Analytic Mechanics Homework Assignment 7

Todd Hirtler

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Problem 1

(Taylor 8.14) Consider a particle of reduced mass μ orbiting in a central force with $U = kr^n$ where $kn > 0$.

a) Explain what the condition $kn > 0$ tells us about the force. Sketch the effective potential energy U_{eff} for the cases that $n = 2, -1, -3$.

b) Find the radius at which the particle (with given angular momentum l) can orbit at a fixed radius. For what values of n is this circular orbit stable? Do your sketches confirm this conclusion?

c) For the stable case, show that the period of small oscillations about the circular orbit is $\tau_{osc} = \tau_{orb}/\sqrt{n+2}$. Argue that if $\sqrt{n+2}$ is a rational number, these orbits are closed. Sketch them for the cases that $n = 2, -1, 7$.

Solution

a) The force can be found by

$$\boldsymbol{F} = -\boldsymbol{\nabla} U = -nkr^{n-1}$$

With $nk > 0$, this force is negative. In other words, the force is a restoring force. The effective potential will be

$$U_{eff} = kr^n + \frac{l^2}{2\mu r^2}$$

For $nk > 0$, both n and k need to have the same sign. Making our constant coefficients have a magnitude of 1, this gives us these effective potentials to plot.

$$U_{eff}(n = 2) = r^2 + \frac{1}{r^2}$$

$$U_{eff}(n = -1) = -\frac{1}{r} + \frac{1}{r^2}$$

$$U_{eff}(n=-3) = -\frac{1}{r^3} + \frac{1}{r^2}$$

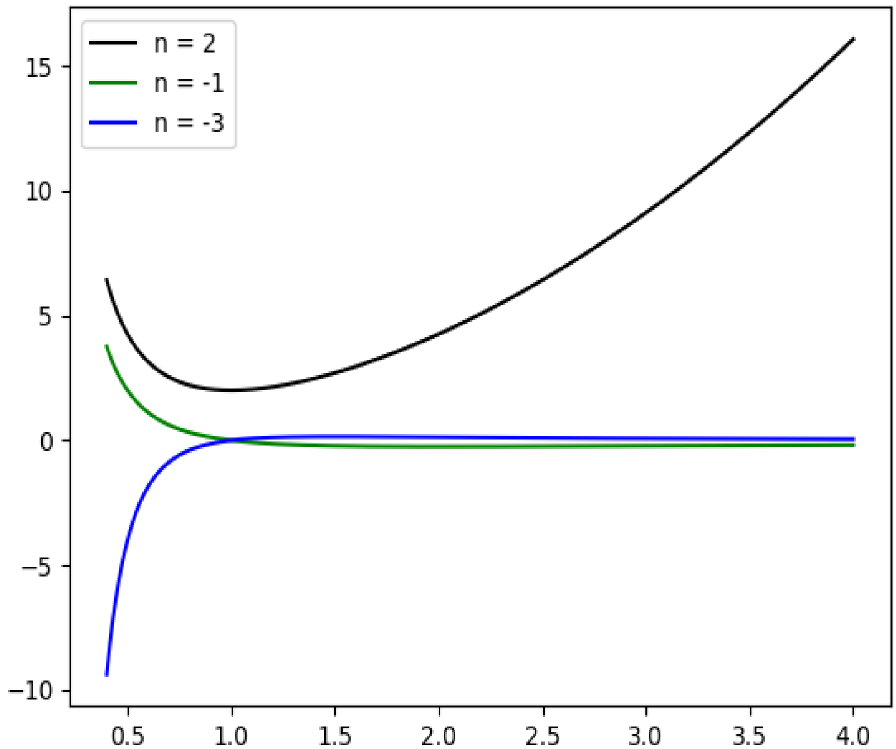

Figure 1

b) For a fixed radius orbit, the gradient of the effective potential with respect to radius must be zero. Solving for r gives us

$$\nabla U_{eff} = nkr^{n-1} - \frac{l^2}{\mu r^3} = 0$$

$$nkr^{n-1} = \frac{l^2}{\mu r^3}$$

$$r^{n+2} = \frac{l^2}{nk\mu}$$

$$r = \left(\frac{l^2}{nk\mu}\right)^{\frac{1}{n+2}}$$

Just looking at this we can see the case where $n = -2$. This value is only finite if $l = \sqrt{nk\mu}$ since $1/(n+2)$ is incredibly large, but any r will be a circular, and stable, orbit. An angular momentum less than this will have a radius of zero, which isn't an orbit. The second partial derivative of the effective potential will give us the concavity of our solution, as well as the stability of our orbits.

$$\frac{\partial^2 U_{eff}}{\partial r^2} = n(n-1)kr^{n-2} + \frac{3l^2}{\mu r^4} > 0$$

Using what we know about nkr^{n+2}, we can simplify this inequality.

$$(n-1)\frac{l^2}{\mu r^4} + \frac{3l^2}{\mu r^4} > 0$$

$$\frac{(n+2)l^2}{\mu r^4} > 0$$

This is positive only when $n > -2$. This matches our sketches, recognizing that there is a minimum with positive concavity for $n = 2, -1$ while there is not for $n = -3$.

c) Let's start off by taylor expanding the effective potential.

$$U_{eff} = U_{eff}(r_0) + (r - r_0)U'(r_0) + \frac{1}{2}(r - r_0)^2 U''(r_0) + O(r^3)$$

We can set the energy at $r = r_0$ to be zero and we know the first derivative is zero. This gives us

$$U_{eff} \approx \frac{1}{2}(r - r_0)^2 U''(r_0)$$

We know that

$$\ddot{r} = -\frac{1}{\mu}\frac{\partial U_{eff}}{\partial r}$$

Which yields us

$$\ddot{r} = -\frac{(r - r_0)(n+2)l^2}{\mu^2 r_0^4}$$

We can add a small perturbation to the radius in the form of

$$r = r_0 + \epsilon$$

which gives us

$$\ddot{\epsilon} = \ddot{r} = -\frac{\epsilon(n+2)l^2}{\mu^2 r_0^4}$$

Our orbit oscillates with frequency

$$\omega_{osc} = \frac{l\sqrt{n+2}}{r_0^2\mu}$$

or a period of oscillation of

$$\tau_{osc} = \frac{r_0^2\mu}{l\sqrt{n+2}}$$

We can now look at the orbital frequency which is

$$\omega_{orb} = \frac{v}{r_0} = \frac{r_0\dot{\phi}}{r_0} = \frac{l}{\mu r_0^2}$$

or an orbital period of

$$\tau_{orb} = \frac{\mu r_0^2}{l}$$

Which gives us the relationship we are after.

$$\tau_{osc} = \frac{\tau_{orb}}{\sqrt{n+2}}$$

If $\sqrt{n+2}$ is a rational number, it means that in some amount of total orbits, the oscillations and the orbital travel will meet. There will be some overall periodicity if their ratio is a rational number. If not, there can be no periodicity and there will be no way for the particle to ever be in the same position oriented in the same direction at any time.

Problem 2

(Taylor 8.21) a) If you haven't already done so, do problem 8.20: Consider a comet which passes through its aphelion at a distance r_{max} from the sun. Imagine that, keeping r_{max} fixed, we somehow make the angular momentum l smaller and smaller, though not actually zero; that is, we let $l \to 0$. Use equations (8.48) and (8.50) to show that in this limit the eccentricity ϵ of the elliptical orbit approaches 1 and that the distance of closest approach r_{min} approaches zero. Describe the orbit with r_{max} fixed but l very small. What is the semimajor axis a?

b) Use Kepler's third law (8.55) to find the period of this orbit in terms of r_{max} (and G and M_s).

c) Now consider the extreme case that the comet is released from *rest* at a distance r_{max} from the sun. (In this case l is actually zero.) Use the technique

described in connection with (4.58) to find how long the comet takes to reach the sun. (Take the sun's radius to be zero.)

d) Assuming the comet can somehow pass freely through the sun, describe its overall motion and find its period.

e) Compare your answers in parts b and d.

Solution

a) Let's write down equations (8.48) and (8.50) for later use.

$$c = \frac{l^2}{\gamma\mu}$$

$$r_{min} = \frac{c}{1+\epsilon}$$

$$r_{max} = \frac{c}{1-\epsilon}$$

with γ being the force constant, and c is a length. If we were to let $l \to 0$, then we know that $c \to 0$. In which case

$$r_{max}(1-\epsilon) = 0$$

Now, this can only be zero if $r_{max} = 0$ or $\epsilon = 1$. Since we are letting r_{max} be nonzero and constant, we know $\epsilon = 1$. In the same regard, r_{min} must be zero since $c = 0$ and the denominator is nonzero. The orbit will become a more and more elliptical until it converges an unclosed parabolic path along the major axis on the limit of $l \to 0$. The semimajor axis will converge on this limit to $a = r_{max}/2$. Very small l will yield an elliptical orbit with eccentricity just below 1, enough to keep it a closed orbit.

b) Kepler's third law is

$$\tau^2 = \frac{4\pi^2}{GM_s}a^3$$

Plugging in what we found in part a gives us

$$\tau^2 = \frac{\pi^2}{2GM_s}r_{max}^3$$

c) Equation (4.58) is

$$t = \sqrt{\frac{m}{2}}\int_{r_{max}}^{0}\frac{dr}{\sqrt{E-U(r)}}$$

The energy will be it's initial total energy

$$E = -G\frac{mM_s}{r_{max}}$$

and the potential energy is

$$U(r) = -G\frac{mM_s}{r}$$

This means the time is

$$t = \sqrt{\frac{m}{2}}\int_{r_{max}}^{0} \frac{dr}{\sqrt{GmM_s(\frac{1}{r} - \frac{1}{r_{max}})}}$$

$$= \frac{1}{\sqrt{2GM_s}}\int_{r_{max}}^{0} \frac{dr}{\sqrt{\frac{1}{r} - \frac{1}{r_{max}}}} = \sqrt{\frac{r_{max}}{2GM_s}}\int_{r_{max}}^{0} \sqrt{\frac{r}{r_{max} - r}}dr$$

$$= \frac{\pi}{2}\sqrt{\frac{r_{max}^3}{2GM_s}}$$

d) The comet's "orbit" will be an oscillation between r_{max} and $-r_{max}$ which means that the time we calculated in the previous part is only a quarter of the total time it travels per "orbit". This would mean we have a timer period of

$$\tau = 2\pi\sqrt{\frac{r_{max}^3}{2GM_s}}$$

e) The time period of the comet in part d is twice as long as the time period of the comet in part b.

Problem 3

A particle of mass m moves with angular momentum l in the field of a fixed force center with

$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3}$$

where k and λ are positive.

a) Write down the transformed radial equation (8.41) and prove that the orbit has the form

$$r(\phi) = \frac{c}{1 + \epsilon\cos(\beta\phi)}$$

where c, β and ϵ are positive constants.

b) Find c and β in terms of the given parameters, and describe the orbit for the case that $0 < \epsilon < 1$.

c) For what values of β is the orbit closed? What happens to your results as $\lambda \to 0$?

Solution

a) We can start with our transformed radial equation.

$$u'' = -u - \frac{\mu}{l^2u^2}(-ku^2 + \lambda u^3)$$

$$u'' = -(1 + \frac{\mu\lambda}{l^2})u + \frac{\mu k}{l^2}$$

We can rewrite this as

$$\frac{u''}{1 + \frac{\mu\lambda}{l^2}} = -u + \frac{\mu k}{l^2 + \mu\lambda}$$

By using a change in variables of

$$w = u - \frac{\mu k}{l^2 + \mu\lambda}$$

We can rewrite our equation

$$w'' = -\left(1 + \frac{\mu\lambda}{l^2}\right)w = -\beta^2 w$$

With

$$\beta = \sqrt{1 + \frac{\mu\lambda}{l^2}}$$

This gives us the solution

$$w = A\cos(\beta\phi)$$

Plugging this into our equation relating w and u, we can solve for u.

$$A\cos(\beta\phi) = u - \frac{\mu k}{l^2 + \mu\lambda}$$

$$u = A\cos(\beta\phi) + \frac{\mu k}{l^2 + \mu\lambda}$$

$$= \frac{\mu k}{l^2 + \mu\lambda}(\epsilon\cos(\beta\phi) + 1)$$

Since we know u is the multiplicative inverse of r, this gives us

$$r = \frac{c}{1 + \epsilon\cos(\beta\phi)}$$

With

$$c = \frac{l^2 + \mu\lambda}{\mu k} = \frac{l^2}{\mu k} + \frac{\lambda}{k}$$

$$\epsilon = \frac{A(l^2 + \mu\lambda)}{\mu k}$$

b) As found in the previous section

$$\beta = \sqrt{1 + \frac{\mu\lambda}{l^2}}$$

$$c = \frac{l^2 + \mu\lambda}{\mu k}$$

The orbit with $0 < \epsilon < 1$ will oscillate between its minimum and maximum r values as given by

$$r_{min} = \frac{c}{1 + \epsilon}$$

$$r_{max} = \frac{c}{1 - \epsilon}$$

c) The orbit is closed when $\epsilon < 1$.

$$\epsilon = \frac{A(l^2 + \mu\lambda)}{\mu k} < 1$$

Now, we just need to get β in there.

$$\beta^2 = \frac{l^2 + \mu\lambda}{l^2}$$

$$l^2\beta^2 = l^2 + \mu\lambda$$

Plugging this into our expression for ϵ gives us

$$\frac{Al^2\beta^2}{\mu k} < 1$$

$$\beta^2 < \frac{\mu k}{Al^2}$$

Which is our condition for β for a closed orbit. Next, we look at the limit as $\lambda \to 0$.

$$\epsilon = \frac{A(l^2 + \mu\lambda)}{\mu k} = \frac{Al^2}{\mu k}$$

$$\beta = \sqrt{\frac{l^2 + \mu\lambda}{l^2}} = 1$$

$$c = \frac{l^2 + \mu\lambda}{\mu k} = \frac{l^2}{\mu k}$$

This leads us to our radial equation to be

$$r = \frac{l^2}{\mu k + Al^2\cos(\phi)}$$

Problem 4

(Taylor 8.31) Consider the motion of two particles subject to a *repulsive* inverse-square force (for example, two positive charges). Show that this system has no states with $E < 0$ (as measured in the CM frame), and that in all states with $E > 0$, the relative motion follows a hyperbola. Sketch a typical orbit. [Hint: You can follow closely the analysis of sections 8.6 and 8.7 except that you must reverse the force; probably the simplest way to do this is to change the sign of γ in (8.44) and all subsequent equations(so that $F(r) = +\gamma/r^2$) and then keep γ itself positive. Assume $l \neq 0$.]

Solution

We start with our differential equation

$$u'' = -u - \frac{\mu}{l^2u^2}F$$

with

$$F = \gamma u^2$$

This gives us

$$u'' = -u - \frac{\mu\gamma}{l^2}$$

Using a change in variables

$$w = u + \frac{\mu\gamma}{l^2}$$

the new differential equation is

$$w'' = -w$$

which yields solution

$$w = A\cos(\phi + \delta)$$

With proper selection of our starting point of ϕ can allow us to get rid of δ so

$$w = A\cos(\phi)$$

Solving for u gives us

$$u = w - \frac{\mu\gamma}{l^2} = A\cos(\phi) - \frac{\mu\gamma}{l^2}$$

We can then get this as a function r by inverting it and simplifying to get

$$r = \frac{l^2}{\mu\gamma(\epsilon\cos(\phi) - 1)} = \frac{c}{\epsilon\cos(\phi) - 1}$$

with

$$c = \frac{l^2}{\mu\gamma}$$

$$\epsilon = \frac{Al^2}{\mu\gamma}$$

The energy is

$$E = \frac{\gamma^2\mu}{2l^2}(\epsilon^2 - 1)$$

This energy is only positive if $\epsilon > 1$, which would mean that all positive energy states have a hyperbolic path. If $\epsilon < 1$, r would be negative, which isn't possible, so there are no states with negative energy. The math will look similar to the following graph.

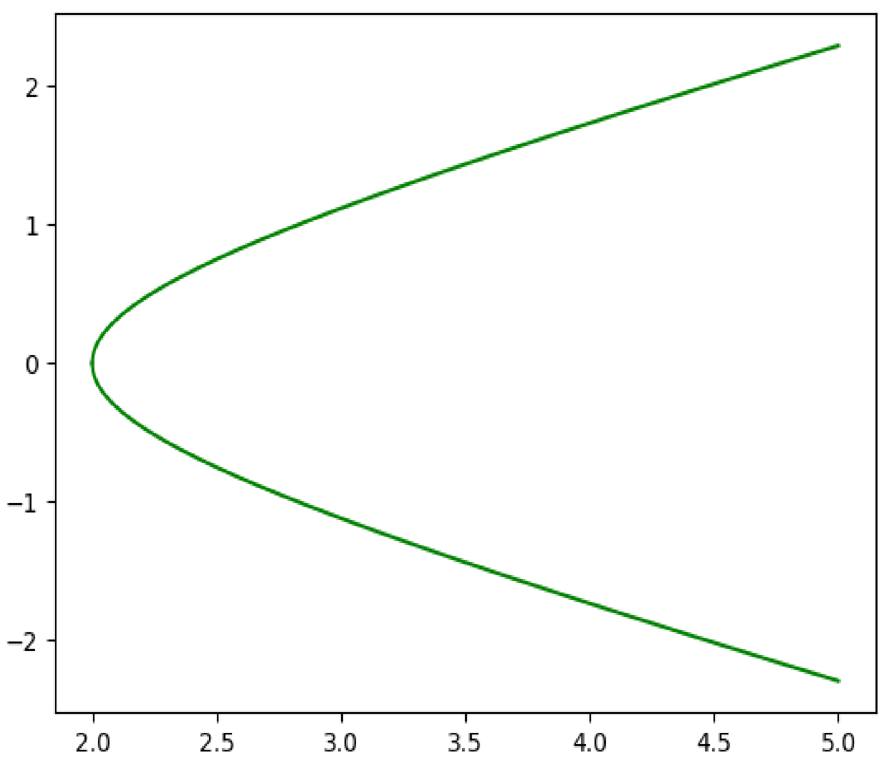

Figure 2

Problem 5

(Taylor 8.35) A spacecraft in a circular orbit wishes to transfer to another circular orbit of quarter the radius by means of a tangential thrust to move into an elliptical orbit and a second tangential thrust at the opposite end of the ellipse to move into the desired circular orbit. (The picture looks like figure 8.13 but run backwards.) Find the thrust factors required and show that the speed in the final orbit is two times *greater* than the initial speed.

Solution

We start off with a circular orbit of radius R_1. This can be related to the elliptical path that will take us to the smaller orbit after the first thrust by

$$R_1 = c_1 = \frac{c_2}{1-\epsilon} = \frac{\lambda^2 R_1}{1-\epsilon}$$

This interesting change of sign from our typical equation is due to us mapping a perigee to an apogee, so a minimum of the circular path is the maximum of the elliptical path. We can solve this equation for ϵ to get

$$1-\epsilon = \lambda_1^2$$
$$\epsilon = 1-\lambda_1^2$$

Now, we want to map the perigee of this ellipse to the desired circular orbit and we can do that with this equation

$$R_2 = \frac{R_1}{4} = \frac{c_2}{1+\epsilon} = \frac{\lambda_1^2 R_1}{2-\lambda_1^2}$$
$$\frac{1}{2} - \frac{\lambda_1^2}{4} = \lambda_1^2$$
$$\lambda_1^2 = \frac{2}{5}$$

Now, we need to see what happens after the second thrust to return ourselves to a circular orbit.

$$\frac{R}{4} = \lambda_2^2 c_2 = \lambda_1^2 \lambda_2^2$$

We can solve this for our new λ_2.

$$\lambda_2^2 = \frac{1}{4\lambda_1^2} = \frac{5}{8}$$

The final velocity is then

$$v_3 = \frac{v_{2a}}{v_{2p}} \lambda_1 \lambda_2 v_1$$

$$= 4\sqrt{\frac{5}{8}}\sqrt{\frac{2}{5}}v_1 = 2v_1$$

So, the final velocity is twice that of the initial velocity.

Problem 6

A particle with mass m is scattered by a fixed force center according to the force given by $F(r) = kr^{-3}$, where k is a constant and r is the distance between the particle and the force center. If the initial speed of the particle is v_0, and it is moving from far away toward the force center, show that the differential scattering cross section is

$$\frac{d\sigma}{d\theta} = \frac{k\pi^2(\pi - \theta)}{mv_0^2\theta^2(2\pi - \theta)^2 \sin(\theta)}$$

where θ is the scattering angle.

Solution

The scattering angle θ is

$$\theta = \pi - 2\psi$$

with ψ being the angle of the particle from a unit vector that points towards the point of closest approach from the target. We can find ψ by solving this integral

$$\psi = \int_{r_{min}}^{\infty} \frac{\frac{b}{r^2}dr}{\sqrt{1 - \frac{b^2}{r^2} - \frac{U}{T}}}$$

The potential energy is

$$U = -\int F dr = \frac{k}{r^2}$$

The initial kinetic energy is

$$T = \frac{1}{2}mv_0^2$$

We can also use the change of variable

$$u = \frac{1}{r}$$

The integral will then be

$$\psi = \int_0^{u_{max}} \frac{b du}{\sqrt{1 - b^2u^2 - \frac{ku^2}{mv_0^2}}}$$

$$= \frac{b}{2\sqrt{b^2 + \frac{k}{mv_0^2}}} \arcsin\left(\sqrt{b^2 + \frac{k}{mv_0^2}}u\right)\Big|_0^{u_{max}}$$

We know that u_{max} can be determined by the non-1 piece in the denominator of our integral.

$$u_{max} = \frac{1}{\sqrt{b^2 + \frac{k}{mv_0^2}}}$$

This makes

$$\psi = \frac{\pi b}{2\sqrt{b^2 + \frac{k}{mv_0^2}}}$$

Now, we need solve this for b.

$$2\psi\sqrt{b^2 + \frac{k}{mv_0^2}} = \pi b$$

$$4\psi^2(b^2 + \frac{k}{mv_0^2}) = \pi^2 b^2$$

$$b^2 = \frac{4\psi^2 k}{mv_0^2(\pi^2 - 4\psi^2)}$$

We know that

$$\psi = \frac{1}{2}(\pi - \theta)$$

Plugging this in gives us

$$b^2 = \frac{k(\pi - \theta)^2}{mv_0^2(2\pi\theta - \theta^2)}$$

The differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin(\theta)}\Big|\frac{db}{d\theta}\Big|$$

Let's start with finding the differential

$$\frac{db}{d\theta} = -\frac{1}{v_0}\sqrt{\frac{k}{m\theta(2\pi - \theta)}} - \frac{(\pi - \theta)}{v_0}\sqrt{\frac{k}{m}}\Big(\frac{\pi - \theta}{(\theta(2\pi - \theta))^{3/2}}\Big)$$

The absolute value will remove the negatives and let's multiply this by b.

$$b\Big|\frac{db}{d\theta}\Big| = \frac{(\pi - \theta)}{v_0^2}\Big(\frac{k}{m\theta(2\pi - \theta)}\Big) + \frac{k(\pi - \theta)^3}{v_0^2 m(\theta(2\pi - \theta))^2}$$

After some simplifying gives us

$$d\left|\frac{db}{d\theta}\right| = \frac{k\pi^2(\pi-\theta)}{mv_0^2\theta^2(2\pi-\theta)^2}$$

Finally, bringing in the sine gives us our differential cross-section.

$$\frac{d\sigma}{d\Omega} = \frac{k\pi^2(\pi-\theta)}{mv_0^2\theta^2\sin(\theta)(2\pi-\theta)^2}$$

Which is what we were after.