gamma z}) = 0 \to \sin\alpha a = 0$

$\alpha a = n\pi$

$\boxed{\alpha = \frac{n\pi}{a}}$

$\Phi_p(x,b,z) = A\sin\alpha x\sin\beta b(e^{\gamma z} - e^{-\gamma z}) = 0$

$\sin\beta b = 0$

$\beta b = m\pi$

$\boxed{\beta = \frac{m\pi}{b}}$

$$\Phi_p = A\sin\alpha_n x\sin\beta_m y(e^{\gamma_{nm}z} - e^{-\gamma_{nm}z})$$

$$\Phi = \sum \Phi_p = \sum_{\substack{n=1\\m=1}}^{\infty} A_{nm}\sin\alpha_n x\sin\beta_m y(e^{\gamma_{nm}z} - e^{-\gamma_{nm}z})$$

$$\gamma = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}$$

$$\Phi(x,y,c) = \sum_{n,m} A_{nm}\sin\alpha_n x\sin\beta_m y(e^{\gamma_{nm}c} - e^{-\gamma_{nm}c}) = V(x,y)$$

$$\Phi(x,y,c) = \sum_{n,m} A_{nm} \sin\alpha_n x \sin\beta_m y \left(e^{\gamma_{nm} c} - e^{-\gamma_{nm} c}\right) = V(x,y)$$

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n'\pi x}{a}\right) dx = \int_0^a \frac{1}{2} \cancel{(\cos(a-b))}$$

$$= \int_0^a \frac{1}{2}\left(\underbrace{\cos\left(\frac{\pi x}{a}(n'-n)\right)}_{\sin\left(\frac{\pi x}{a}(n'-n)\right)} - \underbrace{\cos\left(\frac{\pi x}{a}(n'+n)\right)}_{\sin\left(\frac{\pi x}{a}(n'+n)\right)}\right) dx$$

$$\int_0^a V(x,y) \sin(\alpha_{n'} x)\, dx = \int_0^a \sum_{n,m} A_{nm} \sin\alpha_n x \sin\beta_y y \left(e^{\gamma_{nm} c} - e^{-\gamma_{nm} c}\right) \sin\alpha_{n'} x\, dx$$

$$\sum_{n,m} A_{nm} \frac{a}{2} \delta_{n'm} \frac{b}{2} \delta_{m'm} \left(e^{\gamma c} - e^{-\gamma c}\right)$$

$$= A_{n'm'} \left(\frac{ab}{4}\right)\left(e^{\gamma_{n'm'} c} - e^{-\gamma_{n'm'} c}\right) = \int_0^b \int_0^a V(x,y) \sin\alpha_n x \sin\beta_n y\, dx\, dy$$

$$A_{n'm'} = \frac{4}{ab} \frac{1}{e^{\gamma_{nm'} c} - e^{-\gamma_{nm'} c}} \int_0^b \int_0^a V(x,y) \sin\alpha_n x \sin\beta_m y\, dx\, dy$$

Let's look $V(x,y) = V_0$

$$\therefore A_{nm} = \frac{4}{ab} \frac{1}{e^{\gamma_{nm} c} - e^{-\gamma_{nm} c}} V_0 \cancel{\int \sin\alpha_n x\, dx} \left(\frac{-1}{\alpha_n} \cos(\alpha_n x)\right)\Big|_0^a \left(\frac{-1}{\beta_m} \cos(\beta_n y)\right)\Big|_0^b$$

$\frac{a}{n\pi}$ $\quad 1 - \cos n\pi \quad$ $\frac{b}{m\pi}$ $\quad 1 - \cos \pi m$

$$= \frac{4V_0}{nm\pi^2} \frac{1}{e^{\gamma_{nm} c} - e^{-\gamma_{nm} c}} \underbrace{(1 - \cos n\pi)}_{\substack{2 \text{ odd } n \\ 0 \text{ even } n}} \underbrace{(1 - \cos m\pi)}_{\substack{2 \text{ odd } m \\ 0 \text{ even } m}}$$

$$= \begin{cases} \dfrac{16 V_0}{nm} \dfrac{1}{e^{\gamma_{nm} c} - e^{-\gamma_{nm} c}} & n,m \text{ odd} \\ 0 & n,m \text{ even} \end{cases}$$

Phys 110A 27 Feb 2019
Discussion

Laplace's Equation in Spherical Coordinates
+ Multipole Expansion

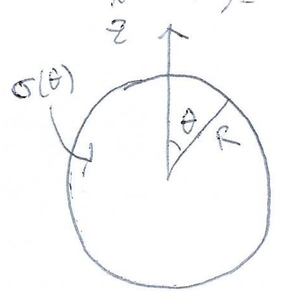
Prob 3.49 modified: Find the exact potential of a spherical shell with a surface charge density $\sigma(\theta) = P_0 \cos\theta$ inside and outside as well as multipole expansion outside the sphere. Show it is a ~~pu d~~ pure dipole

$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$, $\vec{E} = -\vec{\nabla} V$ $V = \Phi$

$\vec{\nabla}^2 \Phi = \frac{-\rho}{\varepsilon_0} = 0$ $\Phi(x,y,z) = X(x)\, Y(y)\, Z(z)$

$\Phi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$ $\Phi(\phi)$ is constant

$R(r) = A r^{\ell} + \frac{B}{r^{\ell+1}}$, $\Theta(\theta) = P_\ell(\cos\theta) = \begin{cases} 1 & \ell = 0 \\ \cos\theta & \ell = 1 \\ \frac{1}{2}(3\cos^2\theta - 1) & \ell = 2 \end{cases}$



$$\Phi_{in} = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta) \qquad B_\ell = 0$$

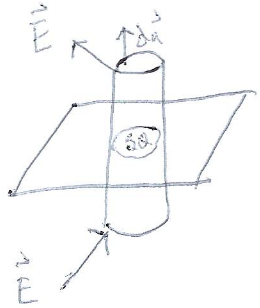
$$\Phi_{out} = \sum_{\ell=0}^{\infty}\left(C_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta) \qquad C_\ell = 0$$

$$\Phi_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \quad , \quad \Phi_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

continuity at $r=R$ $\Phi_{in}(R,\theta) = \Phi_{out}(R,\theta)$

$$\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$\sum_{\ell=0}^{\infty}\left[A_\ell R^\ell - \frac{B_\ell}{R^{\ell+1}}\right] P_\ell(\cos\theta) = 0 \qquad A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \qquad A_\ell = \frac{B_\ell}{R^{2\ell+1}}$$



$$\oint \vec{E}\cdot d\vec{a} = E_{\uparrow,\perp}\,\delta a - E_{\downarrow,\perp}\,\delta a = \frac{\delta Q}{\epsilon_0}$$

$$E_{\uparrow,\perp}\,\delta a - E_{\downarrow,\perp}\,\delta a = \frac{\sigma}{\epsilon_0}$$

evaluate at surface

$$\hat{n}\cdot\vec{E} = \hat{n}\left(-\vec{\nabla}\Phi\right)$$

$$E_n = -\frac{\partial\Phi}{\partial n} \qquad -\frac{\partial\Phi_{out}}{\partial r} - \left(-\frac{\partial\Phi_{in}}{\partial r}\right)\Bigg|_{surface} = \frac{\sigma}{\epsilon_0}$$

$$\Phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$\frac{\partial\Phi_{in}}{\partial r}\Bigg|_{r=R} = \sum l A_l R^{l-1} P_l(\cos\theta)$$

$$\frac{\partial\Phi}{\partial r}\Bigg|_{r=R} = \sum -(l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta)$$

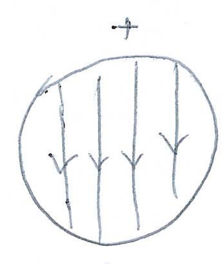
$$\sum_{l=0}^{\infty}\Big((l+1)\frac{B_l}{R^{l+2}} + l \underbrace{A_l}_{\frac{B_l}{R^{2l+1}}} R^{l-1}\Big)P_l(\cos\theta) = \frac{\sigma(\theta)}{\epsilon_0}$$

$$\sum_{l=0}^{\infty}\left((l+1)\frac{B_l}{R^{l+2}} + \frac{lB_l}{R^{l+2}}\right)P_l(\cos\theta) = \sum_{l=0}^{\infty}\frac{B_l(2l+1)}{R^{l+2}}P_l(\cos\theta) = \frac{\sigma(\theta)}{\epsilon_0} = \frac{P_0\cos\theta}{\epsilon_0} = \frac{P_0 P_1(\cos\theta)}{\epsilon_0}$$

$$B_l = 0 \;\; \forall\, l \neq 1 \qquad \frac{B_1(3)}{R^3} = \frac{P_0}{\epsilon_0} \qquad B_1 = \frac{P_0 R^3}{3\epsilon_0},\; A_1 = \frac{B_l}{R^{2l+1}} = \frac{P_0}{3\epsilon_0}$$

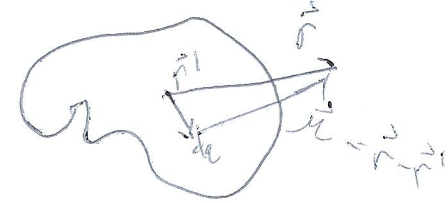
$$\Phi_{in} = A_1 r\cos\theta = \frac{P_0 r}{3\epsilon_0}\cos\theta \qquad \Phi_{out} = \frac{B_1}{r^2}\cos\theta = \frac{P_0 R^3}{3\epsilon_0 r^2}\cos\theta$$

$$-\vec{\nabla}\Phi_{in} = \vec{E} = -\frac{\partial\Phi}{\partial z}\hat{z} = -\frac{P_0}{3\epsilon_0}\hat{z}$$



$$\Phi_{out} = \frac{P_0 R^3}{3\epsilon_0 r^2}\cos\theta \qquad V_{mono} = \frac{Q}{4\pi\epsilon_0 r} \qquad V_{dip} = \frac{\vec{p}\cdot\hat{\mathscr{r}}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

$$\frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \frac{P_0 R^3 \cos\theta}{3\epsilon_0 r^2} \qquad p = P_0 \frac{4}{3}\pi R^3$$



$$V = \int \frac{dq}{4\pi\epsilon_0 \mathscr{r}} = \int \frac{dq}{4\pi\epsilon_0 r}\frac{1}{\sqrt{1+x^2-2x\cos\theta}}$$

$$V_{tot} = \int_{charge} \frac{dq}{4\pi\epsilon_0 \mathscr{r}}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta')$$

$$\mathscr{r} = \sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}} = \sqrt{(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')}$$

$$= \sqrt{r^2 + \vec{r}'^2 - 2\vec{r}\cdot\vec{r}'}$$

$$= \sqrt{r^2 + \vec{r}'^2 - 2rr'\cos\theta}$$

$$\frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta' + \dots\right)$$

$$V(r) = \int \frac{dq}{4\pi\epsilon_0}\left(\frac{1}{r} + \frac{r'\cos\theta'}{r^2}\right)$$

$$\frac{r'}{r} = x \ll 1$$

$$= \sqrt{r^2\left(\left(\frac{r'}{r}\right)^2 + 1 - 2\left(\frac{r'}{r}\right)\cos\theta\right)}$$

$$= r\sqrt{1 + x^2 - 2x\cos\theta}$$

$$= \int \frac{dq}{4\pi\epsilon_0 r} + \int \frac{dq\, r'\cos\theta'}{4\pi\epsilon_0 r^2}$$

$\downarrow$ monopole $= \frac{Q_{tot}}{4\pi\epsilon_0 r}$

$\downarrow$ $\frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}$ $\vec{p} = \int dq\, r'\cos\theta'$

$$V_{mono} = 0 \qquad V_{dip} = \int \frac{dq\, r'\cos\theta'}{4\pi\epsilon_0 r^2} \qquad dq = \sigma(\theta)\, da$$

$$= \sigma(R^2 \sin\theta\, d\theta'\, d\phi')$$

$$= \delta(r'-R)\,\sigma(\theta)\, dr^2 \sin\theta'\, d\theta'\, d\phi'$$

$$\int \sigma(\theta) \qquad \int \frac{(P_0\cos\theta')(R^2\sin\theta'\, d\theta'\, d\phi')(R\cos\theta')}{4\pi\epsilon_0 r^2}$$

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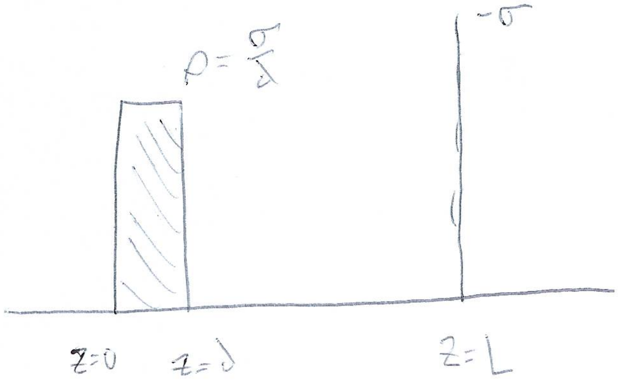
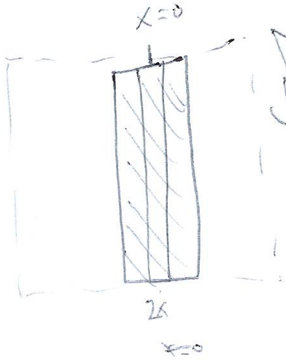
Discussion



da deals only with flux surfaces



$$\oint \vec{E}\cdot d\vec{a} = E\,2A = \frac{Q_{in}}{\epsilon_0} = \frac{\rho A(2x)}{\epsilon_0}$$

$$\vec{E} = \frac{\rho \vec{x}}{\epsilon_0} = \frac{\sigma \vec{x}}{d\epsilon_0} \quad \text{inside: } \left(-\frac{d}{2} < x < \frac{d}{2}\right)$$

Outside: $\oint \vec{E}\cdot d\vec{a} = E\,2A = \frac{\rho A d}{\epsilon_0}$

$$\vec{E} = \frac{\rho d}{2\epsilon_0} \quad \text{outward}$$

~~$Z<0$~~

$$\vec{E}_{sheet} = \frac{-\sigma}{2\epsilon_0} \quad (\text{inwards})$$

$$z<0 \qquad \vec{x} = \vec{z} - \frac{\vec{d}}{2}$$

$$\frac{-\sigma}{2\epsilon_0}\hat{z} + \frac{\sigma}{2\epsilon_0}\hat{z} = \vec{0}$$

$0<z<d$ $\quad \frac{\sigma}{d\epsilon_0}\left(\vec{z} - \frac{\vec{d}}{2}\right) + \frac{\sigma}{2\epsilon_0}\hat{z} = \frac{\sigma \hat{z}\, z}{d\epsilon_0}$

$d<z<L$ $\quad \frac{\sigma\hat{z}}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0}\hat{z} = \frac{\sigma}{\epsilon_0}\hat{z}$

$z>L$

$$\frac{\sigma\hat{z}}{2\epsilon_0} - \frac{\sigma\hat{z}}{2\epsilon_0} = \vec{0}$$

b) Energy gain of an electron from $z < 0$ to $z > L$

$$\Delta E = W_{\substack{\text{done on} \\ \text{electron}}} = \int_{z<0}^{z>L} \vec{F} \cdot d\vec{s} = -e \int \vec{E} \cdot d\vec{s}$$

$$= -e \left(\int_0^d \frac{\sigma \vec{z}}{\epsilon_0 d} \cdot d\vec{z} + \int_d^L \frac{\sigma}{\epsilon_0} \hat{z} \cdot d\vec{z} \right)$$

$$= -e \left(\frac{\sigma}{\epsilon_0 d} \left(\frac{1}{2} d^2 \right) + \frac{\sigma}{\epsilon_0} (L - d) \right)$$

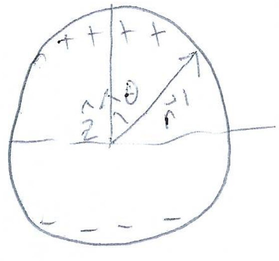
$$= \frac{-e\sigma}{\epsilon_0} \left[L - \frac{d}{2} \right]$$

c) Pressure that wants to expand the slab thickness

$$P = \frac{du}{dV} = \frac{d}{dV} \left[\frac{\epsilon_0}{2} \int \vec{E}^2 dV \right] = \frac{\epsilon_0}{2} |\vec{E}|^2 \Big|_{\text{surface}}$$

$$= \frac{\epsilon_0}{2} \left(\frac{\sigma}{\epsilon_0} \right)^2 = \frac{\sigma^2}{2\epsilon_0}$$

2.



$\sigma(\theta) = \sigma_0 \cos\theta$

a) dipole moment

$$\vec{p} = \int_{\text{charges}} dq\, \vec{r} = \int \sigma\, da\, \vec{r}\,'$$

$$\vec{p} = \vec{p}\cdot\hat{z}\,\hat{z} = \hat{z}\,\hat{z}\cdot\int \sigma\, da\, \vec{r}\,' = \vec{p} = \hat{z}\int \sigma\, da\, R\cos\theta'$$

$$= \hat{z}\int(\sigma_0\cos\theta')(R^2\sin\theta'\, d\theta' d\phi')(R\cos\theta')$$

$$= 2\pi\,\hat{z}\,\sigma_0 R^3 \int_0^{\pi} \cos^2\theta' \sin\theta'\, d\theta'$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$= \hat{z}\,\sigma_0 R^3\, 2\pi \int_1^{-1} -u^2\, du = \frac{-2\pi\sigma_0 R^3 \hat{z}}{3}\, u^3\Big|_1^{-1} = \frac{4\pi\sigma_0 R^3\hat{z}}{3}$$

b)

i) Find $\vec{F}$ on physical dipole

$$\vec{F} = -\vec{\nabla}U = -\vec{\nabla}(-\vec{p}\cdot\vec{E}) = (\vec{p}\cdot\vec{\nabla})\vec{E}_{\text{image}}$$

$$\vec{E}_{dip} = -\vec{\nabla}\left(\frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}\right) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

$$\vec{F} = \frac{q}{4\pi\epsilon_0}\left[\frac{-q}{(2L-d)^2} + \frac{q}{(2L)^2} + \frac{q}{(2L)^2} - \frac{q}{(2L+d)^2}\right]$$

$$\frac{-q^2\hat{z}}{4\pi\epsilon_0(2L)^2}\left[\frac{1}{\left(1-\frac{d}{2L}\right)^2} + \frac{1}{\left(1+\frac{d}{2L}\right)^2} - 2\right] = \frac{-q^2\hat{z}}{16\pi\epsilon_0 L^2}\left[\left(1-2\left(-\frac{d}{2L}\right)+3\left(\frac{d}{2L}\right)^2\right) + \left(1-2\left(\frac{d}{2L}\right)+3\left(\frac{d}{2L}\right)^2 - 2\right)\right]$$

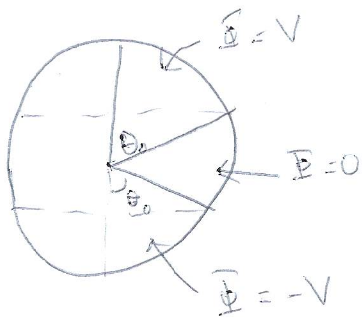
$f(x) = (1+x)^{-2} = 1 - 2x + \frac{6x^2}{2}$

$f'(x) = -2(1+x)^{-3} = -2$

$f''(x) = 6(1+x)^{-4} = 6$

$$= \frac{-q^2\hat{z}}{16\pi\epsilon_0 L^2}\left(\frac{3d^2}{2L^2}\right) = \frac{-3p^2\hat{z}}{32\pi\epsilon_0 L^4}$$

3)



a) find A_1

$$\Phi = \sum A_l r^l P_l(\cos\theta)$$

$$V(\theta) = \begin{cases} V & 0 < \theta_0 \\ 0 & \theta_0 < \theta < \pi - \theta_0 \\ -V & \pi - \theta_0 < \theta < \pi \end{cases}$$

$$\Phi(R,\theta) = \sum A_l R^l P_l(\cos\theta) = V(\theta)$$

$$P_m(\cos\theta)\sin\theta\, d\theta \sum A_l R^l P_l(\cos\theta) = P_m(\cos\theta)\sin\theta\, d\theta\, V(\theta)$$

$$= \sum \frac{2}{2l+1} A_l R^l \delta_{lm} = \int_0^{\pi} V(\theta) P_m(\cos\theta)\, d\theta \sin\theta$$

$$A_m = \frac{2m+1}{2R^m} \int_0^{\pi} V(\theta) P_m(\cos\theta)\sin\theta\, d\theta$$

$$A_1 = \frac{3}{2R}\int_0^{\pi} V(\theta)\cos\theta\sin\theta\, d\theta$$

$$= \frac{3V}{2R}\left(\int_0^{\theta_0} \cos\theta\sin\theta\, d\theta - \int_{\pi-\theta_0}^{\pi} \cos\theta\sin\theta\, d\theta\right) \qquad \begin{matrix} u = \cos\theta \\ du = \sin\theta \end{matrix}$$

$$= \frac{3V}{2R}\left(\int_1^{\cos\theta_0} -u\,du - \int_{\cos(\pi-\theta_0)}^{-1} -u\,du\right) = \frac{3V}{2R}\left(\frac{1}{2}(1-\cos^2\theta_0) - \frac{1}{2}(\cos^2\theta_0 - 1)\right)$$

$$= \frac{3V}{2R}\sin^2\theta_0$$

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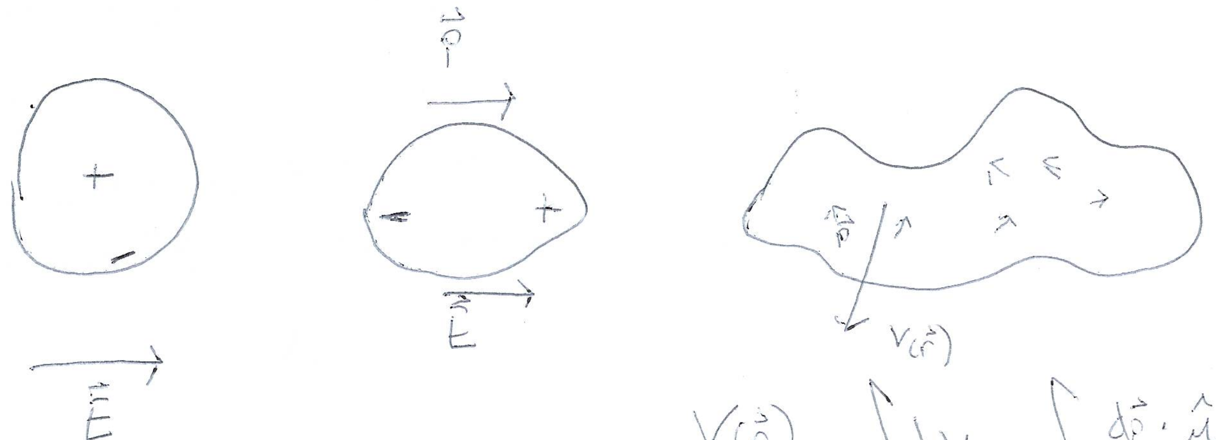
Discussion

Polarization and Electric Displacement



$$dV_{dip} = \frac{d\vec{p}\cdot\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}$$

$$\vec{P} = \frac{d\vec{p}}{d\tau}$$

dipole moment per unit volume

$$d\vec{p} = \vec{P}\,d\tau$$

$$V(\vec{r}) = \int dV_{dip} = \int_{\text{dipoles}} \frac{d\vec{p}\cdot\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}$$

$$= \int \frac{\vec{P}\cdot\hat{\mathscr{r}}\,d\tau}{4\pi\epsilon_0 \mathscr{r}^2}$$

$$\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \int_{\text{dielectric}} \frac{\vec{P}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)}{4\pi\epsilon_0}\,d\tau$$

$$\cancel{V_{\text{dielectric}} = \frac{1}{4\pi\epsilon_0}\left(\int \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)dV - \int\frac{1}{\mathscr{r}}\vec{\nabla}\right.}$$

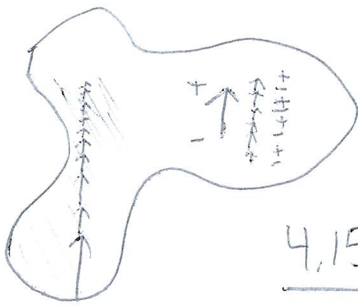
$$V_{\text{dielectric}} = \frac{1}{4\pi\epsilon_0}\left(\int_{\text{volume}} \vec{\nabla}'\cdot\left(\frac{\vec{P}}{\mathscr{r}}\right)d\tau - \int_{\text{volume}} \frac{1}{\mathscr{r}}\,\vec{\nabla}'\cdot\vec{P}(\vec{r}')\,d\tau\right)$$

$$= \frac{1}{4\pi\epsilon_0}\left(\oint_{\text{surface}} \frac{\vec{P}\cdot d\vec{a}}{\mathscr{r}} + \int_{\text{volume}} \frac{1}{\mathscr{r}}\left(-\vec{\nabla}'\cdot\vec{P}\right)d\tau\right)$$

$$V = \int \frac{dq}{4\pi\epsilon_0 \mathscr{r}} \qquad dq = \sigma\, da \text{ or } \rho\, d\tau$$

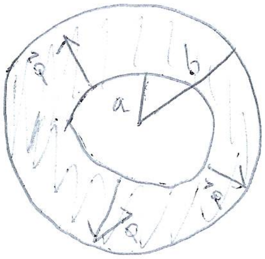
$$\vec{P}\cdot d\vec{a} = \vec{P}\cdot\hat{n}\, da$$

$$\boxed{\vec{P}\cdot\hat{n} = \sigma_B} \qquad \boxed{-\vec{\nabla}\cdot\vec{P} = \rho_B}$$



4.15 Thick spherical shell of inner radius a and outer radius b is made of a dielectric material with "Frozen-in" polarization $\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$

Find $\vec{E}$ everywhere using 2 different methods



$$\sigma_B = \vec{P}\cdot\hat{n} \qquad \sigma_B(r=a) = \vec{P}(a)\cdot(-\hat{r}) = -\frac{k}{a}$$

$$\sigma_B(r=b) = \vec{P}(b)\cdot\hat{r} = \frac{k}{b}$$

$$\rho_B = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left(\frac{k}{r}\hat{r}\right) = -\frac{k}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{r}\right) = -\frac{k}{r^2}$$

$$\cancel{V = \frac{1}{4\pi\epsilon_0}\int}$$

$$\oint \vec{E}\cdot d\vec{a} = E(r)4\pi r^2 = \frac{q_{in}(r)}{\epsilon_0} \qquad q_{in} = ? \quad r<a \quad q_{in} = 0$$

$r > b$

$$q_{in} = -4\pi k b + \frac{k}{b}4\pi b^2 = 0$$

$a<r<b$ $\quad q_{in} = \left(-\frac{k}{a}\right)(4\pi a^2) + \int_a^r -\frac{k}{r'^2}$

$$q_{in} = \left(-\frac{k}{a}\right)(4\pi a^2) + \int_a^r \frac{-k}{r'^2} r'^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= -4k\pi a + \int_a^r -4\pi k(r-a)$$

$$= -4\pi k r$$

$$\vec{E}(\vec{r}) = \begin{cases} \vec{0} & r<a \\ -\frac{k}{r\epsilon_0}\hat{r} & a<r<b \\ \vec{0} & r>b \end{cases}$$

$$\vec\nabla\cdot\vec E = \frac{\rho}{\epsilon_0} \qquad \epsilon_0\,\vec\nabla\cdot\vec E = \rho = \rho_B + \rho_f = -\vec\nabla\cdot\vec P + \rho_f$$

$$\sigma_B = \vec P\cdot\hat r \qquad \rho_B = -\vec\nabla\cdot\vec P$$

$$\vec\nabla\cdot\underbrace{(\epsilon_0\vec E + \vec P)}_{\vec D} = \rho_f$$

$$\vec D = \epsilon_0\vec E + \vec P$$ Electric displacement

$$\vec\nabla\cdot\vec D = \rho_f \;\rightarrow\; \oint \vec D\cdot d\vec a = \frac{Q_{\text{free inside}}}{\cancel{\epsilon_0}}$$

$$\oint \vec D\cdot d\vec a = D(\vec r)\,4\pi r^2 = Q_{\text{free}}(r) = 0 \;\forall r$$

$$\vec D = 0 = \epsilon_0\vec E + \vec P$$

$$\vec E(\vec r) = -\frac{\vec P(\vec r)}{\epsilon_0} = \begin{cases} \vec 0 & r<a \\ -\frac{k}{r\epsilon_0}\hat r & a<r<b \\ \vec 0 & r>b \end{cases}$$

$$\vec D = \epsilon_0\vec E + \vec P$$

linear dielectric $\quad \vec P \propto \vec E \qquad \vec P = \epsilon_0\,\chi\,\vec E$

$\uparrow$ electric susceptibility

$$\vec D = \epsilon_0\vec E + \epsilon_0\chi\vec E = \underbrace{(1+\chi)\epsilon_0}_{\epsilon = \epsilon_0(1+\chi)}\vec E$$

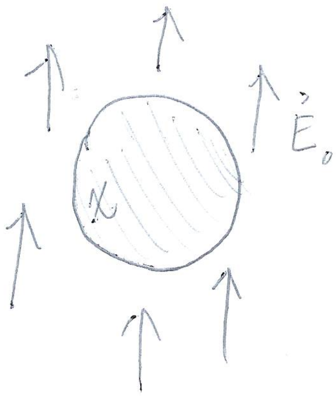
$d\tau$

$$\epsilon = \epsilon_0(1+\chi) = \epsilon_0\,\epsilon_r$$

$$\epsilon_r = (1+\chi)$$

Ex 4.7 but much easier

A sphere of homogeneous linear dielectric is placed in a uniform field $\vec{E}_0$. Find $\vec{E}$ inside sphere.



$$\vec{E}_{in} = \vec{E}_0 - \frac{\vec{P}}{3\epsilon_0} = \vec{E}_0 - \frac{\epsilon_0 \chi \vec{E}_{in}}{3\epsilon_0}$$

$$\vec{E}_{in}\left(1 + \frac{\chi}{3}\right) = \vec{E}_0$$

$$\vec{E}_{in} = \frac{\vec{E}_0}{1+\frac{\chi}{3}} = \frac{3E_0}{2+\epsilon_r}$$

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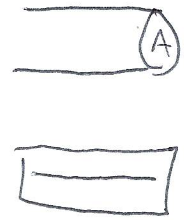
Discussion

Magnetic Fields in Matter

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\nabla^2\vec{A} = -\mu_0\vec{J} \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$



J

K

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \qquad \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}')\,d\tau}{\mathscr{r}}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')\,d\tau}{\mathscr{r}} \qquad \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')\,d\tau}{\mathscr{r}} = \frac{\mu_0}{4\pi}\int\frac{\vec{K}(\vec{r}')\,dA'}{\mathscr{r}}$$

$$= \frac{\mu_0}{4\pi}\int\frac{\vec{I}(\vec{r}')\,d\vec{s}'}{\mathscr{r}}$$

$\vec{I}$

$\cos\alpha = \hat{r}\cdot\hat{r}'$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\oint\frac{I(\vec{r}')\,d\vec{s}}{\mathscr{r}} \quad , \quad \frac{1}{\mathscr{r}} = \begin{cases}\dfrac{1}{r}\displaystyle\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\alpha) & r > r' \\ \dfrac{1}{r'}\displaystyle\sum_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l P_l(\cos\alpha) & r < r'\end{cases}$$

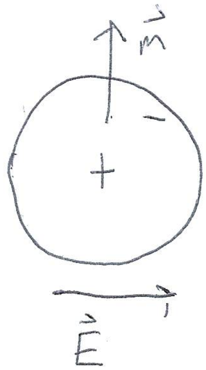
$$\cancel{\frac{\mu_0}{4\pi}\oint\frac{I(\vec{r}')\,d\vec{s}}{\mathscr{r}}}$$

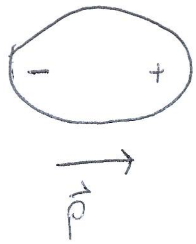
$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\Big(\underbrace{\frac{1}{r}\oint d\vec{s}}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\oint r'\cos\alpha\,d\vec{s}}_{\text{dipole}} + \dots\Big)$$

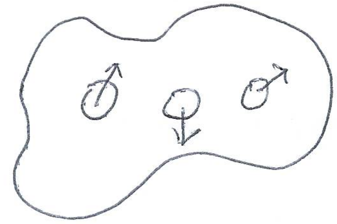
$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint \overbrace{r'\cos\alpha}^{\vec{r}'\cdot\hat{r}}\,d\vec{s}' \qquad \text{Eq 1.108}$$

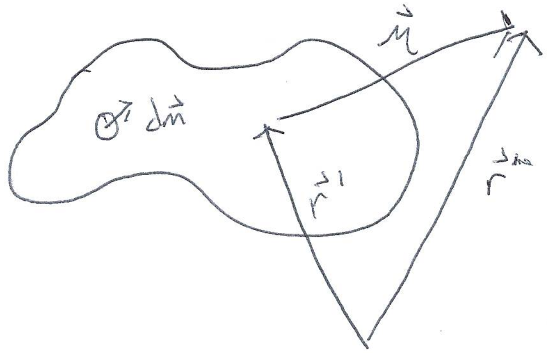
$$= \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} \; , \; \vec{m} = I\int d\vec{A}$$

$$V_{dip}(\vec{r}) = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}$$









$\vec{A}(\vec{r}) = ?$

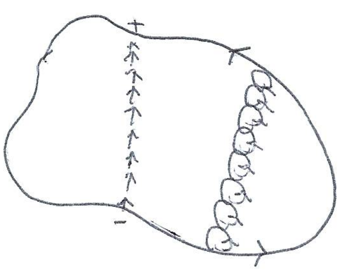
$$\vec{M} = \frac{d\vec{m}}{d\tau} = \text{magnetization}$$

$$\int d\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{d\vec{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \vec{A}(\vec{r})$$

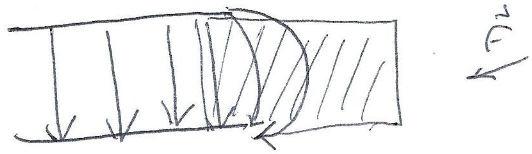
$$= \int \frac{\mu_0}{4\pi} \frac{\vec{M}(\vec{r}') \times \hat{\mathscr{r}}\, d\tau}{\mathscr{r}^2}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{1}{\mathscr{r}}\left(\vec{\nabla}' \times \vec{M}(\vec{r}')\right) d\tau + \frac{\mu_0}{4\pi}\oint \left(\vec{M}(\vec{r}') \times \hat{n}\right) \cdot da'$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \qquad \vec{K}_b = \vec{M} \times \hat{n}\Big|_{\text{surface}}$$

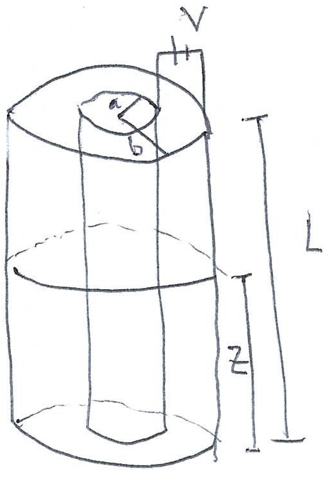


Phys 110A 17Apr19
Midterm Review



$U = -\vec{p}\cdot\vec{E}$

$-\vec{\nabla}U = \vec{F} = \cancel{p}\ \vec{\nabla}(-\vec{p}\cdot\vec{E})$



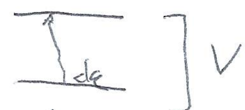
$dW = \vec{F}_{me}\cdot dz + dW_{battery}$

$= dW_{me} + dW_{batter}$

$= F_{me}\,dz + V\,dq = -F_{ele}\,dz + V\,dq$

$W = U = \frac{1}{2}CV^2$ $\quad q = CV(q)$

$\frac{dq}{dz} = \frac{dC}{dz}V$

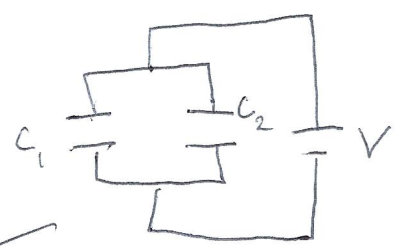


$\frac{dW}{dz} = \frac{1}{2}\frac{dC}{dz}V^2, \quad \frac{dq}{dz} = \frac{dC}{dz}V$

$dW = V\,dq = \frac{q}{C}dq$

$C = \frac{q}{V(q)}$

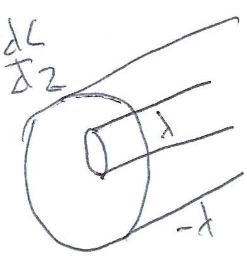
$W = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}CV^2$



$\frac{dW}{dz} = -F_{ele} + V\frac{dq}{dz}$

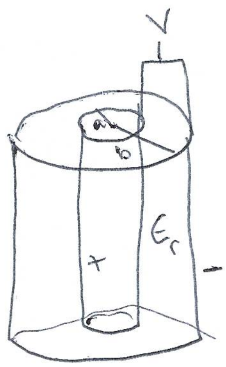
$\frac{1}{2}\frac{dC}{dz}V^2 = -F_{ele} + V^2\frac{dC}{dz}$

$F_{ele} = \frac{1}{2}\frac{dC}{dz}V^2$



$\vec{\nabla}\cdot\vec{D} = \rho_f$

$\oint \vec{D}\cdot d\vec{a} = Q_{f,enc}$



$C = \frac{Q}{V}$

$D(\vec{r})\,2\pi r l = \lambda l \qquad \vec{D}(\vec{r}) = \frac{\lambda}{2\pi r}\hat{s}$

$\vec{E} = \frac{\lambda\hat{s}}{2\pi\epsilon_0\epsilon_r s}$

$= \epsilon_0\epsilon_r\vec{E}$

$V(a) - V(b) = V = -\int_b^a \vec{E}\cdot d\vec{s} = -\int_b^a \frac{\lambda\,ds}{2\pi\epsilon_0\epsilon_r s} = \frac{\lambda}{2\pi\epsilon_0\epsilon_r}\ln\left(\frac{b}{a}\right)$

$C = \frac{\lambda l}{\left(\frac{\lambda}{2\pi\epsilon_0\epsilon_r}\ln\left(\frac{b}{a}\right)\right)} = \frac{2\pi\epsilon_0\epsilon_r l}{\ln\left(\frac{b}{a}\right)}$

$$C_{total} = C_1 + C_2 = \frac{2\pi\epsilon_0\epsilon_r z}{\ln\left(\frac{b}{a}\right)} + \frac{(L-z)\,2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$$

$$= \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}\left(z\epsilon_r + (L-z)\right)$$

$$\vec{F}_{elc} = \frac{1}{2}\frac{dC}{dz}V^2\hat{z} = \frac{1}{2}V^2\left(\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}(\epsilon_r - 1)\right)\hat{z}$$

Equilibrium $\vec{F}_{ele} = \vec{F}_{gravity}$

$\rho\pi(b^2-a^2)z$

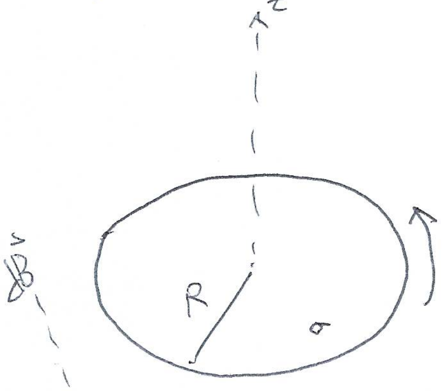
$$\frac{\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}V^2(\epsilon_r - 1)\hat{z} - m(z)g\hat{z} = \vec{0}$$

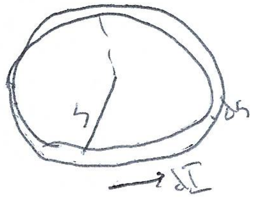
$$z = \frac{\frac{\pi\epsilon_0 V^2}{\ln\left(\frac{b}{a}\right)}(\epsilon_r - 1)}{\rho g\pi(b^2-a^2)} = \frac{\epsilon_0 V^2(\epsilon_r - 1)}{\ln\left(\frac{b}{a}\right)\rho g}$$

$\epsilon_r - 1 = \chi$

$$= \frac{\epsilon_0 V^2 \chi}{\ln\left(\frac{b}{a}\right)\rho g}$$

6) 5.48





$\vec{B}(z)$, $z >> R$

$\vec{m} = ? \rightarrow \vec{B}_{dip}$

$\vec{B}_{ring}(z) = \frac{\mu_0 I}{2} \frac{R^3 \hat{z}}{(R^2+z^2)^{3/2}}$

$\vec{B}_{dip} = \vec{\nabla} \times \vec{A}_{dip}$

$= \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$

$d\vec{B}(z) = \frac{\mu_0}{2} \frac{s^2 \hat{z}\, dI}{(s^2+z^2)^{3/2}}$

$= \hat{z} \frac{\mu_0 s^2 K ds}{2\ (s^2+z^2)^{3/2}}$ $\quad K \rightarrow \sigma v$

$v = \omega s$

$dI = \vec{J} \cdot d\vec{a}$

$= \vec{K} \cdot d\vec{l}$

$\vec{J} = \frac{d\vec{I}}{da}$, $\vec{K} = \frac{d\vec{I}}{dx}$

$= \rho\vec{v}$, $= \sigma\vec{v}$

$\vec{I} = \lambda\vec{v}$

$d\vec{B} = \frac{\hat{z}\ \mu_0 s^3 \omega \sigma\, ds}{2(s^2+z^2)^{3/2}}$

$\vec{B}(z) = \int d\vec{B} = \frac{\hat{z}\ \sigma\omega\mu_0}{2}\int_0^R \frac{s^3 ds}{(s^2+z^2)^{3/2}} = \frac{\hat{z}\ \mu_0\sigma\omega}{2}\left(\frac{R^2+2z^2}{\sqrt{z^2+R^2}} - 2z\right)$

Taylor expansion

$g = \cancel{\frac{R^2+2z^2}{\sqrt{z^2+R^2}}}$

$g = \frac{1}{\sqrt{z^2+R^2}} = \frac{1}{\sqrt{z^2(1+\frac{R^2}{z^2})}} = \frac{1}{z}(1+x^2)^{-1/2}$ $\quad x = \frac{R}{z}$

$f(x) = (1+x^2)^{-1/2} = 1 - \frac{1}{2}\left(\frac{R}{z}\right)^2$

$f(0) = 1$, $f'(x) = -\cancel{\tfrac{1}{2}}\,x(1+x^2)^{-3/2}\,\cancel{2x}\Big|_0 = 0$

$f''(x) = -(1+x^2)^{-3/2} - x\left(-\frac{3}{2}\right)(1+x^2)^{-5/2}\,2x\Big|_0 = -1$

let $f(y) = (1+y)^{-1/2}$, $y = \left(\frac{R}{z}\right)^2$

$f(0) = 1$

$$f'(y) = -\frac{1}{2}(1+y)^{-3/2}\Big|_{y=0} = -\frac{1}{2}$$

$$f''(y) = \frac{3}{4}(1+y)^{-5/2}\Big|_{y=0} = \frac{3}{4}$$

$$f(y) = 1 - \frac{1}{2}y + \frac{1}{2}\frac{3}{4}y^2$$

$$= 1 - \frac{1}{2}\left(\frac{R}{z}\right)^2 + \frac{3}{8}\left(\frac{R}{z}\right)^4$$

$$\vec{B}(z) = \hat{z}\,\frac{\mu_0 \sigma \omega}{2}\left((R^2 + 2z^2)\frac{1}{z}\left(1 - \frac{1}{2}\left(\frac{R}{z}\right)^2 + \frac{3}{8}\left(\frac{R}{z}\right)^4 + \cdots\right) - 2z\right)$$

$$= \hat{z}\,\frac{\mu_0 \sigma \omega}{2}\left(\frac{1}{z}\left(\cancel{R^2} - \frac{1}{2}\frac{R^4}{z^2} + \cancel{O(z^{-4})}\right) + \cancel{2z^2} - \cancel{R^2} + \frac{3}{4}\frac{R^4}{z^2}\right) - \cancel{2z}$$

$$= \hat{z}\,\frac{\mu_0 \sigma \omega}{2}\left(\frac{1}{z}\left(\frac{R^4}{4z^2}\right)\right) = \boxed{\hat{z}\,\frac{\mu_0 \sigma \omega R^4}{8z^3}}$$

$$d\vec{m} = dI\,\pi s^2 \hat{z} = dI\,(A)$$

$$= K\,ds\,\pi s^2 \hat{z}$$

$$d\vec{m} = \sigma \omega s\,ds\,\pi s^2 \hat{z} \qquad \vec{m} = \int d\vec{m} = \int_0^R \pi \sigma \omega s^3\,ds\,\hat{z}$$

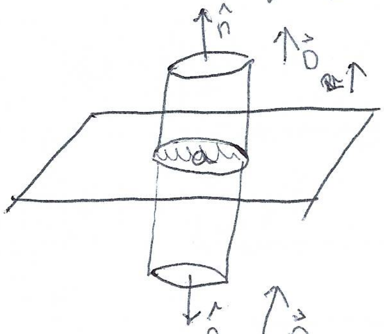
$$= \frac{1}{4}\pi \sigma \omega R^4 \hat{z}$$

$$\vec{B}_{dip} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) =$$

$$= \frac{\mu_0 m}{4\pi z^3}(2\hat{z}) = \frac{\mu_0}{2\pi z^3}\left(\frac{\pi}{4}R^4 \sigma \omega\right)$$

$$= \boxed{\frac{\mu_0 \sigma \omega R^4}{8z^3}}$$

Boundaries $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \rightarrow \vec{\nabla}\cdot\vec{D} = \rho_f \rightarrow \int \vec{\nabla}\cdot\vec{D}\,d^3x = \int \rho_f\, d^3x$



$$\oint \vec{D}\cdot d\vec{a} = Q_{f_{enc}}$$

$$\cancel{\oint \vec{D}\cdot d\vec{a} = D_{2\perp}\,\Delta a}$$

$$\oint \vec{D}\cdot d\vec{a} = D_{\uparrow\perp}\,a - D_{\downarrow\perp}\,a = Q_f$$

$$\boxed{D_\uparrow \rightarrow D_\downarrow = \sigma_f} \xrightarrow{\text{linear}} \epsilon_\uparrow E_{\uparrow\perp} - \epsilon_\downarrow E_{\downarrow\perp} = \sigma_f$$

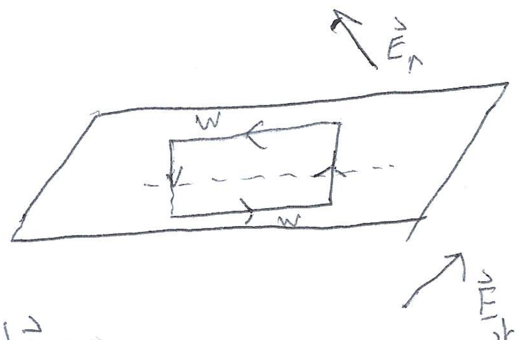
$$E_{\uparrow\perp} - E_{\downarrow\perp} = \frac{\sigma_{total}}{\epsilon_0}$$

<u>Linear media</u>

$$\vec{D} = \epsilon\vec{E} = \epsilon_0\epsilon_r\vec{E} = \epsilon_0(1+\chi_e)\vec{E}$$

$D_{\uparrow\perp} - D_{\downarrow\perp} = \sigma_f$

$\epsilon_\uparrow E_{\uparrow\perp} - \epsilon_\downarrow E_{\downarrow\perp} = \sigma_f$



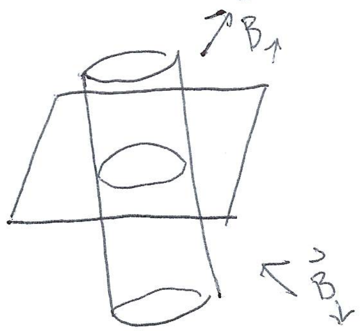
$\vec{\nabla}\times\vec{E} = \vec{0}$

$$\oint \vec{E}\cdot d\vec{s} = 0$$

$$\vec{E}_{\uparrow\parallel}\cdot\vec{w} - \vec{E}_{\downarrow\parallel}\cdot\vec{w} = \vec{0}$$

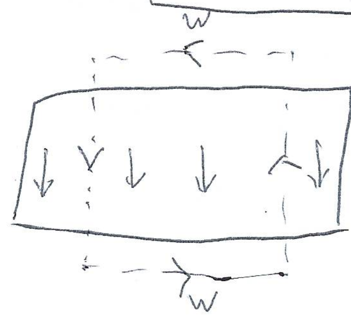
$$\vec{E}_{\uparrow\parallel} - \vec{E}_{\downarrow\parallel} = 0 \Rightarrow \boxed{\vec{E}_{\uparrow\parallel} = \vec{E}_{\downarrow\parallel}}$$

$\vec{\nabla}\cdot\vec{B} = 0$



$\oint \vec{B}\cdot d\vec{a} = 0 \rightarrow B_{\uparrow\perp} a - B_{\downarrow\perp} a = 0$

$$\boxed{B_{\uparrow\perp} = B_{\downarrow\perp}}$$



$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} \rightarrow \vec{\nabla}\times\vec{H} = \vec{J}_f$

$\oint \vec{H}\cdot d\vec{s} = I_f$

$H_{\uparrow\parallel} w - H_{\downarrow\parallel} w = \cancel{K_f}\, I_f$

$H_{\uparrow\parallel} - H_{\downarrow\parallel} = K_f$

$\vec{H}_{\uparrow\parallel} - \vec{H}_{\downarrow\parallel} = \vec{K}_f \times \hat{n}$

$$\boxed{\frac{\vec{B}_{\uparrow\parallel}}{\mu_\uparrow} - \frac{\vec{B}_{\downarrow\parallel}}{\mu_\downarrow} = \vec{K}_f \times \hat{n}}$$

$\vec{\nabla}\times\vec{H} = \vec{J}_f$

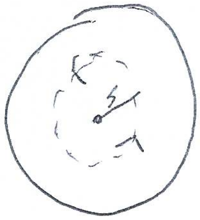
$\int \vec{\nabla}\times\vec{H}\cdot d\vec{a} = \int \vec{J}\cdot d\vec{a} = I_f$

$\oint \vec{H}\cdot d\vec{s} = I_f$

Linear med $\vec{B} = \mu \vec{H}$

$\hookrightarrow \mu = \mu_0(1+\chi_0)$

6.17 Current I flowing down a straight wire of radius a and susceptibility χ_m. Current distributed unifomly. Free current distributed uniformly, what is $\vec{B}(s)$ and find bound currents and net bound current flowing downward



$\hat{z}\,\odot$, $\vec{J}$ is constant $\vec{J} = \frac{I}{\pi a^2}\hat{z}$

$\vec{H} = \frac{\vec{B}}{\mu}$ $\oint \vec{H}\cdot d\vec{s} = I_f(s)$

$H(2\pi s) = I_f = \int_{arc} \vec{J}\cdot d\vec{a} = J\pi s^2$

$= \frac{I}{\pi a^2}\pi s^2 = \frac{I s^2}{a^2}$

$$\vec{H}(s) = \begin{cases} \frac{I s}{2\pi a^2}\hat{\phi} & 0 < s < a \\ \frac{I}{2\pi s}\hat{\phi} & s > a \end{cases} \qquad \vec{B} = \begin{cases} \frac{\mu I \left(\frac{s^2}{a^2}\right)}{2\pi s}\hat{\phi} \\ \frac{\mu_0 I}{2\pi s}\hat{\phi} \end{cases}$$

$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

$\vec{M} = \frac{\vec{B}}{\mu_0} - \vec{H} = \left(\frac{\mu}{\mu_0} - 1\right)\vec{H}$ $\vec{K}_b = \vec{M}\times\hat{n}\Big|_{surface}$

$\vec{J}_b = \vec{\nabla}\times\vec{M}$

$\cancel{\vec{M} = \left(\frac{\mu}{\mu_0} - 1\right)}$

$$\vec{M} = \chi_m \vec{H} = \begin{cases} \frac{I\frac{s^2}{a^2}\chi_m}{2\pi s}\hat{\phi} & 0 \le s < a \\ \vec{0} & s > a \end{cases}$$

$\vec{K}_b = \frac{-I\chi_m}{2\pi a^2}\hat{z}$ $\to I_B(a) = -I\chi_m$

$\vec{J}_b = \frac{\chi_m I}{\pi a^2}\hat{z} \to I_b = \chi_m I$

Phys 110A 17Apr19
Discussion

Magnetic Fields in Matter pt 2

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = \mu_0 (\vec{J}_f + \vec{J}_b)$ $\qquad \vec{J}_b = \vec{\nabla} \times \vec{M}$

$= \mu_0 (\vec{J}_f + \vec{\nabla} \times \vec{M})$ $\qquad \vec{K}_b = \vec{M} \times \hat{n} \Big|_{surface}$

$\vec{\nabla} \times (\vec{B} - \mu_0 \vec{M}) = \mu_0 \vec{J}_f$

$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_f = \vec{\nabla} \times \vec{H}$, $\vec{H} = \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right)$

for linear materials

magnetic susceptibility

$\vec{M} \propto \vec{H}$ $\qquad \vec{M} = \chi_m \vec{H}$

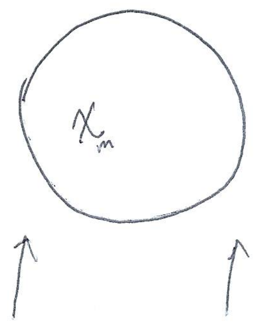
$\cancel{\vec{H} = \frac{\vec{B}}{\mu_0} - \chi_m \vec{B}}$

μ magnetic ~~permeativity~~ permeability

$\vec{H} = \frac{\vec{B}}{\mu_0} - \chi_m \vec{H}$ $\qquad \vec{B} = \overbrace{\mu_0 (1 + \chi_m)}^{\mu} \vec{H}$

$$\vec{M} = \chi_m \vec{H} \qquad \vec{B} = \mu_0(1+\chi_m)\vec{H} = \mu \vec{H}$$

6.18



$\uparrow \vec{B}_0 = B_0 \hat{z}$

Find $\vec{B}_{in}$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r})\,d\tau}{\mathscr{r}}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f = \vec{0}$$

$$\vec{H} = -\vec{\nabla} W$$

$$\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{\nabla}\cdot\left(\frac{\vec{B}}{\mu}\right)$$

$$\vec{\nabla}\cdot\vec{H} \Rightarrow \nabla^2 W = -\vec{\nabla}\cdot\vec{H}$$

$$= \frac{\vec{\nabla}\cdot\vec{B}}{\mu} + \vec{B}\cdot\vec{\nabla}\left(\frac{1}{\mu}\right) = 0$$ inside and out but not surface

$$W_{out}(r,\theta) = \sum\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$W_{in}(r,\theta) = \sum\left(C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$W_{out} \xrightarrow{r\to\infty} \sum A_\ell r^\ell P_\ell(\cos\theta) = -\frac{B_0}{\mu_0} r P_1(\cos\theta)$$

Want $\vec{B}_{out} \to B_0\hat{z} = \mu_0 \vec{H} B_0$

$$\vec{H}_{out} = \frac{B_0}{\mu_0}\hat{z}$$

$$A_1 = \frac{-B_0}{\mu_0}, \quad A_\ell = 0 \quad \ell \neq 1$$

$$-\vec{\nabla} W = \vec{H}$$

$$W_{out} = \frac{-B_0}{\mu_0} z = \frac{-B_0 r P_1(\cos\theta)}{\mu_0}$$

$$W_{out} = \left(A_1 r P_1(\cos\theta) + \left(\sum_{\ell=0}^{\infty}\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)\right)$$

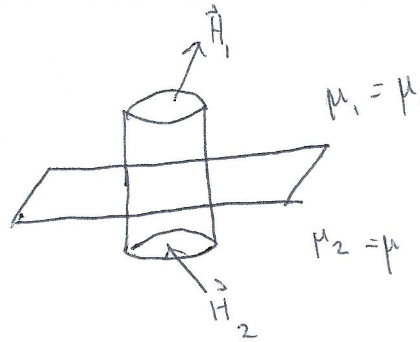
$$W_{in} = \sum C_\ell r^\ell P_\ell(\cos\theta) \qquad W_{in}(R,\theta) = W_{out}(R,\theta)$$

$\ell=1$: $$A_1 R + \frac{B_1}{R^2} = C_1 R$$

$\ell\neq1$: $$\frac{B_\ell}{R^{\ell+1}} = C_\ell R^\ell$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \qquad \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}$$

$$\oint \vec{H}\cdot d\vec{a} = -\oint \vec{M}\cdot d\vec{a}$$



$$W_{out}(r,\theta) = A_1 r P_1(\cos\theta) + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$W_{in}(r,\theta) = \sum C_l r^l P_l(\cos\theta)$$

$$H_{1\perp}A - H_{2\perp}A = -(M_{1\perp}A - M_{2\perp}A)$$

$$H_{1\perp} - H_{2\perp} = -(\chi_1 H_{1\perp} - \chi_2 H_{2\perp})$$

$$\mu_0 H_{1\perp}(1+\chi_1) = \mu_0 H_{2\perp}(1+\chi_2)$$

$$H_{1\perp}\mu_1 = H_{2\perp}\mu_2$$

$l = 1$

$\frac{\partial W}{\partial r}$

$\cancel{A_1}$

$$\left(A_1 - \frac{2B_1}{R^3}\right)\mu_0 = C_1 \mu$$

$$\hat{n}\cdot\vec{H} = -\vec{\nabla}W\cdot\hat{n}$$

$l \neq 1$

$$-(l+1)\frac{B_l}{R^{l+2}}\mu_0 = l C_l R^{l-1}\mu$$

$$-\frac{\partial W_{out}}{\partial n}\mu_{out} = -\frac{\partial W_{in}}{\partial n}\mu_{in}\Big|_{surface}$$

$$A_1 R + \frac{B_1}{R^2} = C_1 R$$

$$-\frac{(l+1)B_l}{R^{2l+1}}\mu_0 = \frac{\mu l B_l}{R^{2l+1}} \Rightarrow \frac{B_l}{R^{2l+1}}\left(\mu l + \mu_0(l+1)\right) = 0$$

$$\frac{B_l}{R^{l+1}} = C_l l \qquad A_1 + \frac{B_1}{R^3} = C_1$$

$$B_l = 0 \ \forall l \neq 1$$

$$C_l = 0 \ \forall l \neq 1$$

$$\mu\left(A_1 + \frac{B_1}{R^3}\right) = \left(A_1 - 2\frac{B_1}{R^3}\right)\mu_0$$

$$\frac{B_1}{R^3}(\mu + 2\mu_0) = A_1(\mu_0 - \mu) \qquad A_1\left(1 + \frac{(\mu_0-\mu)}{\mu+2\mu_0}\right) = C_1$$

$$B_1 = \frac{A_1 R^3(\mu_0 - \mu)}{\mu + 2\mu_0}$$

$$C_1 = \frac{A_1 3\mu_0}{\mu + 2\mu_0}$$

W_{ou}

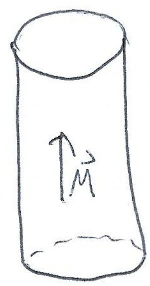
$$W_{out}(r,\theta) = A_1 r P_1(\cos\theta) + \frac{A_1 R^3 (\mu_0 - \mu)}{r^2(\mu + 2\mu_0)} P_1(\cos\theta)$$

$$W_{in}(r,\theta) = \frac{A_1 3\mu_0 r}{\mu + 2\mu_0} P_1(\cos\theta)$$

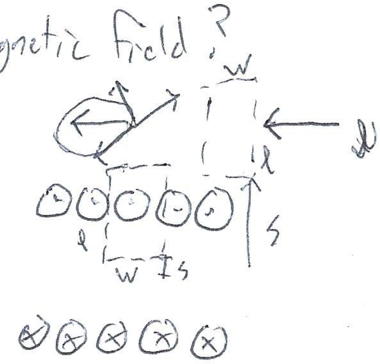
$$\vec{H}_{in} = -\vec{\nabla} W_{in} = -\frac{A_1 3\mu}{\mu + 2\mu_0}\hat{z} = \frac{B_0 3}{\mu + 2\mu_0}\hat{z}$$

$$\vec{B}_{in} = \frac{B_0 3(1+\chi_m)}{3+\chi_m}\hat{z}$$

6.7)



What is magnetic field?



$= m\hat{z}$

$$\vec{K}_b = (M\hat{z}) \times (\hat{s}) = M\hat{\phi}$$

$$\vec{J}_b = \vec{\nabla} \times (M\hat{z}) = \vec{0}$$

$$\vec{B} = \begin{cases} \vec{0} & \text{outside} \\ \mu_0 M\hat{z} & \text{inside} \end{cases}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 J_{\text{encl}}$$

$$B(s)w - \cancel{B(s)l}\ B(s+l)w = 0$$

at big l

$$B(s)w \to 0$$

$$B(s) = 0$$

$$B(s)w = \mu_0 \underbrace{Kw}_{I} \qquad B = \mu_0 K\hat{z}$$

6.8)

$$\vec{M} = ks^2\hat{\phi}$$

$$\vec{K}_b = (kR^2\hat{\phi}) \times (\hat{s}) = -kR^2\hat{z}$$

$$\vec{J}_b = \vec{\nabla} \times (ks^2\hat{\phi}) = 3ks\,\hat{z}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enc}} \qquad B(s)\,2\pi s = \mu_0 \int \vec{J} \cdot d\vec{a} = \cancel{\mu_0 \int 3ks'\hat{z} \cdot (s'ds'}$$

$$= \mu_0 \iint 3ks'\hat{z} \cdot (s'ds'd\phi'\hat{z})$$

$$= 2\pi\mu_0 ks^3$$

$$\vec{B}(s) = \begin{cases} \mu_0 ks^2\,\hat{\phi} & \text{inside} \\ \vec{0} & \text{outside} \end{cases}$$

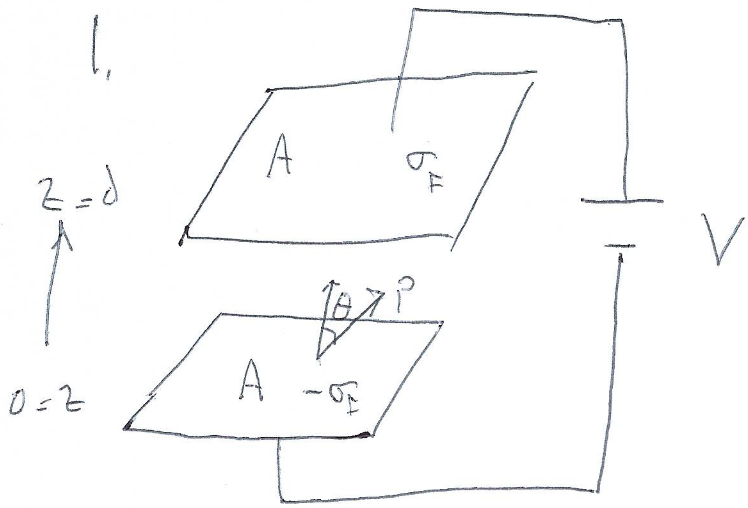
$$2\pi\mu_0 kR^3 + \mu_0(-kR^2 \cdot 2\pi R)$$

$$= 0$$

Phys 110A 24Apr2019
Discussion

Midterm Solutions

1.



a) $\theta = 60°$

Find σ_F and σ_b in terms of V, d, P. Assume

$V \gg \frac{Pd}{\epsilon_0}$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0 \qquad \sigma_b = \vec{P}\cdot\hat{n}\Big|_{\text{surface}} = \begin{cases} \vec{P}\cdot\hat{z} = P\cos\theta = P\cos 60 = \frac{P}{2} & \text{at } z=d \\ -\vec{P}\cdot\hat{z} = -\frac{P}{2} & \text{at } z=0 \end{cases}$$

$$\vec{E} = \frac{\sigma_b+\sigma_F}{\epsilon_0}(-\hat{z}) \rightarrow V(d)-V(0) = V = -\int_0^d \vec{E}\cdot d\vec{s} = \left(\frac{\sigma_b+\sigma_F}{\epsilon_0}\right)d$$

$$\sigma_F = \frac{V\epsilon_0}{d} - \sigma_b = \frac{V\epsilon_0}{d} - \frac{P}{2}$$

$$\sigma_F = \begin{cases} \frac{V\epsilon_0}{d} - \frac{P}{2} = \frac{\epsilon_0}{d}\left(V - \frac{Pd}{2\epsilon_0}\right) \simeq \frac{\epsilon_0 V}{d} & z=d \\ -\frac{\epsilon_0 V}{d} & z=0 \end{cases}$$

$$\sigma_b(d) = P\cos\theta = P/2 \qquad \sigma_F(d) = \frac{\epsilon_0}{d}\left(V - \frac{Pd}{2\epsilon_0}\right) \approx \frac{\epsilon_0 V}{d}$$

b) Disconnect battery charge stays constant

$$\sigma_F \to \sigma_F \qquad \sigma_b = \vec{P}\cdot\hat{n} = \pm P\cos\theta =$$

$$\vec{E} = \frac{\sigma_b + \sigma_F}{\epsilon_0}(-\hat{z}) = \begin{cases} \theta = 0 & \dfrac{\frac{P}{2} + \frac{\epsilon_0 V}{d}}{\epsilon_0}(-\hat{z}) \\[2ex] \theta = \pi & \dfrac{-\frac{3P}{2} + \frac{\epsilon_0 V}{d}}{\epsilon_0}(-\hat{z}) \end{cases}$$

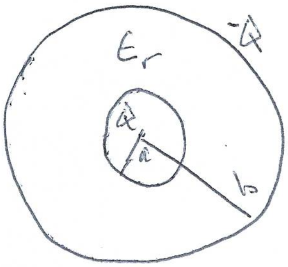
c) ΔU between two states Which is lower energy

$$U = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} E^2\, d\tau = \begin{cases} \dfrac{\epsilon_0}{2}\dfrac{1}{\epsilon_0^2}\left(\dfrac{P}{2} + \dfrac{\epsilon_0 V}{d}\right)^2 Ad & \theta = 0 \\[2ex] \dfrac{\epsilon_0}{2}\dfrac{1}{\epsilon_0^2}\left(-\dfrac{3P}{2} + \dfrac{\epsilon_0 V}{d}\right)^2 Ad & \theta = \pi \end{cases}$$

$$\Delta U = \cancel{\frac{1}{2\epsilon_0}Ad(2P) = \frac{PAd}{\epsilon_0}}$$

$$= \frac{Ad}{2\epsilon_0}\left(\frac{P\epsilon_0 V}{d} - \left(-\frac{3P\epsilon_0 V}{d}\right)\right) = 2APV$$

2)



a) $\vec{E}$ and $\vec{D}$ inside insulator

$$\oint \vec{D}\cdot d\vec{a} = D(r)\,4\pi r^2 = Q$$

$$D = \frac{Q}{4\pi r^2}\hat{r}$$

$$\vec{E} = \frac{\vec{D}}{\epsilon_0 \epsilon_r} = \frac{Q}{4\pi \epsilon_0 \epsilon_r r^2}\hat{r} \quad a < r < b$$

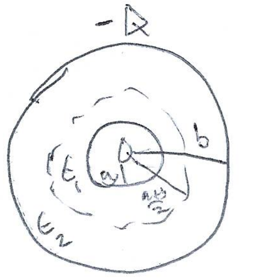
b) Capacitance

$$V(a) - V(b) = V = -\int_a^b \vec{E}\cdot d\vec{s} = -\int_a^b \frac{Q}{4\pi\epsilon_0\epsilon_r r^2}\,dr$$

$$= \frac{Q}{4\pi\epsilon_0\epsilon_r}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\epsilon_0\epsilon_r}\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{4\pi\epsilon_0\epsilon_r}{\left(\frac{1}{a}-\frac{1}{b}\right)}$$

c)



Find Q_b at interface

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r} \text{ still } \quad a < r < b$$

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} \Rightarrow \vec{P} = \vec{D} - \epsilon_0\vec{E} = $$

$$= \frac{\epsilon_0 \vec{D}}{\epsilon_0\epsilon_r} + \vec{P} \Rightarrow \vec{P} = \vec{D}\left(1 - \frac{1}{\epsilon_r}\right)$$

For dielectric 1

$$\sigma_b = \vec{P}_1\cdot\hat{r} = \frac{Q}{\cancel{4\pi r^2}}$$

$$= \frac{Q}{4\pi\left(\frac{a+b}{2}\right)^2}\left(1 - \frac{1}{\epsilon_1}\right)$$

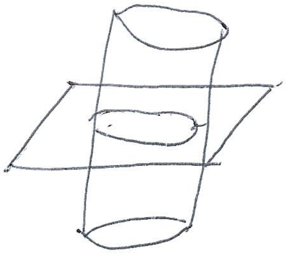
for dielectric 2

$$\sigma_b = \vec{P}_2\cdot(-\hat{r})$$

$$= \frac{-Q}{4\pi\left(\frac{a+b}{2}\right)^2}\left(1 - \frac{1}{\epsilon_2}\right)$$

$$Q_b = Q\left(1 - \frac{1}{\epsilon_1}\right) - Q\left(1 - \frac{1}{\epsilon_2}\right)$$

$$= Q\left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right)$$



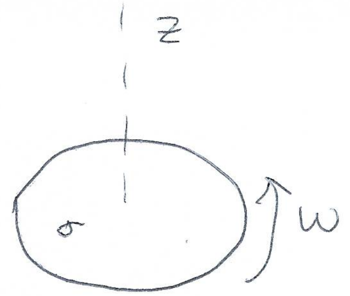
$$\oint \vec{D}\cdot d\vec{a} \Rightarrow D_{2\perp}A - D_{1\perp}A = Q_f$$

$$\vec{D}_1 = \vec{D}_2 \text{ interface}$$

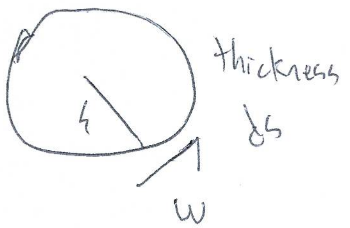
$$\cancel{\epsilon_1\vec{E}_1 = \epsilon_2\vec{E}_2}$$

$$\oint \vec{E}\cdot d\vec{a} \Rightarrow E_{2\perp} - E_{1\perp} = \sigma = \sigma_b$$

$$\frac{D}{\epsilon_2} - \frac{D}{\epsilon_1} = \sigma_b \qquad \frac{Q}{\epsilon_2}\left(\frac{1}{4\pi(\frac{a+b}{2})^2}\right) - \frac{Q}{\epsilon_1}\left(\frac{1}{4\pi(\frac{a+b}{2})^2}\right) = \sigma_b$$



a) what is $\vec{m}$



$$d\vec{m} = \pi s^2 \, dI \, \hat{z}$$

$$= \pi s^2 (K \, ds)\hat{z} = \pi s^2 \sigma v \, ds \, \hat{z}$$

$$= \pi s^2 (\sigma \omega s \, ds)\hat{z}$$

$$\vec{m} = \int d\vec{m} = \int \pi \sigma \omega s^3 \, ds \, \hat{z}$$

$$\boxed{\vec{m} = \frac{1}{4}\pi \sigma \omega a^4 \hat{z}}$$

b) Find asymptotic form of $\vec{B}$ for $z >> a$

$$\vec{B}_{dip} = \frac{\mu_0}{4\pi r^3}\left(3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right)$$

$$\vec{B}_{dip}(\vec{r} = \vec{z}) = \frac{\mu_0}{4\pi z^3}\left(3(\vec{m}\cdot\hat{z})\hat{z} - \vec{m}\right) = \frac{2\vec{m}\mu_0}{4\pi z^3} = \frac{\mu_0 \sigma \omega a^4 \hat{z}}{8 z^3}$$

Phys 110A 01May19

Discussion

Maxwell's Equations and Ohm's Law

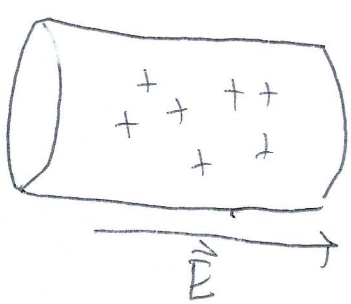
$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \oint \vec{E}\cdot d\vec{a} = \frac{Q}{\epsilon_0}$$

$$\vec{\nabla}\times\vec{E} = \frac{-\partial\vec{B}}{\partial t} \qquad \int \frac{-\partial\vec{B}}{\partial t}\cdot d\vec{a} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} = \oint \vec{E}\cdot d\vec{s} = \frac{-d}{dt}\Phi_B$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \oint \vec{B}\cdot d\vec{a} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc} + \mu_0\epsilon_0\frac{d}{dt}\Phi_E$$

$$= \mu_0\left(\vec{J} + \underbrace{\epsilon_0\frac{\partial\vec{E}}{\partial t}}_{\vec{J}_d}\right)$$



conductivity

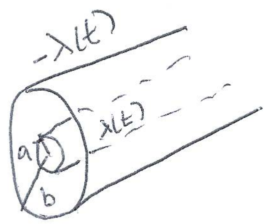
$$\vec{J} \propto \vec{E} \qquad \vec{J} = \sigma\vec{E} \quad \text{Ohm's Law}$$

17) Two concentric cylinders of inner radius a and outer radius b with a material of conductivity σ between them. Let there be a uniform charge/length $\lambda(t)$ on the inner surface and $-\lambda(t)$ on the outer surface

a) what is $\vec{E}$ inside

b) Find $\vec{J}$ and $\vec{I}$ from where to where

c) What is $\vec{\nabla}\cdot\vec{B}$, $\vec{\nabla}\times\vec{B}$ and $\vec{B}$ inside



a) $\oint \vec{E}\cdot d\vec{a} = E(s)\,2\pi s\,l = \frac{\lambda l}{\epsilon_0}$

$$\boxed{\vec{E}(s,t) = \frac{\lambda(t)}{2\pi\epsilon_0 s}\hat{s}}$$

b) $\vec{J} = \sigma\vec{E}$

$$\boxed{\vec{J}(s,t) = \frac{\sigma\lambda(t)}{2\pi\epsilon_0 s}\hat{s}}$$

$$\vec{I} = \int_{\text{cylinder}} \vec{J}\cdot d\vec{a} = \int \frac{\sigma\lambda}{2\pi\epsilon_0 s}\cdot(s\,dz\,d\phi)$$

$$= \frac{\sigma\lambda\,l(2\pi)}{2\pi\epsilon_0} = \boxed{\frac{\sigma\lambda l}{\epsilon_0} = I}$$

c)

$\vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = \frac{\sigma\lambda\mu_0}{2\pi\epsilon_0 s}\hat{s} - \frac{\sigma\mu_0\lambda}{2\pi\epsilon_0 s}\hat{s} = \vec{0}$$

$$\frac{\partial\vec{E}}{\partial t} = \frac{\dot{\lambda}}{2\pi\epsilon_0 s}\hat{s} = \frac{-\sigma\lambda}{2\pi\epsilon_0^2 s}$$

$$\dot{\lambda}(t) = \frac{\dot{Q}(t)}{l} = \frac{-I}{l} = -\frac{\sigma\lambda}{\epsilon_0}$$

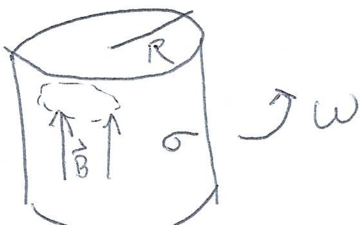
7.33

An infinite cylinder of radius R carries a uniform surface charge σ. We want to set it spinning about its axis to a final angular velocity ω_f. How much work will it take per unit length

a) Find $\vec{B}$ and the induced $\vec{E}$ inside and outside in terms of ω, $\dot{\omega}$ and s. Calculate the torque we must exert and find W.

$$W = \int \tau \, d\theta$$

$$\vec{K} = \sigma \vec{v} = \sigma \omega R \hat{\phi}$$



$$\oint \vec{B} \cdot d\vec{s} = \mu_0 K l = B l$$

$$\vec{B} = \mu_0 K \hat{z}$$

$$\boxed{\vec{B}(t) = \mu_0 K(t) \hat{z}, \quad K(t) = \sigma \omega R}$$

Loop radius s inside

$$\Phi_B = \int \vec{B} \cdot d\vec{a} = B(t) \pi s^2$$

$$\dot{\Phi}_B = \dot{B} \pi s^2 \qquad \oint \vec{E} \cdot d\vec{s} = E(s) 2\pi s$$

$$E(s) 2\pi s = -\dot{B} \pi s^2 \Rightarrow \boxed{\vec{E}(s)_{in} = -\frac{\dot{\vec{B}} s}{2} \hat{\phi}}$$

Outside

$$\int \vec{B} \cdot d\vec{a} = B \pi R^2 \qquad -\dot{\Phi}_B = -\dot{B} \pi R^2 = 2\pi s E$$

$$\boxed{\vec{E}_{out} = \frac{-\dot{\vec{B}} \pi R^2}{2\pi s} \hat{\phi}}$$

$$\vec{E}_{ind}(R) = -\frac{\dot{B} \pi R}{2\pi} \hat{\phi} = -\frac{\mu_0 \sigma \dot{\omega} \pi R^2}{2\pi} \hat{\phi}$$

$$d\vec{\tau}_{exert} = \int \vec{R} \times (-dq \vec{E}_{ind})$$