Analytic Mechanics Homework Assignment 11

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Problem 1: Orthonormality of Eigenvectors

Let $\{\omega_i, i = 1, \ldots, n\}$ and $\boldsymbol{A_i}, i = 1, \ldots, n$ be a set of eigenvalues (characteristic frequencies) and the corresponding eigenvectors such that A_{ji} is the j-th components of the eigenvector $\boldsymbol{A_i}$. This means the r-th eigenvalue ω_r and eigenvector $\boldsymbol{A_r}$ satisfy the condition

$$\sum_j (K_{kj} - \omega^2 M_{kj}) A_{jr} = 0$$

where K_{kj} and M_{kj} are the matrix elements of the spring constant matrix $\boldsymbol{K}$ and mass matrix $\boldsymbol{M}$ respectively. Use this condition to show that the eigenvectors are orthogonal. That is, the product of the components of any eigenvectors $\boldsymbol{A_r}$ and $\boldsymbol{A_s}$ fulfills

$$\sum_{j,k} M_{kj} A_{jr} A_{ks} = 0$$

$$r \neq s$$

By subjecting the components of the eigenvector $\boldsymbol{A_r}$ to the requirement

$$\sum_{j,k} M_{kj} A_{jr} A_{kr} = 1$$

$\boldsymbol{A_r}$ is then normalized.

Solution

We can start by moving the mass matrix term to the RHS.

$$\sum_j K_{kj} A_{jr} = \sum_j \omega_1^2 M_{kj} A_{jr}$$

We can multiply everything by a term from a vector with a different eigenvalue.

$$\sum_{j,k} K_{kj} A_{ks} A_{jr} = \sum_{j,k} \omega_1^2 M_{kj} A_{ks} A_{kr}$$

We can also set this up by switching the roles of the initial and secondary vector.

$$\sum_{j,k} K_{kj} A_{jr} A_{ks} = \sum_{j,k} \omega_s^2 M_{kj} A_{jr} A_{ks}$$

Subtracting these two terms gives us

$$\sum_{j,k} (\omega_r^2 - \omega_s^2) M_{kj} A_{jr} A_{ks} = 0$$

If $r \neq s$, the eigenvalue term is nonzero which means that

$$\sum_{j,k} M_{kj} A_{jr} A_{ks} = 0$$

This means that the eigenvectors of different eigenvalues are orthogonal to each other. This isn't necessarily true if there are any degeneracies, but you can always find orthogonal eigenvectors to map that degenerate space.

Problem 2

(Taylor 11.11) a) Write down the equations of motion corresponding to (11.2) for the equal-mass carts of section 11.2 with three identical springs, but with each cart subject to a linear resistive force $-b\boldsymbol{v}$ (same coefficient b for both carts) and with a driving force $F(t) = F_0 \cos(\omega t)$ applied to cart 1.

b) Show that if you change variables to the normal coordinates $\xi_1 = \frac{1}{2}(x_1 + x_2)$ and $\xi_2 = \frac{1}{2}(x_1 - x_2)$, the equations of motion for ξ_1 and ξ_2 are uncoupled.

c) Using the methods of section 5.5, write down the general solutions.

d) Assuming that $\beta = b/2m \ll \omega_0$, show that ξ_1 resonates when $\omega \approx \omega_0 = \sqrt{k/m}$ and likewise ξ_2 when $\omega \approx \sqrt{3}\omega_0$.

e) Prove, on the other hand, that if both carts are driven in phase with the same force $F_0 \cos(\omega t)$, only ξ_1 shows a resonance. Explain.

Solution

a) The equations of motion will be

$$m\ddot{x}_1 = -2kx_1 + kx_2 - b\dot{x}_1 + F_0 \cos(\omega t)$$

$$m\ddot{x}_2 = kx_1 - 2kx_2 - b\dot{x}_2$$

b) By adding the two equations together and dividing by two, we get

$$\frac{1}{2}m(\ddot{x}_1 + \ddot{x}_2) = \frac{1}{2}(-kx_1 - k_2 - b\dot{x}_1 - b\dot{x}_2 + F_0\cos(\omega t))$$

$$\ddot{\xi}_1 + \frac{b}{m}\dot{\xi}_1 + \frac{k}{m}\xi_1 = \frac{F_0}{2m}\cos(\omega t)$$

We get something similar when we subtract equation 2 from equation 1 and then divide by two.

$$\frac{1}{2}m(\ddot{x}_1 - \ddot{x}_2) = \frac{1}{2}(-3kx_1 + 3kx_2 - b\dot{x}_1 + b\dot{x}_2 + F_0\cos(\omega t)$$

$$\ddot{\xi}_2 + \frac{b}{m}\dot{\xi}_2 + \frac{3k}{m}\xi_2 = \frac{F_0}{2m}\cos(\omega t)$$

As can be seen, these two equations are no longer coupled in this new basis.

c) We are going to let $\beta = b/2m$, $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$ so our equations can be rewritten as

$$\ddot{\xi}_1 + 2\beta\dot{\xi}_1 + \omega_1^2\xi_1 = \frac{F_0}{2m}\cos(\omega t)$$

$$\ddot{\xi}_2 + 2\beta\dot{\xi}_2 + \omega_2^2\xi_2 = \frac{F_0}{2m}\cos(\omega t)$$

We can start y finding the homogeneous solutions. We have characteristic equations of the form

$$r^2 + 2\beta r + \omega_i^2 = 0$$

with roots being

$$r = -\beta \pm \sqrt{\beta^2 - \omega_i^2}$$

This gives us the homogeneous solutions for each equation.

$$\xi_{1h} = C_{11}e^{(-\beta+\sqrt{\beta^2-\omega_1^2})t} + C_{12}e^{(-\beta-\sqrt{\beta^2-\omega_1^2})t}$$

$$\xi_{2h} = C_{21}e^{(-\beta+\sqrt{\beta^2-\omega_2^2})t} + C_{22}e^{(-\beta-\sqrt{\beta^2-\omega_2^2})t}$$

Next, we want the particular solution. We can solve this for one and it will give us the solution for both modes. At first, we will extend our problem into a complex equation, because they are easier to solve, and then the real part will be the physical part. Let $A = F_0/2m$ to make it easier to write and let η be the frequency of a mode.

$$\ddot{z} + 2\beta\dot{z} + \eta^2 z = Ae^{i\omega t}$$

We can make a guess that our solution will be of the form

$$z = Be^{i(\omega t - \delta)}$$

Plugging this into our equation gives

$$-B\omega^2 e^{i(\omega t-\delta)} + 2\beta B i\omega e^{i(\omega t-\delta)} + B\eta^2 e^{i(\omega t-\delta)} = Ae^{i\omega t}$$

Simplifying it further gives us

$$Be^{-i\delta}(\eta^2 + 2\beta\omega i - \omega^2) = A$$

$$Be^{-i\delta} = \frac{A}{\eta^2 + 2\beta\omega i - \omega^2} = \frac{A(\eta^2 - \omega^2 - 2\beta\omega i)}{(\eta^2-\omega^2)^2 + 4\beta^2\omega^2}$$

We can break this complex exponent into its parts using Euler's identity which gives us

$$B\cos(\delta) = \frac{A(\eta^2-\omega^2)}{(\eta^2-\omega^2)^2 + 4\beta^2\omega^2}$$

$$B\sin(\delta) = \frac{2\beta A\omega}{(\eta^2-\omega^2)^2 + 4\beta^2\omega^2}$$

We can use the Pythagorean identity to solve for B.

$$B^2 = \frac{A^2((\eta^2-\omega^2)^2 + 4\beta^2\omega^2)}{((\eta^2-\omega^2)^2 + 4\beta^2\omega^2)^2}$$

which gives us

$$B = \frac{A}{\sqrt{(\eta^2-\omega^2)^2 + 4\beta^2\omega^2}}$$

We can find delta by dividing our trigonometric expressions

$$\tan(\delta) = \frac{2\beta\omega A}{A(\eta^2-\omega^2)}$$

$$\delta = \arctan\left(\frac{2\beta\omega}{\eta^2-\omega^2}\right)$$

We only care about the real parts of this so

$$\xi_{1p} = \frac{F_0}{2m\sqrt{(\omega_1^2-\omega^2)^2 + 4\beta^2\omega^2}}\cos\left(\omega t - \arctan\left(\frac{2\beta\omega}{\omega_1^2-\omega^2}\right)\right)$$

$$\xi_{2p} = \frac{F_0}{2m\sqrt{(\omega_2^2-\omega^2)^2 + 4\beta^2\omega^2}}\cos\left(\omega t - \arctan\left(\frac{2\beta\omega}{\omega_2^2-\omega^2}\right)\right)$$

The generalized solution will just be the sum of the respective parts.

$$\xi_1 = C_{11}e^{(-\beta+\sqrt{\beta^2-\omega_1^2})t}+C_{12}e^{(-\beta-\sqrt{\beta^2-\omega_1^2})t}+\frac{F_0}{2m\sqrt{(\omega_1^2-\omega^2)^2+4\beta^2\omega^2}}\cos\left(\omega t-\arctan\left(\frac{2\beta\omega}{\omega_1^2-\omega^2}\right)\right)$$

$$\xi_2 = C_{21}e^{(-\beta+\sqrt{\beta^2-\omega_2^2})t}+C_{22}e^{(-\beta-\sqrt{\beta^2-\omega_2^2})t}+\frac{F_0}{2m\sqrt{(\omega_2^2-\omega^2)^2+4\beta^2\omega^2}}\cos\left(\omega t-\arctan\left(\frac{2\beta\omega}{\omega_2^2-\omega^2}\right)\right)$$

d) To notice the resonance, we want to look at the amplitude of the cosine part in each equation. The $4\beta^2\omega^2$ piece can be ignored since it will be really small compared to the other piece in the square roots. This gives us an amplitude of

$$A = \frac{F_0}{2m(\eta^2-\omega^2)}$$

If we let $\omega_0 = \omega_1 = \omega$, the $\eta = \omega$ for the first equation ξ_1, and the magnitude will be infinite. For the second equation ξ_2, the magnitude won't be infinite because $\eta \neq \omega_2$. This shows that ξ_1 resonates with that choice of frequency. On the otherhand, if we let $\omega = \sqrt{3}\omega_0 = \omega_2$, $\eta = \omega$ for the second equation ξ_2 and the magnitude is infinite, while the first equation ξ_1 will not be infinite in magnitude since $\eta \neq \omega_1$. This shows that ξ_2 resonates with that choice of frequency.

e) The new equations of motion will be

$$m\ddot{x}_1 = -2kx_1 + kx_2 - b\dot{x}_1 + F_0\cos(\omega t)$$

$$m\ddot{x}_2 = kx_1 - 2kx_2 - b\dot{x}_2 + F_0\cos(\omega t)$$

which can be converted into our normal mode eigenbasis

$$\ddot{\xi}_1 + 2\beta\dot{\xi}_1 + \omega_1^2\xi_1 = \frac{F_0}{m}\cos(\omega t)$$

$$\ddot{\xi}_2 + 2\beta\dot{\xi}_2 + \omega_2^2\xi_2 = 0$$

This means that only ξ_1 has a resonance since it is the only one that has a nonhomogeneous solution. Since both carts are being pushed with the same driving force, there isn't much opportunity for them to be pushed out of phase with each other, so over time they will move together and the difference of their motion, $\xi_2 = x_1 - x_2$, will decay to zero.

Problem 3

(Taylor 11.12) Here is a different way to couple two oscillators. The two carts in Figure 11.16 have equal masses m (though different shapes). They are joined

by identical but separate springs (force constant k) to separate walls. Cart 2 rides in cart 1, as shown, and cart 1 is filled with molasses, whose viscous drag supplies the couplign between the carts.

a) Assuming that the drag force has magnitude $\beta m v$ where $\boldsymbol{v}$ is the relative velocity of the two carts, write down the equations of motion of the two carts using as coordinates x_1 and x_2, the displacements of the carts from their equilibrium positions. Show that they can be written in matrix form as $\ddot{\boldsymbol{x}} + \beta\boldsymbol{D}\dot{\boldsymbol{x}} + \omega_0^2\boldsymbol{x} = 0$, where $\boldsymbol{x}$ is the 2×1 column made up of x_1 and x_2, $\omega_0 = \sqrt{k/m}$, and $\boldsymbol{D}$ is a certain 2×2 square matrix.

b) There is nothing to stop you from seeking a solution of the form $\boldsymbol{x}(t) = Re\boldsymbol{z}(t)$, with $\boldsymbol{z}(t) = \boldsymbol{a}e^{rt}$. Show that you do indeed get two solutions of this form with $r = i\omega_0$ or $r = -\beta + i\omega_1$ where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. (Assume that the viscous force is weak, so that $\beta < \omega_0$.)

c) Describe the corresponding motions. Explain why one of these modes is damped but the other is not.

Solution

a) The equations of motion will be

$$m\ddot{x}_1 = -kx_1 - \beta m(\dot{x}_1 - \dot{x}_2)$$

$$m\ddot{x}_2 = -kx_2 - \beta m(\dot{x}_2 - \dot{x}_1)$$

or

$$\ddot{x}_1 + \omega_0^2 + \beta(\dot{x}_1 - \dot{x}_2) = 0$$
$$\ddot{x}_2 + \omega_0^2 - \beta(\dot{x}_1 - \dot{x}_2) = 0$$

This gives us a matrix $\boldsymbol{D}$ of

$$\boldsymbol{D} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

That satisfies

$$\ddot{\boldsymbol{x}} + \beta\boldsymbol{D}\dot{\boldsymbol{x}} + \omega_0^2\boldsymbol{x} = 0$$

b) With the choice of solution to be $\boldsymbol{x} = Re\boldsymbol{z}$ with $\boldsymbol{z} = \boldsymbol{a}e^{rt}$, our differential equation becomes

$$e^{rt}(r^2\boldsymbol{I} + \beta r\boldsymbol{D} + \omega_0^2\boldsymbol{I})\boldsymbol{a} = 0$$

The eigenmatrix will then be

$$((r^2 + \omega_0^2)\boldsymbol{I} + \beta r\boldsymbol{D}) = \begin{bmatrix} r^2 + \beta r + \omega_0^2 & -\beta r \\ -\beta r & r^2 + \beta r + \omega_0^2 \end{bmatrix}$$

which yields the eigenvalue polynomial equation

$$(r^2 + \beta r + \omega_0^2) - \beta^2 r^2 = 0$$

$$(r^2 + \omega_0^2)(r^2 + 2\beta r + \omega_0^2) = 0$$

$$(r^2 + \omega_0^2)(r + \beta - i\omega_1)(r + \beta + i\omega_1) = 0$$

This gives us the eigenvalues

$$(r_1, r_2, r_3, r_4) = (i\omega_0, -i\omega_0, -\beta + i\omega_1, -\beta - i\omega_1)$$

We reject the second and fourth values because they are physically the same as the first and the third values. Looking at the case for $r = i\omega_0$, we find the eigenvector

$$\hat{\boldsymbol{e_1}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Since the physical part is the real part of this solution, the equation of motion will be

$$\boldsymbol{x} = \begin{bmatrix} A \\ A \end{bmatrix} \cos(\omega_0 t - \delta)$$

For the second mode, we will want to solve that long polynomial $r^2 + \beta r + \omega_0$ with our $r = -\beta + i\omega_1$.

$$\beta^2 - 2\beta\omega_1 i - \omega_1^2 - \beta^2 + \beta\omega_1 i + \omega_0^2$$
$$= \omega_0^2 - \omega_1^2 - \beta\omega_1 i$$
$$= \omega_0^2 - \omega_0^2 + \beta^2 - \beta\omega_1 i$$
$$= \beta^2 - \beta\omega_1 i$$

This will yield us an eigenvector

$$\hat{\boldsymbol{e_2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The resulting solutions will then be

$$\boldsymbol{x} = e^{-\beta t} \begin{bmatrix} A \\ -A \end{bmatrix} \cos(\omega_1 t - \delta)$$

c) The first mode corresponds to the carts oscillating together with the same frequency, amplitude, and phase. The second mode corresponds to the carts oscillating with the same frequency, amplitude and opposite phase. Since there is a dampening force attributed to the difference in displacement of the two carts, this second mode has a dampening effect and will die off eventually. The carts in the first mode stay together, so the difference in positions of the two remains zero, and the dampening effect never comes into play.

Problem 4

(Taylor 11.19) A simple pendulum (mass M and length L) is suspended from a cart (mass m) that can oscillate on the end of a spring of force constant k, as shown in Figure 11.18.

a) Assuming that the angle ϕ remains small, write down the system's Lagrangian and the equations of motion for x and ϕ.

b) Assuming that $m = M = L = g = 1$ and $k = 2$ (all in appropriate units) find the normal frequencies, and for each normal frequency find and describe the motion of the corresponding normal mode.

Solution

a) The kinetic energy of the system is

$$T = \frac{1}{2}ML^2\dot{\phi}^2 + \frac{1}{2}(m+M)\dot{x}^2 + ML\dot{x}\dot{\phi}\cos(\phi)$$

And the potential energy is

$$U = mgL(1-\cos(\phi)) + \frac{1}{2}kx^2$$

This gives us the Lagrangian

$$\mathcal{L} = \frac{1}{2}ML^2\dot{\phi}^2 + \frac{1}{2}(m+M)\dot{x}^2 + ML\dot{x}\dot{\phi}\cos(\phi) - mgL(1-\cos(\phi)) - \frac{1}{2}kx^2$$

This yields the two Lagrange equations of motion are

$$ML^2\ddot{\phi} + MLg\ddot{x}\cos(\phi) + MLg\sin(\phi) = 0$$

$$(m+M)\ddot{x} + ML\ddot{\phi}\cos(\phi) - ML\dot{\phi}^2\sin(\phi) + kx = 0$$

b) With our given assumptions, the new equations of motion will be

$$\ddot{\phi} + \ddot{x} + \phi = 0$$

$$2\ddot{x} + \ddot{\phi} + 2x = 0$$

We can let our mass and spring matrices be

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

with our vector being

$$\boldsymbol{x} = \begin{bmatrix} \phi \\ x \end{bmatrix}$$

The eigenmatrix will then be

$$(\boldsymbol{K} - \omega^2 \boldsymbol{M}) = \begin{bmatrix} 1-\omega^2 & -\omega^2 \\ -\omega^2 & 2-2\omega^2 \end{bmatrix}$$

The determinant of this matrix will yield the eigenvalue polynomial

$$(\omega^2 - 2 - \sqrt{2})(\omega^2 - 2 + \sqrt{2}) = 0$$

which yields the eigenvalues

$$(\omega_1^2, \omega_2^2) = (2 + \sqrt{2}, 2 - \sqrt{2})$$

The first mode with eigenvalue ω_1^2 will have the eigenvector

$$\hat{\boldsymbol{e}_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} \left(\frac{2+\sqrt{2}}{1+\sqrt{2}}\right) \\ 1 \end{bmatrix}$$

This corresponds to the cart and the pendulum oscillating together with the same frequency and direction, but the pendulum will have a larger amplitude oscillation than the cart. The mode with eigenvalue ω_2^2 will have the eigenvector

$$\hat{\boldsymbol{e}_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\left(\frac{2-\sqrt{2}}{1-\sqrt{2}}\right) \\ 1 \end{bmatrix}$$

This corresponds to the pendulum and the cart oscillating at the same frequency but opposite phase and the amplitude of the pendulum's oscillation is larger.

Problem 5

(Taylor 11.31) Consider a frictionless rigid horizontal hoop of radius R. Onto this hoop I thread three beads with masses $2m$, m and m, and between the beads, three identical springs, each with force constant k. Solve for the three normal frequencies and find and describe the three normal modes.

Solution

The three equations of motion are

$$m\ddot{\phi}_1 = -2k\phi_1 + k\phi_2 + k\phi_3$$
$$m\ddot{\phi}_2 = k\phi_1 - 2k\phi_2 + k\phi_3$$
$$2m\ddot{\phi}_3 = k\phi_1 + k\phi_2 - 2k\phi_3$$

Which gives us our mass and spring matrices

$$\boldsymbol{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{bmatrix}$$

The matrix we will want to find the determinant of will be

$$(\boldsymbol{K} - \omega^2\boldsymbol{M}) = \begin{bmatrix} 2k - m\omega^2 & -k & -k \\ -k & 2k - m\omega^2 & -k \\ -k & -k & 2k - 2m\omega^2 \end{bmatrix}$$

The determinant yields the eigenvalue polynomial of

$$2m\omega^2(2k - m\omega^2)(3k - m\omega^2) = 0$$

With eigenvalues

$$(\omega_1^2, \omega_2^2, \omega_3^2) = (0, \frac{2k}{m}, \frac{3k}{m})$$

For eigenvalue ω_1^2 the eigenvector will be

$$\hat{\boldsymbol{e_1}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

This mode is equivalent to all the beads moving together, and the springs are neither compressed nor stretched. The second mode will have eigenvalue ω_2^2 and will have an eigenvector

$$\hat{\boldsymbol{e_2}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

This mode corresponds to the more massive bead oscillating with the opposite phase of the other two beads but the same amplitude and frequency. The third mode will have eigenvalue ω_3^2 and will have an eigenvector

$$\hat{\boldsymbol{e_3}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

This mode corresponds to the more massive bead not moving and the other two beads oscillating with the same amplitude and frequency but opposite phase.

Problem 6

(Taylor 11.32) As a model of a linear triatomic molecule (such as CO_2), consider the system shown in Figure 11.21, with two identical atoms each of mass m connected by two identical springs to a single atom of mass M. To simplify matters, assume that the system is confined to move in one dimension.

a) Write down the Lagrangian and find the normal frequencies of the system. Show that one of the normal frequencies is zero.

b) Find and describe the motion in the normal modes whose frequencies are nonzero.

c) Do the same for the mode with zero frequency. [*Hint:* See the comments at the end of problem 11.27.]

Solution

a) The equations of motion are

$$m\ddot{x}_1 = -kx_1 + kx_2$$
$$M\ddot{x}_2 = kx_1 - 2kx_2 + kx_3$$
$$m\ddot{x}_3 = kx_2 - kx_3$$

which gives the mass and spring matrices

$$\boldsymbol{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

This gives us an eigenmatrix

$$(\boldsymbol{K} - \omega^2 \boldsymbol{M}) = \begin{bmatrix} k - \omega^2 m & -k & 0 \\ -k & 2k - \omega^2 M & -k \\ 0 & -k & k - \omega^2 m \end{bmatrix}$$

This gives the eigenvalue polynomial

$$(k - \omega^2 m)(2k - \omega^2 M)(k - \omega^2 m) - k^2) + k(\omega^2 mk - k^2) = 0$$

$$(k - \omega^2 m)(2k^2 - 2k\omega^2 m - \omega^2 kM + \omega^4 mM - 2k^2) = 0$$

$$(k - \omega^2 m)(\omega^4 mM - (2km + kM)\omega^2) = 0$$

$$\omega^2 (k - \omega^2 m)(\omega^2 - \frac{2km + kM}{mM}) = 0$$

which gives the eigenvalues

$$(\omega_1^2, \omega_2^2, \omega_3^2) = (0, \frac{k}{m}, \frac{2k}{M} + \frac{k}{m})$$

b) The second mode will have the eigenvalue ω_2^2 which will yield the eigenvector

$$\hat{\boldsymbol{e}_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

or with the equation of motion

$$\boldsymbol{x} = \begin{bmatrix} A \\ 0 \\ -A \end{bmatrix} \cos(\omega_2 t - \delta)$$

This corresponds to the outer masses oscillating at the same amplitude and frequency as each other, but opposite phases while the middle mass is not moving. The third mode will have the eigenvalue ω_3^2 which will yield the eigenvector

$$\hat{\boldsymbol{e}_3} = \frac{1}{\sqrt{\frac{4m^2}{M^2} + 1 + 1}} \begin{bmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{bmatrix}$$

or with the equation of motion

$$\boldsymbol{x} = \begin{bmatrix} A \\ -\frac{2mA}{M} \\ A \end{bmatrix} \cos(\omega_3 t - \delta)$$

This mode corresponds to the outer masses oscillating with the same frequency and amplitude and phase as each other while the middle mass is oscillating with the same frequency, opposite phase and a scaled amplitude based on the ratio of the smaller masses to the bigger mass.

c) The first mode will have an eigenvalue of ω_1^2 which will yield the eigenvector

$$\hat{\boldsymbol{e}_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

or with the equation of motion

$$\boldsymbol{x} = \begin{bmatrix} A \\ A \\ A \end{bmatrix} t + \begin{bmatrix} C \\ C \\ C \end{bmatrix}$$

This mode corresponds to the translation of the molecule in a direction without compressing or stretching any of the springs.