Analytic Mechanics Homework Assignment 12

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Problem 1

(Taylor 13.6) In discussing the oscillation of a cart on the end of a spring, we almost always ignore the mass of the spring. Set up the Hamiltonian $\mathcal{H}$ for a cart of mass m on a spring (force constant k) whose mass M is *not* negligible, using the extension x of the spring as the generalized coordinate. Solve Hamilton's equations and show that the mass oscillates with the angular frequency $\omega = \sqrt{k/(m+M/3)}$. That is, the effect of the spring's mass is to add $M/3$ to m. (Assume that the spring's mass is distributed uniformly and that it stretches uniformly.)

Solution

The kinetic energy will get an extra term

$$T_{spr} = \frac{1}{2}\int_0^L u^2 dM$$

with $u = \frac{\dot{x}x}{L}$. Each mass element is

$$dM = \frac{Mdx}{L}$$

which makes this integral

$$T_{spr} = \frac{M\dot{x}^2}{2L^2}\int_0^L x^2 dx = \frac{M\dot{x}^2}{6}$$

This gives a total kinetic energy of

$$T = \frac{1}{2}\Big(m + \frac{M}{3}\Big)\dot{x}^2$$

The potential energy is

$$U = \frac{1}{2}kx^2$$

The Lagrangian for this system is then

$$\mathcal{L} = \frac{1}{2}\Big(m + \frac{M}{3}\Big)\dot{x}^2 - \frac{1}{2}kx^2$$

The generalized momentum will then be

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \Big(m + \frac{M}{3}\Big)\dot{x} = p$$

The Hamiltonian for this system is then

$$\mathcal{H} = p\dot{x} - \frac{1}{2}p\dot{x} + \frac{1}{2}kx^2 = \frac{p^2}{2(m + \frac{M}{3})} + \frac{1}{2}kx^2$$

The Hamilton equations will then be

$$-\frac{\partial \mathcal{H}}{\partial x} = -kx = \dot{p}$$

$$\frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m + \frac{M}{3}} = \dot{x}$$

Combing these together gives us

$$\Big(m + \frac{M}{3}\Big)\ddot{x} = -kx$$

which has solutions of the form

$$x = A\cos(\omega t - \delta) + B\sin(\omega t - \delta)$$

with

$$\omega = \sqrt{\frac{k}{m + \frac{M}{3}}}$$

Problem 2

(Taylor 13.17) Consider the mass confined to the surface of a cone described in Example 13.4 (page 533). We saw that there are solutions for which the mass remains at the fixed height $z = z_0$, with fixed angular velocity $\dot{\phi}_0$ say.

a) For any chosen value of p_ϕ, use (13.34) to get an equation that gives the corresponding value of the height z_0.

b) Use the equations of motion to show that this motion is stable. That is, show that if the orbit has $z = z_0 + \epsilon$ with ϵ small, then ϵ will oscillate about zero.

c) Show that the angular frequency of these oscillations is $\omega = \sqrt{3}\dot{\phi}_0 \sin(\alpha)$, where α is the half angle of the cone ($\tan(\alpha) = c$ where c is the constant in $\rho = cz$).

d) Find the angle α for which the frequency of oscillation ω is equal to the orbital angular velocity $\dot{\phi}_0$, and describe the motion for this case.

Solution

a) The equations given by 13.34 are

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{m(c^2+1)}$$

$$\dot{p}_z = -\frac{\partial \mathcal{H}}{\partial z} = \frac{p_\phi^2}{mc^2 z^3} - mg$$

Which can be used to make the equation of motion

$$m(c^2+1)\ddot{z} = \frac{p_\phi^2}{mc^2 z^3} - mg = 0$$

All we need to do is solve for z

$$\frac{1}{z^3} = \frac{m^2 g c^2}{p_\phi^2}$$

$$z_0 = \sqrt[3]{\frac{p_\phi^2}{m^2 g c^2}}$$

b) The equations of motion are

$$m(c^2+1)\ddot{z} = \frac{p_\phi^2}{mc^2 z^3} - mg$$

We can then use variation of parameters to get

$$m(c^2+1)\ddot{\epsilon} = \frac{p_\phi^2}{mc^2 (z_0+\epsilon)^3} - mg$$

$$= \frac{p_\phi^2}{mc^2 z_0^3}\frac{1}{(1+\frac{\epsilon}{z_0})^3} - mg$$

Using a Taylor expansion and ignoring all but the first two terms gives

$$m(c^2+1)\ddot{\epsilon} = mg\Big(1 - \frac{3\epsilon}{z_0}\Big) - mg$$

$$= -\frac{3mg\epsilon}{z_0}$$

$$\ddot{\epsilon} = -\frac{3g}{z_0(c^2+1)}\epsilon$$

This oscillation is stable.

c) We can solve for z_0 replacing p_ϕ with $mc^2z_0^2\dot{\phi}$ which gives us

$$z_0 = \sqrt[3]{\frac{m^2c^4z_0^2\dot{\phi}_0^2}{m^2c^2g}} = z_0\sqrt[3]{\frac{c^2z_0\dot{\phi}_0^2}{g}}$$

$$g = c^2z_0\dot{\phi}_0^2$$

$$z_0 = \frac{g}{c^2\dot{\phi}_0^2}$$

This gives us the frequency

$$\omega = \sqrt{\frac{3g}{z_0(c^2+1)}} = \sqrt{\frac{3c^2\dot{\phi}_0^2}{c^2+1}} = \sqrt{3}\dot{\phi}_0\sin(\alpha)$$

d) The angle α will have to be

$$\alpha = \arcsin\Big(\frac{1}{\sqrt{3}}\Big) \approx 35.26°$$

In this case, the path of the mass is a closed orbit and that there is only one unique position per given ϕ.

Problem 3

(Taylor 13.18) All of the examples in this chapter and all of the problems (except this one) treat forces that come from a potential energy $U(\boldsymbol{r})$ [or occasionally $U(\boldsymbol{r}, t)$]. However, the proof of Hamilton's equations given in Section 13.3 applies to any system for which Lagrange's equations hold, and this can include forces not derivable from a potential energy. An important example of such a force is the magnetic force on a charged particle.

a) Use the Lagrangian (7.103) to show that the Hamiltonian for a charge q in an electromagnetic field is

$$\mathcal{H} = \frac{(\boldsymbol{p} - q\boldsymbol{A})^2}{2m} + qV$$

(This Hamiltonian plays an important role in the quantum mechanics of charged particles.)

b) Show that Hamilton's equations are equivalent to the familiar Lorentz force equation $m\ddot{\boldsymbol{r}} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$.

Solution

a) The Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{\boldsymbol{r}}^{\boldsymbol{2}} - q(V - \dot{\boldsymbol{r}} \cdot \boldsymbol{A})$$

The generalized momentum will be

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{\boldsymbol{r}} + q\boldsymbol{A} = \boldsymbol{p}$$

and

$$\dot{\boldsymbol{r}} = \frac{\boldsymbol{p} - q\boldsymbol{A}}{m}$$

We can then express the Hamiltonian as

$$\mathcal{H} = \boldsymbol{p} \cdot \dot{\boldsymbol{r}} - \mathcal{L}$$

$$\begin{aligned}
\mathcal{H} = \boldsymbol{p} \cdot \boldsymbol{q} - \frac{m}{2}\Big(\frac{\boldsymbol{p}^{\boldsymbol{2}} - 2q\boldsymbol{A} \cdot \boldsymbol{p} + q^2\boldsymbol{A}^{\boldsymbol{2}}}{m^2}\Big) + qV - q\Big(\frac{\boldsymbol{p} \cdot \boldsymbol{A} - 1\boldsymbol{A}^{\boldsymbol{2}}}{m}\Big) \\
= \frac{\boldsymbol{p}^{\boldsymbol{2}} - q\boldsymbol{A} \cdot \boldsymbol{p}}{m} - \frac{\boldsymbol{p}^{\boldsymbol{2}} - 2q\boldsymbol{A} \cdot \boldsymbol{p} + q^2\boldsymbol{A}^{\boldsymbol{2}}}{2m} + qV - q\Big(\frac{\boldsymbol{p} \cdot \boldsymbol{A} - q\boldsymbol{A}^{\boldsymbol{2}}}{m}\Big) \\
= \frac{\boldsymbol{p}^{\boldsymbol{2}} - 2q\boldsymbol{A} \cdot \boldsymbol{p}}{2m} + \frac{q^2\boldsymbol{A}^{\boldsymbol{2}}}{2m} + qV \\
\mathcal{H} = \frac{(\boldsymbol{p} - q\boldsymbol{A})^2}{2m} + qV
\end{aligned}$$

b) The Hamilton equations will be

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{r}_i = \frac{p_i - qA_i}{m}$$

$$-\frac{\partial \mathcal{H}}{\partial r_i} = \dot{p}_i = \frac{q}{m}(p_j - qA_j)\partial_i A_j - q\partial_i V$$

We can plug the first equation into the second equation.

$$\dot{p}_i = q\dot{r}_j\partial_i A_j - q\partial_i V = qv_j\partial_i A_j - q\partial_i V$$

Let's multiply the first Hamilton equation by m and differentiate it in respect to time.

$$m\ddot{r}_i = \dot{p}_i - q\dot{A}_i$$

We can plug in what we found for $\dot{p}$ into this equation to get

$$m\ddot{r}_i = qv_j\partial_i A_j - q\partial_i V - q\dot{A}_i$$

$$= qv_j\partial_i A_j - q\partial_i - q\Big(\frac{\partial A_i}{\partial t} + v_j\partial_j A_i\Big)$$

$$= q(v_j\partial_i A_j - v_j\partial_j A_i) + q\Big(- \partial_i V - \frac{\partial A_i}{\partial t}\Big)$$

This last piece is

$$-\partial_i V - \frac{\partial A_i}{\partial t} = E_i$$

Let's find out what $\boldsymbol{v} \times \boldsymbol{B}$ looks like.

$$\boldsymbol{v} \times \boldsymbol{B} = \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$$

$$= \epsilon_{ijk}v_j(\boldsymbol{\nabla} \times \boldsymbol{A})_k = \epsilon_{ijk}v_j\epsilon_{klm}\partial_l A_m$$

$$= \epsilon_{ijk}\epsilon_{lmk}v_j\partial_l A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})v_j\partial_l A_m$$

$$= v_j\partial_i A_j - v_j\partial_j A_i$$

Which, luckily, is exactly what we are looking for. So, we end up with the Lorentz force

$$m\ddot{\boldsymbol{r}} = q(\boldsymbol{v} \times \boldsymbol{B} + \boldsymbol{E})$$

Problem 4

(Taylor 13.25) Here is another example of a canonical transformation, which is still to simple to be of any real use, but does nevertheless illustrate the power of these changes of coordinates.

a) Consider a system with one degree of freedom and Hamiltonian $\mathcal{H} = \mathcal{H}(q,p)$ and a new pair of coordinates Q and P defined so that

$$q = \sqrt{2P}\sin(Q), \quad p = \sqrt{2P}\cos(Q)$$

Prove that if $\partial\mathcal{H}/\partial q = -\dot{p}$ and $\partial\mathcal{H}/\partial p = \dot{q}$, it automatically follows that $\partial\mathcal{H}/\partial Q = -\dot{P}$ and $\partial\mathcal{H}/\partial P = \dot{Q}$. In other words, the Hamiltonian formalism applies just as well to the new coordinates as to the old.

b) Show that the Hamiltonian of a one-dimensional harmonic oscillator with mass $m = 1$ and force constant $k = 1$ is $\mathcal{H} = \frac{1}{2}(q^2 + p^2)$.

c) Show that if you rewrite this Hamiltonian in terms of the coordinates Q and P defined in (13.62), then Q is ignorable. [The change of coordinates (13.62) was cunningly chosen to produce this elegant result.] What is P?

d) Solve the Hamiltonian equation for $Q(t)$ and verify that, when rewritten for q, your solution gives the expected behavior.

Solution

a) Let's start by differentiating the Hamiltonian in respect to Q.

$$\frac{\partial\mathcal{H}}{\partial Q} = \frac{\partial\mathcal{H}}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial\mathcal{H}}{\partial p}\frac{\partial p}{\partial Q} = -\dot{P}$$

$$= -\frac{\partial P}{\partial q}\frac{\partial q}{\partial t} - \frac{\partial P}{\partial p}\frac{\partial p}{\partial t} = \frac{\partial P}{\partial p}\frac{\partial\mathcal{H}}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial\mathcal{H}}{\partial p}$$

This gives us the equalities

$$\frac{\partial P}{\partial p} = \frac{\partial q}{\partial Q}, \quad \frac{\partial p}{\partial Q} = -\frac{\partial P}{\partial q}$$

We can do the same thing with the Hamiltonian but in respect to P.

$$\frac{\partial\mathcal{H}}{\partial P} = \frac{\partial\mathcal{H}}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial\mathcal{H}}{\partial p}\frac{\partial p}{\partial P} = \dot{Q}$$

$$= \frac{\partial Q}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial Q}{\partial p}\frac{\partial p}{\partial t} = \frac{\partial Q}{\partial q}\frac{\partial\mathcal{H}}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial\mathcal{H}}{\partial q}$$

This gives us the equalities

$$\frac{\partial p}{\partial P} = \frac{\partial Q}{\partial q}, \quad \frac{\partial q}{\partial P} = -\frac{\partial Q}{\partial p}$$

which agrees with the previous found equalities. Now we need to find some of our partial derivatives

$$\frac{\partial q}{\partial P} = \frac{1}{\sqrt{2P}}\sin(Q) = \frac{q}{2P}$$

$$\frac{\partial q}{\partial Q} = \sqrt{2P}\cos(Q) = p$$

$$\frac{\partial p}{\partial P} = \frac{1}{\sqrt{2P}}\cos(Q) = \frac{p}{2P}$$
$$\frac{\partial p}{\partial Q} = -\sqrt{2P}\sin(Q) = -q$$

We can use a couple tricks to find Q and P in terms of q and p.

$$q^2 + p^2 = 2P$$
$$P = \frac{1}{2}(q^2 + p^2)$$
$$\frac{q}{p} = \tan(Q)$$
$$Q = \arctan\left(\frac{q}{p}\right)$$

We can then differentiate these new equations and find

$$\frac{\partial P}{\partial p} = p$$
$$\frac{\partial P}{\partial q} = q$$
$$\frac{\partial Q}{\partial p} = -\frac{q}{p^2 + q^2} = -\frac{q}{2P}$$
$$\frac{\partial Q}{\partial q} = \frac{p}{q^2 + p^2} = \frac{p}{2P}$$

These partial derivatives satisfy the equalities that we found earlier, so the Hamiltonian formalism applies to the new coordinates as well as the old coordinates.

b) The kinetic energy is

$$T = \frac{p^2}{2m} = \frac{p^2}{2}$$

and the potential energy is

$$U = \frac{kq^2}{2} = \frac{q^2}{2}$$

which means the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}(q^2 + p^2)$$

c) If we rewrite this Hamiltonian with our new coordinates, we have

$$\mathcal{H} = \frac{1}{2}(2P\sin^2(Q) + 2P\cos^2(Q) = P$$

Since the Hamiltonian is only dependent on P and not Q, Q is ignorable.

d) We can start off by finding the Hamilton equations.

$$\frac{\partial \mathcal{H}}{\partial P} = 1 = \dot{Q}$$

$$\frac{\mathcal{H}}{\partial Q} = 0 = -\dot{P}$$

These gives us

$$Q = t + C$$

Plugging this into our original equations gives us

$$q = \sqrt{2P}\sin(t + C)$$

$$p = \sqrt{2P}\cos(t + C)$$

which shows the oscillatory behavior we expect.

Problem 5

(Taylor 13.35) A beam of particles is moving along an accelerator pipe in the z direction. The particles are uniformly distributed in a cylindrical volume of length L_0 (in the z direction) and radius R_0. The particles have momenta uniformly distributed with p_z in an interval $p_0 \pm \Delta p_z$ and the transverse momentum $p_\perp$ inside a circle of radius $\Delta p_\perp$. To increase the particles' spatial density, the beam is focused by electric and magnetic fields, so that the radius shrinks to a smaller value R. What does Liouville's theorem tell you about the spread in the transverse momentum $p_\perp$ and the subsequent behavior of the radius R? (Assume that the focusing does not affect either L_0 or Δp_z.)

Solution

Liouville's theorem says that the volume of a system in phase space remains constant. The volume before the beam is focused is

$$V = \pi L_0 R_0^2 \Delta p_z \Delta p_{\perp 0}^2$$

The volume after the beam is focused is

$$V = \pi L_0 R_1^2 \Delta p_z \Delta p_{\perp 1}^2$$

Equating these two equations gives us

$$\frac{R_1^2}{R_0^2} = \frac{\Delta p_{\perp 0}^2}{\Delta p_{\perp 1}^2}$$

or

$$\frac{R_1}{R_0} = \frac{\Delta p_{\perp 0}}{\Delta p_{\perp 1}}$$

This means that if we focus the beam into a smaller radius, the transverse momentum spread increases.

Problem 6: Poisson Bracket

Recall the Poisson bracket of a two functions $F(q_i, p_i)$ and $G(q_i, p_i)$ is defined to be

$$\{F, G\} = \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

summing over i.

a) Suppose that q_i is ignorable, then $\{p_i, H\} = 0$.

b) If two quantities R and S are constants of motion, use the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

to show that $\{R, S\}$ is also a constant of motion.

c) Let $\boldsymbol{L}$ be an angular momentum with three components L_1, L_2 and L_3. Show that if L_1 and L_2 are conserved, then L_3 is also conserved. Also, show that L^2 and any component of $\boldsymbol{L}$ are Poisson commute. Here, L^2 is $\boldsymbol{L} \cdot \boldsymbol{L}$.

Solution

b)

$$\{H, \{R, S\}\} + \{R, \{S, H\}\} + \{S, \{H, R\}\} = 0$$
$$= -\{\{R, S\}, H\} + \{S, -\{R, H\}\} = 0$$
$$= -\{\{R, S\}, H\} = \{\{R, S\}, H\} = 0$$

So, if R and S are constants of motion, $\{R, S\}$ is also a constant of motion.

c) We can put L_1 and L_2 into Poisson brackets and see what happens.

$$\{L_1, L_2\} = \frac{\partial L_1}{\partial \boldsymbol{r}}\frac{\partial L_2}{\partial \boldsymbol{p}} - \frac{\partial L_1}{\partial \boldsymbol{p}}\frac{\partial L_2}{\boldsymbol{r}} = r_1 p_2 - r_2 p_1 = L_3$$

Which means if both L_1 and L_2 are conserved, so is L_3. We can do something similar for $\boldsymbol{L}^2$.

$$\begin{aligned}
\{L^2, L_1\} &= \{L_1^2 + L_2^2 + L_3^2, L_1\} \\
&= \{L_1^2, L_1\} + \{L_2^2, L_1\} + \{L_3^2\} \\
&= 2L_1\{L_1, L_1\} + 2L_2\{L_2, L_1\} + 2L_3\{L_3, L_1\} \\
&= -2L_2L_3 + 2L_2L_3 = 0
\end{aligned}$$

We can do the same thing for each component of angular momentum with the same result. The only difference will be which term is zero, negative or positive. This means that $\boldsymbol{L^2}$ Poisson commutes with each component of momentum.