Taylor Chapter 11 Summary: Coupled Oscillators and Normal Modes

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Two Masses and Three Springs

Let's look at a coupled system consisting of two carts of mass m_1 and m_2. respectively, next to each other on some plane each attached to their respective walls with a spring of force constant k_1 and k_3,respectively, and attached to each other with a spring of force constant k_2. We can say that these carts are **coupled** in that the motion of one affects the motion of the other. The net force on cart 1 is

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1) = -(k_1 + k_2)x_1 + k_2x_2$$

and on cart 2 is

$$m_2\ddot{x}_2 = k_2x_1 - (k_2 + k_3)x_2 =$$

This can be converted into the matrix form

$$\boldsymbol{M}\ddot{\boldsymbol{x}} = -\boldsymbol{K}\boldsymbol{x}$$

where $\boldsymbol{M}$ is the "mass matrix" and $\boldsymbol{K}$ is the "spring-constant" matrix and in this case

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The solutions to this equation will be of the form

$$x_i(t) = \alpha_i \cos(\omega t - \delta_i)$$

$$y_i(t) = \alpha_i \sin(\omega t - \delta_i)$$

We can use the trick we learned earlier to extend our solution into the complex plane, which is a lot easier to work with. This will give us

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) + iy_1(t) \\ x_2(t) + iy_2(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{i(\omega t - \delta_1)} \\ \alpha_2 e^{i(\omega t - \delta_2)} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 e^{-i\delta_1} e^{i\omega t} \\ \alpha_2 e^{-i\delta_2} e^{i\omega t} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t} = \boldsymbol{a}e^{i\omega t}$$

with $a_i = \alpha_i e^{-i\delta_1}$. Plugging this into our matrix equation gives

$$-\omega^2 \boldsymbol{M}\boldsymbol{a}e^{i\omega t} = -\boldsymbol{K}\boldsymbol{a}e^{i\omega t}$$

or

$$(\boldsymbol{K} - \omega^2\boldsymbol{M})\boldsymbol{a} = 0$$

Which is an eigenvalue equation. The eigenvalues of the determinant of the term in the parentheses are **normal frequencies**.

Identical Springs and Equal Masses

We will look at this system with each spring having the same force constant k and the same mass m. The mass and spring matrices will then be

$$\boldsymbol{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

The matrix we will look at to find the eigenvalues will be

$$(\boldsymbol{K} - \omega^2 \boldsymbol{M}) = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix}$$

The determinant is then

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = (2k - m\omega^2)^2 - k^2 = (k - m\omega^2)(3k - m\omega^2) = 0$$

which yields the eigenvalues

$$(\omega_1, \omega_2) = \left(\sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}}\right)$$

All that we need to do now is find the eigenvectors for each eigenvalue and we will have our normal modes.

The First Normal Mode

The first normal mode will have eigenvalue $\omega_1 = \sqrt{k/m}$. This gives us an eigenmatrix of

$$(\boldsymbol{K} - \omega_1^2 \boldsymbol{M}) = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

This gives us

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

This gives us the eigenvector

$$\hat{\boldsymbol{e_1}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This is equivalent to the carts oscillating together with the same phase, frequency and amplitude. The middle spring never changes length and the carts act as if there is only one cart and one spring. The equations of motion will be

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A \\ A \end{bmatrix} \cos(\omega_1 t - \delta)$$

The Second Normal Mode

The second normal mode is the mode with eigenvalue $\omega_2 = \sqrt{3k/m}$. The eigenmatrix is then

$$(\boldsymbol{K} - \omega_2^2 \boldsymbol{M}) = \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix}$$

This gives us

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

This gives us the eigenvector

$$\hat{\boldsymbol{e}}_{\mathbf{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This is equivalent to the carts oscillating with the same frequency and amplitude but opposite directions. The equations of motion will be

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A \\ -A \end{bmatrix} \cos(\omega_2 t - \delta)$$

The General Solution

The general solution in this new basis will be a superposition of our normal modes.

$$\boldsymbol{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2)$$

We can define new coordinates ξ_1, ξ_2 in this new coordinate system with

$$\xi_1 = \frac{1}{2}(x_1 + x_2), \quad \xi_2 = \frac{1}{2}(x_1 - x_2)$$

The first and second normal modes in this new coordinate system will be, respectively,

$$\boldsymbol{\xi}_{\mathbf{1}} = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} A\cos(\omega_1 t - \delta) \\ 0 \end{bmatrix}$$

$$\boldsymbol{\xi}_{\mathbf{2}} = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ A\cos(\omega_2 t - \delta) \end{bmatrix}$$

The Lagrangian Method

The equations of motion can also be found using the Lagrangian method. The kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

and the potential energy is

$$\begin{aligned} U &= \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3x_2^2 \\ &= \frac{1}{2}(k_1 + k_2)x_1^2 - k_2x_1x_2 + \frac{1}{2}(k_2 + k_3)x_2^2 \end{aligned}$$

This leads to the two Lagrange equations

$$m_1\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2x_2$$

$$m_2\ddot{x}_2 = k_2x_1 - (k_2 + k_3)x_2$$

Two Weakly Coupled Oscillators

We are still dealing with the same two cart, 3 spring system, except the second spring has a force constant much smaller than the others $k_2 \ll k$. The new spring matrix is then

$$\boldsymbol{K} = \begin{bmatrix} k + k_2 & -k_2 \\ -k_2 & k + k_2 \end{bmatrix}$$

And the eigenmatrix is then

$$(\boldsymbol{K} - \omega^2\boldsymbol{M}) = \begin{bmatrix} k + k_2 - m\omega^2 & -k_2 \\ -k_2 & k + k_2 - m\omega^2 \end{bmatrix}$$

This matrix has determinant

$$\begin{vmatrix} k + k_2 - m\omega^2 & -k_2 \\ -k_2 & k + k_2 - m\omega^2 \end{vmatrix} = (k - m\omega^2)(k + 2k_2 - m\omega^2)$$

Which gives us the eigenvalues

$$(\omega_1, \omega_2) = \left(\sqrt{\frac{k}{m}}, \sqrt{\frac{k + 2k_2}{m}}\right)$$

Since these frequencies are very close to each other we can define a new frequency which is the average of the two.

$$\omega_0 = \frac{\omega_1 + \omega_2}{2}$$

We can then say that

$$\omega_1 = \omega_0 - \epsilon, \quad \omega_2 = \omega_0 + \epsilon$$

Using this we have a general solution of

$$\boldsymbol{z}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_0 - \epsilon)} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_0 + \epsilon)t}$$

Over a short time interval, this oscillates relative to $e^{i\omega_0 t}$ since ϵ is so small. Over time, the motion of the carts will deviate from this behavior since $\epsilon \neq 0$. In a maximally mixed state $C_1 = C_2 = A/2$. The equations of motion become

$$\boldsymbol{z}(t) = \frac{A}{2} \begin{bmatrix} e^{-i\epsilon t} + e^{i\epsilon t} \\ e^{-i\epsilon t} - e^{i\epsilon t} \end{bmatrix} e^{i\omega_0 t} = A \begin{bmatrix} \cos(\epsilon t) \\ -i\sin(\epsilon t) \end{bmatrix} e^{i\omega_0 t}$$

The real part of this function is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A \begin{bmatrix} \cos(\epsilon t)\cos(\omega_0 t) \\ \sin(\epsilon t)\sin(\omega_0 t) \end{bmatrix}$$

The Double Pendulum

Let's consider a double pendulum with a mass m_1 being suspended by a massless rigid rod of length L_1 from a fixed pivot. The second mass m_2 is supsended by a massless rigid rod of length L_2 from m_1. The potential energy of m_1 is

$$U_1 = m_1 g L_1 (1 - \cos(\phi_1))$$

The potential energy of m_2 is

$$U_2 = m_2 g (L_1(1 - \cos(\phi_1)) + L_2(1 - \cos(\phi_2)))$$

The total potential energy is then

$$U = (m_1 + m_2) g L_1 (1 - \cos(\phi_1)) + m_2 g L_2 (1 - \cos(\phi_2))$$

The kinetic energy of m_1 is

$$T_1 = \frac{1}{2} m_1 L_1^2 \dot{\phi}_1^2$$

The kinetic energy of m_2 is

$$T_2 = \frac{1}{2} m_2 (L_1^2 \dot{\phi}_1^2 + 2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) + L_2^2 \dot{\phi}_2^2)$$

The total kinetic energy is then

$$T = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\phi}_1^2 + m_2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) + \frac{1}{2} m_2 L_2^2 \dot{\phi}_2^2$$

We are going to use the small angle approximation to reduce this problem to something that is solvable. The kinetic term then reduces to

$$T = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\phi}_1^2 + m_2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 + \frac{1}{2} m_2 L_2^2 \dot{\phi}_2^2$$

The potential term is reduced to

$$U = \frac{1}{2}(m_1 + m_2) g L_1 \phi_1^2 + \frac{1}{2} m_2 g L_2 \phi_2^2$$

The Lagrange equations are then

$$(m_1 + m_2)L_1^2 \ddot{\phi}_1 + m_2 L_1 L_2 \ddot{\phi}_2 = -(m_1 + m_2) g L_1 \phi_1$$

$$m_2 L_1 L_2 \ddot{\phi}_1 + m_2 L_2^2 \ddot{\phi}_2 = -m_2 g L_2 \phi_2$$

This can be put into our matrix equation form

$$\boldsymbol{M}\ddot{\boldsymbol{\phi}} = -\boldsymbol{K}\boldsymbol{\phi}$$

with

$$\boldsymbol{M} = \begin{bmatrix} (m_1 + m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} (m_1 + m_2) g L_1 & 0 \\ 0 & m_2 g L_2 \end{bmatrix}$$

The mass matrix plays the role of inertia and the spring matrix plays a similar role to that of a matrix of spring constants. The solutions will be in the same form as the previous system with

$$\boldsymbol{\phi}(t) = Re\boldsymbol{z}(t)$$

where

$$\boldsymbol{z}(t) = \boldsymbol{a}e^{i\omega t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t}$$

Equal Lengths and Masses

The masses are equal so $m_1 = m_2 = m$ and the lengths are equal and are $L_1 = L_2 = L$ and we have natural frequency $\omega_0 = \sqrt{g/L}$. Our mass and spring matrices are then

$$\boldsymbol{M} = mL^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{K} = mL^2 \begin{bmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{bmatrix}$$

The eigenmatrix is then

$$(\boldsymbol{K} - \omega^2 \boldsymbol{M}) = mL^2 \begin{bmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & (\omega_0^2 - \omega^2) \end{bmatrix}$$

This yields

$$2(\omega_0^2 - \omega^2)^2 - \omega^4 = \omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0$$

This gives the eigenvalues

$$\omega_1^2 = (2 - \sqrt{2})\omega_0^2, \quad \omega_2^2 = (2 - \sqrt{2})\omega_0^2$$

The first mode eigenmatrix will be

$$(\boldsymbol{K} - \omega_1^2 \boldsymbol{M}) = mL^2\omega_0^2(\sqrt{2} - 1) \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$$

Which yields the eigenvector

$$\hat{\boldsymbol{e_1}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

which is equivalent to both pendulums oscillating in phase, but the amplitude of the lower pendulum is higher. The equation of motion is then

$$\boldsymbol{\phi}(t) = Re\boldsymbol{a}e^{i\omega_1 t} = A_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \cos(\omega_1 t - \delta_1)$$

The second mode will have an eigenmatrix

$$(\boldsymbol{K} - \omega_2^2 \boldsymbol{M}) = -mL^2\omega_0^2(\sqrt{2} + 1) \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$$

which yields the eigenvector

$$\hat{\boldsymbol{e_2}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$$

which is equivalent to the pendulums oscillation completely out of phase with each other and the bottom pendulum having a larger amplitude. The equation of motion is then

$$\boldsymbol{\phi}(t) = Re\boldsymbol{a}e^{i\omega_2 t} = A_2 \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \cos(\omega_2 t - \delta_2)$$

The General Case

Using generalized coordinates $\boldsymbol{q}$ with Cartesian coordinates $\boldsymbol{r_\alpha}$, we can generalize normal modes to a system with n degrees of freedom. The kinetic energy is then described as

$$T = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\sum_{j,k} A_{jk}(\boldsymbol{q})\dot{q}_j\dot{q}_k$$

With small oscillations around a stable equilibrium set to $\boldsymbol{q}$, then only small oscillations concern us. We can the Taylor expand the potential energy and ignore the terms we don't like that are of third order or higher.

$$U(\boldsymbol{q}) = U(0) + \sum_j \frac{\partial U}{\partial q_j} q_j + \frac{1}{2}\sum_{j,k} \frac{\partial^2 U}{\partial q_j \partial q_k} q_j q_k + \dots$$

The first term can be set to zero and at the equilibrium point the second term is zero, so

$$U(\boldsymbol{q}) = \frac{1}{2}\sum_{j,k} K_{jk} q_j q_k$$

with

$$\frac{\partial^2 U}{\partial q_j \partial q_k} = K_{jk}$$

The kinetic energy can be simplified by ignoring all terms except the constant term since $\dot{q}_j\dot{q}_k$ is a second order small quantity. This yields

$$T(\dot{\boldsymbol{q}}) = \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k$$

The Lagrangian is then

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T(\dot{\boldsymbol{q}}) - U(\boldsymbol{q})$$

Since the kinetic energy is now dependent only on the generalized velocity and the potential energy is only dependent on the generalized position, the equations of motion will be solvable linear equations.

We can find the equations of motion using

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}, \quad [i = 1, \dots, n]$$

For a simpler system with two degrees of freedom yield a potential energy of

$$U = \frac{1}{2}\sum_{j,k=1}^{2} K_{jk} q_j q_k = \frac{1}{2}(K_{11}q_1^2 + 2K_{12}q_1 q_2 + K_{22}q_2^2)$$

The derivatives will be

$$\frac{\partial U}{\partial q_1} = K_{11}q_1 + K_{12}q_2$$

$$\frac{\partial U}{\partial q_2} = K_{22}q_2 + K_{12}q_1$$

which can be generalized to

$$\frac{\partial U}{\partial q_i} = \sum_j K_{ij}q_j, \quad [i = 1, \dots, n]$$

The kinetic energy works in the same way so the Lagrange equations become

$$\sum_j M_{ij}\ddot{q}_j = -\sum_j K_{ij}q_j, \quad [i = 1, \dots, n]$$

or in matrix form

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = -\boldsymbol{K}\boldsymbol{q}$$

The solutions will be in the form

$$\boldsymbol{q}(t) = Re\boldsymbol{z}(t), \quad \text{with} \quad \boldsymbol{z}(t) = \boldsymbol{a}e^{i\omega t}$$

The eigenvalue equation is

$$(\boldsymbol{K} - \omega^2\boldsymbol{M})\boldsymbol{a} = 0$$

with solutions iff ω satisfies the **characteristic equation**.

$$det(\boldsymbol{K} - \omega^2\boldsymbol{M}) = 0$$

Example: A Bead on a Wire

A bead of mass m is threaded on a frictionless wire in the xy plane that is bent in the shape of $y = f(x)$ with a minimum at the origin. The potential energy is then

$$U = mgy = mgf(x) \approx \frac{1}{2}mgf''(0)x^2$$

The kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(1 + f'(x)^2)\dot{x}^2 \approx \frac{1}{2}m\dot{x}^2$$

Three Coupled Pendulums

We have three pendulums connected by two springs. The generalized coordinates that will be used will be the angles each pendulum makes ϕ_1, ϕ_2, ϕ_3. We can write down the small angle approximations directly without finding the exact expressions for the energy. The kinetic energy will be

$$T = \frac{1}{2}mL^2(\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2)$$

The gravitational potential energy will be

$$U_{grav} = \frac{1}{2}mgL(\phi_1^2 + \phi_2^2 + \phi_3^2)$$

The spring potential energy can be approximated to be

$$U_{spr} = \frac{1}{2}kL^2((\phi_2 - \phi_1)^2 + (\phi_3 - \phi_2)^2)$$

$$= \frac{1}{2}kL^2(\phi_1^2 + 2\phi_2^2 + \phi_3^2 - 2\phi_1\phi_2 - 2\phi_2\phi_3)$$

We can switch over to **natural units** and make annoying constants disappear. We can bring them back later with dimensional analysis. So, let's choose $m = L = 1$. The kinetic energy is then

$$T = \frac{1}{2}(\dot{\phi}_1^2 + \dot{\phi}_2 + \dot{\phi}_3)$$

and the potential energy is

$$U = \frac{1}{2}g(\phi_1^2 + \phi_2^2 + \phi_3^2) + \frac{1}{2}k(\phi_1^2 + 2\phi_2^2 + \phi_3^2 - 2\phi_1\phi_2 - 2\phi_2\phi_3)$$

We can use our matrix equation with

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} g+k & -k & 0 \\ -k & g+2k & -k \\ 0 & -k & g+k \end{bmatrix}$$

The eigenmatrix is then

$$(\boldsymbol{K} - \omega^2\boldsymbol{M}) = \begin{bmatrix} g+k-\omega^2 & -k & 0 \\ -k & g+2k-\omega^2 & -k \\ 0 & -k & g+k-\omega^2 \end{bmatrix}$$

With the eigen-equation being

$$(g - \omega^2)(g + k - \omega^2)(g + 3k - \omega^2) = 0$$

This gives the eigenvalues

$$(\omega_1^2, \omega_2^2, \omega_3^2) = (g, g+k, g+3k)$$

The first mode will yield an eigenvector of

$$\hat{\boldsymbol{e_1}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Which is equivalent to all of the pendulums oscillating together with the same amplitude and same phase and aks like a single pendulum. The second mode will have an eigenvector

$$\hat{\boldsymbol{e_2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

which is equivalent to the outside penulums oscillating with equal amplitude but opposite phase and the middle pendulum is not moving. The third mode will have an eigenvector

$$\hat{\boldsymbol{e_3}} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

which is equivalently the outside pendulums oscillating together with the same phase and amplitude while the middle pendulum is oscillating with twice the amplitude and opposite phase of the other pendulums.