Analytic Mechanics Homework Assignment 8

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Problem 1

(Taylor 7.36) A pendulum is made from a massless spring (force constant k and unstretched length l_0) that is suspended at one end from a fixed pivot O and has a mass m attached to its other end. The spring can stretch and compress but cannot bend, and the whole system is confined to a single vertical plane.

a) Write down the Lagrangian for the pendulum, using as a generalized coordinates the usual angle ϕ and the length r of the spring.

b) Find the two Lagrange equations of the system and interpret them in terms of Newton's second law, as given in Equation (1.48).

c) The equations of part b cannot be solved analytically in general. However, they *can* be solved for small oscillations. Do this and describe the motion. [Hint: let l denote the equilibrium length of the spring with the mass hanging from it and write $r = l + \epsilon$. "Small oscillations" involve only small values of ϵ and ϕ, so you can use the small angle approximations and drop from your equations all terms that involve powers of ϵ or ϕ (or their derivatives) higher than the first power (also products of ϵ and ϕ or their derivatives). This dramatically simplifies and uncouples the equations.]

Solution

a) Let's start by finding the kinetic energy of the system. We have a kinetic energy term related to the angular motion of the pendulum as well as the kinetic energy term relating to the lengthening and shortening of the length of the pendulum.

$$T = \frac{1}{2}mr^2\dot{\phi}^2 + \frac{1}{2}m\dot{r}^2$$

We next look into finding the potential energy of the system. We can define the $E = 0$ to be at $\phi = \pi/2$.

$$U = -mgr\cos(\phi) + \frac{1}{2}k(r - l_0)^2$$

Our Lagrangian is then

$$\mathcal{L} = \frac{1}{2}mr^2\dot{\phi}^2 + \frac{1}{2}m\dot{r}^2 + mgr\cos(\phi) - \frac{1}{2}k(r - l_0)^2$$

b) Let's start with the radial Lagrange equation.

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\phi}^2 + mg\cos(\phi) - k(r - l_0)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = m\ddot{r}$$

The radial Lagrange equation is then

$$m(\ddot{r} - r\dot{\phi}^2) = g\cos(\phi) - k(r - l_0)$$

The left hand side is our Newton's second law equation in polar coordinates. The first term on the right hand side is the component caused by gravity while the second term on the right hand side is the components caused by the compression/decompression of the spring. Let's now look at the ϕ equations.

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mgr\sin(\phi)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = mr^2\ddot{\phi} + 2mr\dot{\phi}\dot{r}$$

The angular Lagrange equation is then

$$m(r\ddot{\phi} + 2\dot{\phi}\dot{r}) = -mg\sin(\phi)$$

The left hand side is Newton's second law equation in polar coordinates. On the right hand side, we have a gravitational term that is proportional to the weight of the mass.

c) Let's first look at the radial equation

$$m(\ddot{r} - r\dot{\phi}^2) = mg\cos(\phi) - k(r - l_0)$$

At small angles, we can assume $\phi \approx 0$. We can use this to try to relate the gravitational component and the spring component. The derivatives of our coordinates can be assumed to eb zero, which gives us

$$0 = mg - k(r - l_0)$$

$$mg = k(r - l_0)$$
$$r = \frac{mg}{k} + l_0 = l$$

We want to now introduce our small perturbation of $r = l + \epsilon$. Its derivatives will then be

$$\dot{r} = \dot{\epsilon}$$

$$\ddot{r} = \ddot{\epsilon}$$

Our equation is then

$$m\ddot{\epsilon} = mg - k(\frac{mg}{k} + l_0 + \epsilon - l_0)$$
$$\ddot{\epsilon} = -\frac{k}{m}\epsilon$$

With $\omega^2 = k/m$ we have sinusoidal oscillations with frequency ω in the radial direction. Let's move on to the angular equation.

$$m(r\ddot{\phi} + 2\dot{\phi}\dot{r}) = -mg\sin(\phi)$$

We can simplify this with small oscillations to give

$$(l + \epsilon)\ddot{\phi} + 2\dot{\phi}\dot{\epsilon} = -g\phi$$

We can further simplify this into

$$l\ddot{\phi} = -g\phi$$
$$\ddot{\phi} = -\frac{g}{l}\phi$$

With $\omega^2 = g/l$, there is an angular sinusoidal oscillation about l at frequency ω.

Problem 2

(Taylor 14.29) An elastic collision is defined as one in which the total kinetic energy of the two particles is the same before and after the collision.

a) Show that in the CM frame, the individual kinetic energies of the two particles are separately conserved in an elastic collision.

b) Explain clearly why the same result is obviously *not* true in the lab frame. (Think about the energy of the target particle.)

c) Let ΔE denote the energy gained by the target particle in the collision (and hence the energy lost by the projectile). Using Figure 14.15, show that the fractional energy lost by the projectile (in the lab frame) is

$$\frac{\Delta E}{E} = \frac{4\lambda}{(1+\lambda)^2}\sin^2\left(\frac{\theta}{2}\right)$$

where, as usual, λ is the mass ratio m_1/m_2. (Note that in Figure 14.15 the line DC represents the recoil momentum of the target.)

d) For a given mass ratio λ, what sort of collision gives the largest fractional energy loss? What value of λ maximizes this energy loss? (Your answer is important in situations where one wants a particle to lose energy as quickly as possible—as in a nuclear reactor, for example.)

Solution

a) In the CM frame, the sum of the momenta before the collision is zero. Since the collision is elastic, momentum is conserved so the sum of the momenta after the collision is also zero.

$$\boldsymbol{p_{1i}} + \boldsymbol{p_{2i}} = \mathbf{0}$$

$$\boldsymbol{p_{1f}} + \boldsymbol{p_{2f}} = \mathbf{0}$$

This gives us

$$\boldsymbol{p_{1i}} = -\boldsymbol{p_{2i}}$$

$$\boldsymbol{p_{1f}} = -\boldsymbol{p_{2f}}$$

The kinetic energy is conserved in an elastic collision so we can equate the kinetic energy before and after the collision.

$$T_i = T_f$$

$$\frac{p_{1i}^2}{2m_1} + \frac{p_{2i}^2}{2m_2} = \frac{p_{1f}^2}{2m_1} + \frac{p_{2f}^2}{2m_2}$$

Using what we found earlier

$$p_{1i}^2\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = p_{1f}^2\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

$$p_{1i}^2 = p_{1f}^2$$

$$p_{2i}^2 = p_{2f}^2$$

This tells us that the kinetic energy is conserved for each particle separately since th magnitude of the momentum for each particle is unchanged after the collision.

b) In the lab frame this can't be true since one of the particles would be initially stationary, and will then get some energy/momentum from the collision.

c) The change in energy is easiest to look at with the CM frame.

$$\Delta E = \frac{(\Delta \boldsymbol{p})^2}{2m_2} = \frac{1}{2m_2}(p_i^2 + p_f^2 - 2\boldsymbol{p_i} \cdot \boldsymbol{p_f})$$

We know that $p_i = p_f$ so

$$\Delta E = \frac{1}{2m_2}(2p_i^2 - 2p_i^2\cos(\theta_{CM}))$$
$$= \frac{2p_i^2\sin^2\left(\frac{\theta_{CM}}{2}\right)}{m_2}$$

The total energy is easier to find in the lab frame, since the second particle won't be moving.

$$E = \frac{\boldsymbol{p_{lab1}^2}}{2m_1} = \frac{(\lambda\boldsymbol{p_i} + \boldsymbol{p_i})^2}{2m_1}$$

The ratio will then be

$$\frac{\Delta E}{E} = \frac{4m_1p_i^2\sin^2\left(\frac{\theta_{CM}}{2}\right)}{m_2p_i^2(\lambda+1)^2} = \frac{4\lambda\sin^2\left(\frac{\theta_{CM}}{2}\right)}{(1+\lambda)^2}$$

Which is what we're after.

d) This fractional energy loss is maximum when the sine is at a maximum, which is an angle of $\theta_{CM} = \pi$, which corresponds to the incoming particle bouncing off in the exact opposite direction from where it came. To find the λ that maximizes the energy loss, we need to differentiate this ratio by λ and set it to zero.

$$\frac{\partial}{\partial\lambda}\left(\frac{\Delta E}{E}\right) = 0 = \frac{4(\lambda+1)^2 - 8\lambda(\lambda+1)}{(\lambda+1)^4}$$
$$= \frac{4(1-\lambda)}{(\lambda+1)^3}$$

This tells us that $\lambda = 1$ maximized this energy loss.

Problem 3

(Taylor 9.11) In this problem you will prove the equation of motion (9.34) for a rotating frame using the Lagrangian approach. As usual, the Lagrangian method is in many ways easier than the Newtonian (except that it calls for some slightly tricky vector gymnastics), but is perhaps less insightful. Let $\mathcal{S}$ be a noninertial frame rotating with constant velocity Ω relative to the inertial frame $\mathcal{S}_0$. Let both frames have the same origin, $O = O'$.

a) Find the Lagrangian $\mathcal{L} = T - U$ in terms of the coordinates $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ of $\mathcal{S}$. [Remember that you must first evaluate T in the inertial frame. In this connection, recall that $\boldsymbol{v_0} = \boldsymbol{v} + \boldsymbol{\Omega} \times \boldsymbol{r}$.]

b) Show that the three Lagrange equations reproduce (9.34) precisely.

Solution

a) To find the Lagrangian we need to first find the kinetic energy.

$$T = \frac{1}{2}mv_0^2 = \frac{1}{2}m\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})^2$$

$$= \frac{1}{2}m(\dot{r}^2 + 2\dot{\boldsymbol{r}} \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}) + (\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}))$$

$$= \frac{1}{2}m(\dot{r}^2 + 2(\dot{\boldsymbol{r}} \times \boldsymbol{\Omega}) \cdot \boldsymbol{r} + r^2\Omega^2 - (\boldsymbol{\Omega} \cdot \boldsymbol{r})^2)$$

This gives us a Lagrangian with some potential U.

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + 2(\dot{\boldsymbol{r}} \times \boldsymbol{\Omega}) \cdot \boldsymbol{r} + r^2\Omega^2 - (\boldsymbol{\Omega} \cdot \boldsymbol{r})^2) - U$$

b) Now we need to find how the Lagrangian changes in respect to the radial position and the radial velocity. The radial position piece is

$$\frac{\partial \mathcal{L}}{\partial r} = m(\dot{\boldsymbol{r}} \times \boldsymbol{\Omega}) + mr\Omega^2 - m(\boldsymbol{\Omega} \cdot \boldsymbol{r})\boldsymbol{\Omega} - \nabla U$$

$$= m(\dot{\boldsymbol{r}} \times \boldsymbol{\Omega}) + m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega} - \nabla U$$

Now, the radial velocity piece is

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{\boldsymbol{r}} + m(\boldsymbol{\Omega} \times \boldsymbol{r})$$

which means

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{r}}\Big) = m\ddot{\boldsymbol{r}} + m(\boldsymbol{\Omega} \times \dot{\boldsymbol{r}})$$

Equating these two pieces gives us

$$m\ddot{\boldsymbol{r}} + m(\boldsymbol{\Omega} \times \dot{\boldsymbol{r}}) = m(\dot{\boldsymbol{r}} \times \boldsymbol{\Omega}) + m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega} - \nabla U$$

$$m\ddot{\boldsymbol{r}} = 2m(\dot{\boldsymbol{r}} \times \boldsymbol{\Omega}) + m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega} + \boldsymbol{F}$$

Which is exactly what we're looking for.

Problem 4

(Taylor 9.20) Consider a frictionless puck on a horizontal turntable that is rotating counterclockwise with angular velocity Ω.

a) Write down Newton's second law for the coordinates x and y of the puck as seen by me standing on the turntable. (Be sure to include the centrifugal and Coriolis forces, but ignore the earth's rotation.)

b) Solve the two equations by the trick of writing $\eta = x + iy$ and guessing a solution of the form $\eta = e^{-i\alpha t}$. [In this case—as in the case of critically damped SHM discussed in section 5.4—you get only one solution this way. The other has the same form (5.43) we found for the second solution in damped SHM.] Write down the general solution.

c) At time $t = 0$, I push the puck from position $\boldsymbol{r_0} = (x_0, 0)$ with velocity $\boldsymbol{v_0} = (v_{x0}, v_{y0})$ (All as measured by me on the turntable). Show that

$$x(t) = (x_0 + v_{x0}t)\cos(\Omega t) + (v_{y0} + \Omega x_0)t\sin(\Omega t)$$

$$y(t) = -(x_0 + v_{x0}t)\sin(\Omega t) + (v_{y0} + \Omega x_0)t\cos(\Omega t)$$

d) Describe and sketch the behavior of the puck for large values of t. [Hint: When t is large the terms proportional to t dominate (except in the case that both their coefficients are zero). With t large, write (9.72) in the form $x(t) = t(B_1\cos(\Omega t) + B_2\sin(\Omega t))$, with similar expression for $y(t)$, and use the trick of (5.11) to combine the sine and cosine into a single cosine—or sine, in the case of $y(t)$. By now you can recognize that the path is the same kind of spiral, whatever the initial conditions (with the one exception mentioned).]

Solution

a) Since we have no additional forces besides the Coriolis force and the centrifugal force, our Newton's second la equation is

$$m\ddot{\boldsymbol{r}} = 2m\dot{\boldsymbol{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega}$$

If we say that the table is the x-y plane, we can define the following vectors

$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$$

$$\dot{\boldsymbol{r}} = \dot{x}\hat{\boldsymbol{x}} + \dot{y}\hat{\boldsymbol{y}}$$

$$\ddot{\boldsymbol{r}} = \ddot{x}\hat{\boldsymbol{x}} + \ddot{y}\hat{\boldsymbol{y}}$$

$$\mathbf{\Omega} = \Omega\hat{\boldsymbol{z}}$$

We can use these to get rid of all of the cross products in the force equation.

$$\dot{\boldsymbol{r}} \times \mathbf{\Omega} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dot{x} & \dot{y} & 0 \\ 0 & 0 & \Omega \end{vmatrix} = \dot{y}\Omega\hat{\boldsymbol{x}} - \dot{x}\Omega\hat{\boldsymbol{y}}$$

$$\mathbf{\Omega} \times \boldsymbol{r} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & 0 & \Omega \\ x & y & 0 \end{vmatrix} = -y\Omega\hat{\boldsymbol{x}} + x\Omega\hat{\boldsymbol{y}}$$

$$(\mathbf{\Omega} \times \boldsymbol{r}) \times \mathbf{\Omega} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ -y\Omega & x\Omega & 0 \\ 0 & 0 & \Omega \end{vmatrix} = x\Omega^2\hat{\boldsymbol{x}} + y\Omega^2\hat{\boldsymbol{y}}$$

Throwing these pieces into our force equation gives us these two equations

$$\ddot{x} = x\Omega^2 + 2\dot{y}\Omega$$

$$\ddot{y} = y\Omega^2 - 2\dot{x}\Omega$$

b) Using the trick, we know that

$$\eta = x + iy$$

$$\dot{\eta} = \dot{x} + i\dot{y}$$

$$\ddot{\eta} = \ddot{x} + i\ddot{y}$$

This gives us

$$\begin{aligned} e\ddot{t}a &= x\Omega^2 + 2\dot{y}\Omega + iy\Omega^2 - 2i\dot{x}\Omega \\ &= \Omega^2(x + iy) + 2\Omega(\dot{y} - i\dot{x}) \\ &= \Omega^2\eta - 2i\Omega\dot{\eta} \end{aligned}$$

Let's try some complex solution to find α.

$$\eta = e^{-i\alpha t}$$

$$\dot{\eta} = -i\alpha e^{-i\alpha t}$$

$$\ddot{\eta} = -\alpha^2 e^{-i\alpha t}$$

This gives us

$$-\alpha^2 = \Omega^2 - 2\alpha\Omega$$

$$\alpha^2 - 2\alpha\Omega + \Omega^2 = 0$$

$$(\alpha - \Omega)^2 = 0$$

$$\alpha = \Omega$$

We need two linearly dependent solutions so we can use

$$\eta = (A + Bt)e^{-i\Omega t}$$

c) At $t = 0$, η will be

$$\eta(0) = A = x_0$$

To find what B is, we need to differentiate η.

$$\dot{\eta} = Be^{-i\Omega t} - i\Omega(x_0 + Bt)e^{-i\Omega t}$$

Setting $t = 0$, this will be

$$\dot{\eta}(0) = B - i\Omega x_0 = v_{x0} + iv_{y0}$$

Which means that

$$B = v_{x0} + iv_{y0} + i\Omega x_0$$

Let's now mess around with η to find what we want.

$$\eta = (x_0 + t(v_{x0} + iv_{y0} + i\Omega x_0))e^{-i\Omega t}$$

$$= (x_0 + v_{x0}t + it(v_{y0} + \Omega x_0))(\cos(\Omega t) - i\sin(\Omega t))$$

$$= (x_0 + v_{x0}t)\cos(\Omega t) + (v_{y0} + \Omega x_0)t\sin(\Omega t) - i(x_0 + v_{x0}t)\sin(\Omega t) + i(v_{y0} + \Omega x_0)t\cos(\Omega t)$$

This gives us

$$x = (x_0 + v_{x0}t)\cos(\Omega t) + (v_{y0} + \Omega x_0)t\sin(\Omega t)$$

$$y = -(x_0 + v_{x0}t)\sin(\Omega t) + (v_{yo} + \Omega x_0)t\cos(\Omega t)$$

Which is what we're after.

d) The path of the puck will be a spiral. Without the push, the puck will be traveling in a circle. The push causes the puck to start traveling in a circular motion but with increasing r. This increasing r is linear, and is caused by the first order terms in our equations.

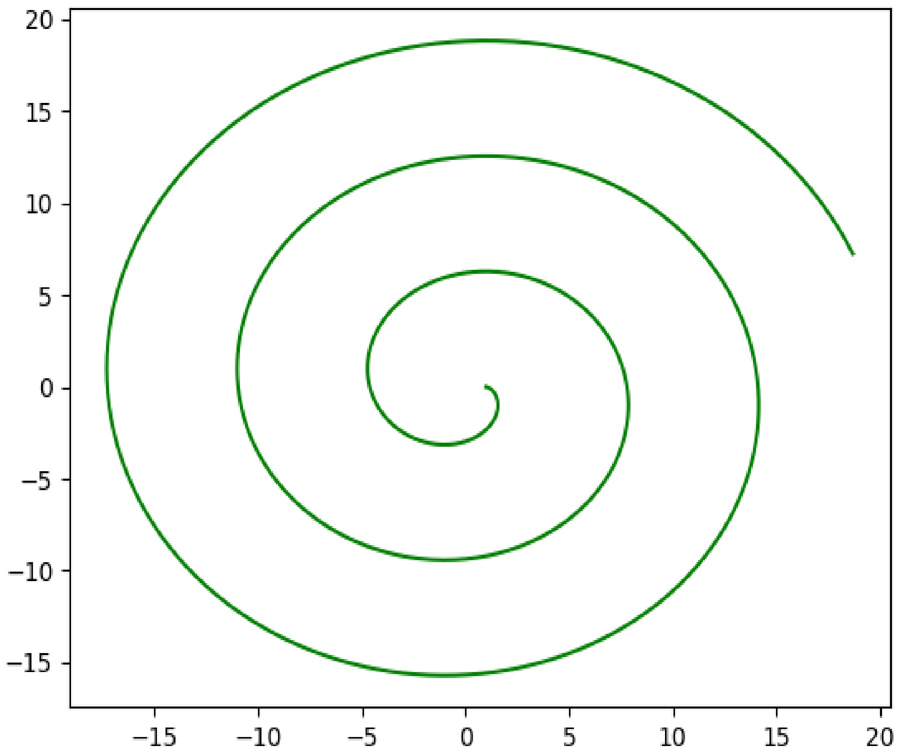

Figure 1

Problem 5

(Taylor 9.30) The Coriolis force can produce a torque on a spinning object. To illustrate this, consider a horizontal hoop of mass m and radius r spinning with angular velocity ω about its vertical axis at colatitude θ. Show that the Coriolis force due to the earth's rotation produces a torque of magnitude $m\omega\Omega r^2\sin(\theta)$ directed to the west, where Ω is the earth's angular velocity. This torque is the basis of the gyrocompass.

Solution

Our force is

$$\boldsymbol{F} = 2m\dot{\boldsymbol{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega}$$

This second term will not contribute to the torque so we can ignore it. This gives us a torque of

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = 2m\boldsymbol{r} \times (\dot{\boldsymbol{r}} \times \boldsymbol{\Omega})$$

Let's look at our hoop in the yz-plane. This gives us

$$\boldsymbol{\Omega} = \Omega \sin(\theta)\hat{\boldsymbol{y}} + \Omega \cos(\theta)\hat{\boldsymbol{z}}$$

We can say that

$$\boldsymbol{r} = R\hat{\boldsymbol{r}}$$

$$\dot{\boldsymbol{r}} = R\dot{\hat{\boldsymbol{r}}} = R\dot{\phi}\hat{\boldsymbol{\phi}}$$

This gives us

$$\boldsymbol{\tau} = 2mR^2\Omega\dot{\phi}\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{\phi}} \times (\sin(\theta)\hat{\boldsymbol{y}} + \cos(\theta)\hat{\boldsymbol{z}}))$$

We need $\hat{\boldsymbol{\phi}}$ in cartesian coordinates which is

$$\hat{\boldsymbol{\phi}} = -\sin(\phi)\hat{\boldsymbol{x}} + \cos(\phi)\hat{\boldsymbol{y}}$$

This first cross product is then

$$(-\sin(\phi)\hat{\boldsymbol{x}} + \cos(\phi)\hat{\boldsymbol{y}}) \times (\sin(\theta)\hat{\boldsymbol{y}} + \cos(\theta)\hat{\boldsymbol{z}}) = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & \sin(\theta) & \cos(\theta) \end{vmatrix}$$

$$= \hat{\boldsymbol{x}}\cos(\phi)\cos(\theta) + \hat{\boldsymbol{y}}\sin(\phi)\cos(\theta) - \hat{\boldsymbol{z}}\sin(\phi)\sin(\theta) = \boldsymbol{A}$$

This is in cartesian coordinates so we need $\hat{\boldsymbol{r}}$ also in cartesian coordinates.

$$\hat{\boldsymbol{r}} = \cos(\phi)\hat{\boldsymbol{x}} + \sin(\phi)\hat{\boldsymbol{y}}$$

Let's take this cross product

$$\hat{\boldsymbol{r}} \times \boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \cos(\phi) & \sin(\phi) & 0 \\ \cos(\phi)\cos(\theta) & \sin(\phi)\cos(\theta) & -\sin(\phi)\sin(\theta) \end{vmatrix}$$

$$= -\hat{\boldsymbol{x}}\sin^2(\phi)\sin(\theta) + \hat{\boldsymbol{y}}\cos(\phi)\sin(\phi)\sin(\theta)$$

This gives us

$$\boldsymbol{\tau} = 2mR^2\Omega\dot{\phi}(-\hat{\boldsymbol{x}}\sin^2(\phi)\sin(\theta) + \hat{\boldsymbol{y}}\cos(\phi)\sin(\phi)\sin(\theta))$$

or

$$d\tau = 2dmR^2\Omega\dot{\phi}(-\hat{\boldsymbol{x}}\sin^2(\phi)\sin(\theta) + \hat{\boldsymbol{y}}\cos(\phi)\sin(\phi)\sin(\theta))$$

$$= \frac{mR^2\Omega\dot{\phi}d\phi}{\pi}(-\hat{\boldsymbol{x}}\sin^2(\phi)\sin(\theta) + \hat{\boldsymbol{y}}\cos(\phi)\sin(\phi)\sin(\theta))$$

Let's integrate this in pieces. We can pull out the $\sin(\theta)$ from the vector piece. The x integral is

$$\int_0^{2\pi} \sin^2(\phi)d\phi = \pi$$

The y integral is

$$\int_0^{2\pi} \sin(\phi)\cos(\phi)d\phi = 0$$

This gives us

$$\tau = mR^2\Omega\dot{\phi}\sin(\theta) = mr^2\Omega\omega\sin(\theta)$$

Which is what we are after.

Problem 6

(Taylor 9.31) The Compton generator is a beautiful demonstration of the Coriolis force due to the earth's rotation, invented by the American physicist A. H. Compton (1892-1962, best known as author of the Compton effect) while he was still an undergraduate. A narrow glass tube in the shape of a torus or ring (radius R of the ring $\gg$ radius of the tube) is filled with water, plus some dust particles to let one see any motion of the water. The ring and water are initially stationary and horizontal, but the ring is then spun through 180^o about its east-west diameter. Explain why this should cause the water to move around the tube. Show that the speed of the water just after the 180^o turn should be $2\Omega R\cos(\theta)$, where Ω is the earth's angular velocity, and θ is the colatitude of the experiment. What would this speed be if $R \approx 1m$ and $\theta = 40^o$? Compton measured this speed with a microscope and got agreement within 3%.

Solution

The ring rotation should cause the water to move because the Earth's rotation causes a Coriolis force to act on the water in the tube. In an inertial reference frame, that tube, and its contents, aren't just rotating about the West-East axis, but also rotating around the central axis of Earth. This gives the water some angular momentum that needs to be conserved as we rotate the tube. We want to first look at the Earth's rotation. With θ being its colatitude, we can define our unit vectors with $\hat{\boldsymbol{e_1}}$ being South, $\hat{\boldsymbol{e_2}}$ being East and $\hat{\boldsymbol{e_3}}$ being radially outward away from the center of Earth. We can then define $\boldsymbol{\Omega}$ to be

$$\boldsymbol{\Omega} = -\Omega\sin(\theta)\hat{\boldsymbol{e_1}} + \Omega\cos(\theta)\hat{\boldsymbol{e_3}}$$

We need to now look at our system with the ring of radius R. We will rotate the ring an angle α around $\hat{\boldsymbol{e_2}}$. We can also define ϕ to be the angle between $\hat{\boldsymbol{e_1}}$ and some position vector $\boldsymbol{r}$ on the ring. We can the define a position vector in its three components as

$$\boldsymbol{r} = R\cos(\alpha)\cos(\phi)\hat{\boldsymbol{e_1}} + R\sin(\phi)\hat{\boldsymbol{e_2}} + R\sin(\alpha)\cos(\phi)\hat{\boldsymbol{e_3}}$$

Since we are rotating this about α and that the ring isn't deforming, the only coordinate that changes over time is α so

$$\dot{\boldsymbol{r}} = -R\dot{\alpha}\sin(\alpha)\cos(\phi)\hat{\boldsymbol{e_1}} + R\dot{\alpha}\cos(\alpha)\cos(\phi)\hat{\boldsymbol{e_3}}$$

The Coriolis force will be the one contributing to the added torque and is

$$\begin{aligned}
\boldsymbol{F_{Cor}} &= 2m(\dot{\boldsymbol{r}} \times \boldsymbol{\Omega}) \\
&= 2m\begin{vmatrix} \hat{\boldsymbol{e_1}} & \hat{\boldsymbol{e_2}} & \hat{\boldsymbol{e_3}} \\ -R\dot{\alpha}\sin(\alpha)\cos(\phi) & 0 & R\dot{\alpha}\cos(\alpha)\cos(\phi) \\ -\Omega\sin(\theta) & 0 & \Omega\cos(\theta) \end{vmatrix} \\
&= 2m(-R\dot{\alpha}\Omega\sin(\alpha)\cos(\phi)\cos(\theta) + R\dot{\alpha}\Omega\cos(\alpha)\cos(\phi)\sin(\theta))\hat{\boldsymbol{e_2}} \\
&= 2mR\dot{\alpha}\Omega\cos(\phi)\sin(\alpha-\theta)\hat{\boldsymbol{e_2}}
\end{aligned}$$

We only care about the part of this force that goes tangentially around the ring. This would be maximal furthest away from the axis of rotation and minimal on the axis of rotation. This gives us the relevant force.

$$\boldsymbol{F} = 2mR\dot{\alpha}\Omega\cos^2(\phi)\sin(\alpha-\theta)\hat{\boldsymbol{e_2}}$$

We can now find the average acceleration

$$a = \frac{R\dot{\alpha}\Omega\sin(\alpha-\theta)}{\pi}\int_0^{2\pi}\cos^2(\phi)d\phi = R\dot{\alpha}\Omega\sin(\alpha-\theta)$$

The velocity is then

$$\begin{aligned}
v &= \int R\Omega\sin(\alpha-\theta)\frac{d\alpha}{dt}dt = R\Omega\int_0^{\pi}\sin(\alpha-\theta)d\alpha \\
&= R\Omega(-\cos(\alpha-\theta)\Big|_0^{\pi} = 2R\Omega\cos(\theta)
\end{aligned}$$

With $R = 1$, $\Omega = 7.27 \times 10^{-5}$ rad/sec and $\theta = 40^o$ we have a velocity of about 0.00011 m/s for the water.