Analytic Mechanics Homework Assignment 5

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Problem 1

(Taylor 6.4) A ray of light travels from point P_1 in a medium of refractive index n_1 to P_2 in a medium of index n_2, by way of the point Q on the plane interface between the two media, as in Figure 6.9. Show that Fermat's principle implies that, on the actual path followed, Q lies in the same vertical plane as P_1 and P_2 and obeys Snell's law, that $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$. [Hint: Let the interface be the xz plane, and let P_1 lie on the y axis at $(0, h_1, 0)$ and P_2 in the x, y plane at $(x_2, -h_2, 0)$. Finally let $Q = (x, 0, z)$. Calculate the time for the light to traverse the path P_1QP_2 and show that it is minimum when Q has $z = 0$ and satisfies Snell's law.]

Solution

We can define some variable S which is the sum of our two lengths weighted by their medium's refractive index.

$$S = n_1 S_1 + n_2 S_2 = n_1 \sqrt{x^2 + h_1^2 + z^2} + n_2 \sqrt{(x_2 - x)^2 + h_2^2 + z^2}$$

For this to be a minimum, the partial derivative in respect to z and x must be zero. Let's look at the z part first.

$$\frac{\partial S}{\partial z} = \frac{n_1 z}{\sqrt{x^2 + h_1^2 + z^2}} + \frac{n_2 z}{\sqrt{(x_2 - x)^2 + h_2^2 + z^2}}$$

$$= z\left(\frac{n_1}{S_1} + \frac{n_2}{S_2}\right) = 0$$

Now, this means z has to be zero, since the term inside the parentheses must be nonzero. Now we can look at the partial derivative in respect to x.

$$\frac{\partial S}{\partial x} = \frac{n_1 x}{\sqrt{x^2 + h_1^2}} - \frac{n_2 (x_2 - x)}{\sqrt{(x_2 - x)^2 + h_2^2}} = 0$$

This yields Snell's Law

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

Problem 2

(Taylor 6.19) A surface of revolution is generated as follows: Two fixed points (x_1, y_1) and (x_2, y_2) in the x, y plane are joined by a curve y = y(x). [Actually, you'll make life easier if you start out writing this as x = x(y).] The whole curve is now rotated about the x axis to generate a surface. Show that the curve for which the area of the surface is stationary has the form $y = y_0 \cosh((x - x_0)/y_0)$, where x_0 and y_0 are constants. (This is often called the soap-bubble problem, since the resulting surface is usually the shape of a soap bubble held by two coaxial rings of radii y_1 and y_2.)

Solution

We start with our surface, which can be defined by the following integral

$$S = 2\pi \int y\sqrt{1 + x'^2}dy$$

We can find where this surface is stationary by using the Euler-Lagrange equations on the function

$$f = 2\pi y\sqrt{1 + x'^2}$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial x'} = \frac{2\pi y x'}{\sqrt{1 + x'^2}} = A$$

We know that A is a constant because the first partial derivative we found is zero. Let's solve for x'.

$$x'^2(4\pi^2 y^2 - A^2) = A^2$$

$$x' = \frac{A}{\sqrt{4\pi^2 y^2 - A^2}}$$

We can integrate this using a change in variables

$$u = 2\pi y$$

$$du = 2\pi dy$$

$$dx = \frac{1}{2\pi} \frac{Adu}{\sqrt{u^2 - A^2}}$$

Integrating this gives us

$$x = \frac{A}{2\pi} \ln\left(\sqrt{u^2 - A^2} + u\right) + B$$

Rearranging this gives us

$$\frac{2\pi(x - B)}{A} = \ln\left(\sqrt{u^2 - A^2} + u\right)$$

$$e^{\frac{2\pi(x-B)}{A}} = \sqrt{u^2 - A^2} + u = \eta$$

We need to solve for u in terms of η to deal with this mess

$$(\eta - u)^2 = u^2 - A^2 = \eta^2 - 2u\eta + u^2$$

$$\eta^2 + A^2 = 2u\eta$$

$$u = \frac{\eta^2 + A^2}{2\eta} = 2\pi y$$

Solving for y we get

$$y = \frac{\eta^2 + A^2}{4\pi\eta} = \frac{e^{\frac{4\pi(x-B)}{A}} + A^2}{4\pi e^{\frac{2\pi(x-B)}{A}}}$$

$$= \frac{1}{4\pi}(e^{\frac{2\pi(x-B)}{A}} + A^2 e^{-\frac{2\pi(x-B)}{A}})$$

For this to be a hyperbolic cosine, A needs to be 1. We can then say that

$$y_0 = \frac{1}{2\pi}$$

$$x_0 = B$$

Which gives us

$$y = y_0 \cosh\left(\frac{(x - x_0)}{y_0}\right)$$

Problem 3

(Taylor 6.23) An aircraft whose airspeed is v_0 has to fly from town O (at the origin) to town P, which is a distance D due east. There is a steady gentle wind shear, such that $\boldsymbol{v} = Vy\hat{\boldsymbol{x}}$, where x and y are measured east and north

respectively. Find the path, $y = y(x)$, which the plane should follow to minimize its flight time, as follows:

a) Find the plane's ground speed in terms of v_0, V, ϕ (the angle by which the plane heads to the north of east), and the plane's position.

b) Write down the time of flight as an integral of the form $\int_0^D f\,dx$. Show that if we assume that y' and ϕ both remain small (as is certainly reasonable if the wind speed is not too large), then the integrand f takes the approximate form $f = (1 + \frac{1}{2}y'^2)/(1 + ky)$ (times an uninteresting constant) where $k = V/v_0$.

c) Write down the Euler-Lagrange equation that determines the best path. To solve it, make the intelligent guess that $y(x) = \lambda x(D - x)$, which clearly passes through the two towns. Show that it satisfies the Euler-Lagrange, provided $\lambda = (\sqrt{4 + 2k^2D^2} - 2)/(kD^2)$. How far north does this path take the plane, if $D = 2000$ miles, $v_0 = 500$ mph, and the wind shear is $V = 0.5$ mph/mi? How much time does the plane save by following this path? [You'll probably want to use a computer to do this integral.]

Solution

a) The total velocity of the airplane can be described by

$$\boldsymbol{v} = (v_0 \cos(\phi) + Vy)\hat{\boldsymbol{x}} + v_0 \sin(\phi)\hat{\boldsymbol{y}}$$

and the speed is

$$v = \sqrt{(v_0 \cos(\phi) + Vy)^2 + v_0^2 \sin^2(\phi)}$$

b) So, we know that

$$\frac{ds}{dt} = v$$

Now, ds is found using the pythagorean theorem

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + y'^2}$$

And since we only care about the horizontal part of our speed, we get

$$dt = \frac{dx\sqrt{1 + y'^2}}{v_0 \cos(\phi) + Vy}$$

Here comes the part where we start approximating stuff. With our given assumptions, we find

$$dt = \frac{dx(1 + \frac{1}{2}y'^2)}{v_0 + Vy}$$

Pulling out v_0, we get an integral of the form

$$dt = \frac{1}{v_0}\int_0^D \frac{1+y'^2}{1+ky}dx$$

Which is what we're after.

c) We are going to be looking for something of the form

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}$$

So the first term will be

$$\frac{\partial f}{\partial y} = -\frac{k(1+y'^2)}{(1+ky)^2}$$

The partial derivative with respect to y' is

$$\frac{\partial f}{\partial y'} = \frac{y'}{1+ky}$$

So the second term is

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{y''(1+ky) - ky'^2}{(1+ky)^2}$$

Our resultant differential equation will then be

$$y''(1+ky) - \frac{k}{2}y'^2 + k = 0$$

Using the cunning solution provided

$$y = \lambda x(D-x)$$

$$y' = \lambda(D-2x)$$

$$y'' = -2x$$

Plugging this into our differential equation yields us

$$-2\lambda(1+k\lambda x(D-x)) - \frac{k}{2}(\lambda D - 2\lambda x)^2 + k = 0$$

$$-2\lambda - 2\lambda 62xDk + 2\lambda^2 x^2 k - \frac{k}{2}(\lambda^2 D^2 - 4\lambda^2 Dx + 4\lambda^2 x^2) + k = 0$$

$$\lambda^2\Big(-\frac{kD^2}{2}\Big) - 2\lambda + k = 0$$

$$\lambda^2 k D^2 + 4\lambda - 2k = 0$$

Using the quadratic formula gives us

$$\lambda = \frac{-2 \pm \sqrt{4 + 2k^2 D^2}}{kD^2}$$

We need to find the maximum of y so

$$y' = 0 = \lambda(D - 2x)$$

$$x = \frac{D}{2}$$

Solving for lambda gives us

$$\lambda = 3.66 \times 10^{-4}$$

These together gives us a maximum y of

$$y = 366.025 \text{ mi}$$

To find the time difference between fastest path and a straight line, we just have to solve the integral

$$\Delta t = \frac{1}{v_0} \int_0^D \frac{1 + \lambda^2 (D - 2x)^2}{1 + k\lambda x(D - x)} dx - \frac{D}{v_0}$$

This will yield a time difference of about 0.4 hours.

Problem 4

(Taylor 6.25) Consider a single loop of the cycloid (6.26) with a fixed value of a, as shown in Figure 6.11. A cart is released from rest at a point P_0 anywhere on the track between O and the lowest point P (that is, P_0 has parameter $0 < \theta_0 < \pi$). Show that the time for the cart to roll from P_0 to P is given by the integral

$$\text{time}(P_0 \to P) = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1 - \cos(\theta)}{\cos(\theta_0) - \cos(\theta)}} d\theta$$

and prove that this time is equal to $\pi\sqrt{a/g}$. Since this is independent of the position of P_0, the cart takes the same time to roll from P_0 to P, whether P_0 is at O, or anywhere between O and P, even infinitessimally close to P. Explain qualitatively how this surprising result can possibly be true. [Hint: To do mathematics, you have to make some cunning changes of variables. One route is this: Write $\theta = \pi - 2\alpha$ and then use the relevant trig identities to replace the cosines of θ by sines of α. Now substitutes $\sin(\alpha) = u$ and do the remaining integral.]

Solution

The parametric equation for a cycloid is

$$x = a(\theta - \sin(\theta))$$

$$y = a(1 - \cos(\theta))$$

Using conservation of energy, we get the relation

$$mga(1 - \cos(\theta)) - mga(1 - \cos(\theta_0)) = \frac{1}{2}mv^2$$

Solving for v gives us

$$v = \sqrt{2ga(\cos(\theta_0) - \cos(\theta))}$$

We can also find that the curve is

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

$$= \sqrt{(a - a\cos(\theta))^2 + (a\sin(\theta))^2} d\theta$$

$$= \sqrt{2a^2(1 - \cos(\theta)} d\theta$$

The time can be determined by

$$t = \frac{ds}{v} = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1 - \cos(\theta)}{\cos(\theta_0) - \cos(\theta)}} d\theta$$

Using the suggested change of variables, we can try to manipulate this integrand into something more easy to deal with.

$$\theta = \pi - 2\alpha$$

$$d\theta = -2d\alpha$$

$$alpha_0 = \frac{\pi - \theta_0}{2}$$

$$\alpha_f = 0$$

$$\cos(\theta) = \cos(\pi - 2\alpha) = -\cos(-2\alpha) = -\cos(2\alpha)$$

$$= \cos^2(\alpha) - \sin^2(\alpha) = 1 - 2\sin^2(\alpha)$$

Plugging these into the main integral yields us

$$t = -2\sqrt{\frac{a}{g}} \int_{\alpha_0}^{0} \sqrt{\frac{2 - 2\sin^2(\alpha)}{\cos(\theta_0) + 1 - 2\sin^2(\alpha)}} d\alpha$$

$$= -2\sqrt{\frac{a}{g}}\int_{\alpha_0}^{0}\frac{\cos(\alpha)}{\sqrt{\cos^2\left(\frac{\theta_0}{2}\right)-\sin^2(\alpha)}}d\alpha$$

We can then use another change of variables

$$u = \sin(\alpha)$$

$$du = \cos(\alpha)d\alpha$$

$$u_0 = \sin(\alpha_0) = \sin\left(\frac{\pi-\theta_0}{2}\right) = -\cos\left(\frac{\theta_0}{2}\right)$$

$$u_f = 0$$

Plugging this into our integral, we have

$$t = -2\sqrt{\frac{a}{g}}\int_{u_0}^{0}\frac{du}{\sqrt{u_0^2-u^2}}$$

Now, we can solve this and get

$$t = -2\sqrt{\frac{a}{g}}\arctan\left(\frac{u}{\sqrt{u_0^2-u^2}}\right)\Big|_{u_0}^{0} = -2\sqrt{\frac{a}{g}}\left(\frac{-\pi}{2}\right) = \pi\sqrt{\frac{a}{g}}$$

Which is what we are after. This can make sense from a qualitative perspective by the increasing slope. The higher up we release the cart, the less of the cycloid is in the way of cart, so it is able to accelerate downwards faster. As the cart travels along the cycloid, it's redirected more in a horizontal direction and isn't as effected by gravity. The angle between the normal force and gravity are changing in such a way along a cycloid, that they are able to maintain the same time of travel, regardless of the starting position.

Problem 5

(Taylor 7.8) a) Write down the Lagrangian $\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2)$ for two particles of equal masses, $m_1 = m_2 = m$, confined to the x axis and connected by a spring with potential energy $U = \frac{1}{2}kx^2$. [Here x is the extension of the spring, $x = (x_1 - x_2 - l)$, where l is the string's unstretched length, and I assume that mass 1 remains to the right of mass 2 at all times.]

b) Rewrite $\mathcal{L}$ in terms of the new variables $X = \frac{1}{2}(x_1 + x_2)$ (the CM position) and x (the extension), and write down the two Lagrange equations for X and x.

c) Solve for $X(t)$ and $x(t)$ and describe the motion.

Solution

a) We can start off on finding our first Lagrangian

$$\mathcal{L}(x_1, x_2, \dot{x_1}, \dot{x_2}) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(x_1^2 - 2x_1x_2 - 2x_1l + 2x_2l + x_2^2 + l^2)$$

b) With our new change in variables, we need to solve for x_1 and x_2 individually. First, we know that

$$x_1 = x_2 + x + l$$

$$x_2 = x_1 - x - l$$

So

$$X = \frac{1}{2}(2x_1 - x - l)$$

$$x_1 = X + \frac{x}{2} + \frac{l}{2}$$

$$\dot{x}_1 = \dot{X} + \frac{\dot{x}}{2}$$

$$X = \frac{1}{2}(2x_2 + x + l)$$

$$x_2 = X - \frac{x}{2} - \frac{l}{2}$$

$$\dot{x_2} = \dot{X} - \frac{\dot{x}}{2}$$

We can now use these to build up a prettier Lagrangian

$$\mathcal{L}(x, X, \dot{x}, \dot{X}) = \frac{1}{2}m((\dot{X} + \frac{\dot{x}}{2})^2 + (\dot{X} - \frac{\dot{x}}{2})^2) - \frac{1}{2}kx^2$$

$$= m\Big(\dot{X}^2 + \frac{\dot{x}^2}{4}\Big) - \frac{1}{2}kx^2$$

This gives us two sets of Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial x} = -kx$$

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big) = \frac{d}{dt}\Big(\frac{m}{2}\dot{x}\Big) = \frac{m\ddot{x}}{2}$$

$$\frac{\partial \mathcal{L}}{\partial X} = 0$$

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{X}}\Big) = \frac{d}{dt}(2m\dot{X}) = 2m\ddot{X}$$

This gives us two differential equations

$$\ddot{x} = -\frac{2k}{m}x$$

$$\ddot{X} = 0$$

c) We can solve the first equation by setting ω to be

$$\omega^2 = \frac{2k}{m}$$

This gives us

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

With A and B being determined by our initial conditions. The spring can be seen to oscillate, or the particles are moving towards each other and away from each other in a periodic fashion. We can now solve the second equation to be of the form

$$X(t) = Ct + D$$

With C and D being determined by the initial conditions. This is a linear function, so the center of mass is moving along at a constant speed in some direction.

Problem 6

(Taylor 7.13) In section 7.4 [Equations (7.41) through (7.51)], I proved Lagrange's equations for a single particle constrained to move on a two-dimensional surface. Go through the same steps to prove Lagrange's equations for a system consisting of two particles subject to various unspecified constraints. [Hint: The net force on particle 1 is the sum of the total constraint force $\boldsymbol{F}_1^{cstr}$ and the total constraint force $\boldsymbol{F}_1$, and likewise for particle 2. The constraint forces come in many guises (the normal force of a surface, the tension force of a string tied between the particles, etc.), but is always true that the net work done by all constraint forces in any displacement consistent with the constraints is zero–this is the defining property of constraint forces. Meanwhile, we take for granted that the nonconstraint forces are derivable from a potential energy $U(\boldsymbol{r_1}, \boldsymbol{r_2}, t)$; that is $\boldsymbol{F_1} = -\nabla_1 U$ and likewise for particle 2. Write down the difference δS between the action integral for the right path given by $\boldsymbol{r_1}(t)$ and $\boldsymbol{r_2}(t)$ and any nearby wrong path given by $\boldsymbol{r_1}(t) + \boldsymbol{\epsilon_1}(t)$ and $\boldsymbol{r_2}(t) + \boldsymbol{\epsilon_2}(t)$. Paralleling the steps of section 7.4, you can show that δS is given by an integral analogous to (7.49), and this is zero by the defining property of constraint forces.]

Solution

We can start by defining our paths. $\boldsymbol{R}$ will represent incorrect paths, ϵ is a deviation from the correct path, and $\boldsymbol{r}$ will represent the correct path.

$$\boldsymbol{R_i}(t) = \boldsymbol{r}_i(t) + \boldsymbol{\epsilon}_i(t)$$

Our paths will start and stop at the same point so the action will be

$$S = \int_0^t \mathcal{L}(\boldsymbol{R}_i, \dot{\boldsymbol{R}}_i, t')dt'$$

With the correct action being

$$S_0 = \int_0^t \mathcal{L}(\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i, t')dt'$$

The difference in action will be

$$\delta S = S - S_0$$

Alternatively, the change in Lagrangian is

$$\delta\mathcal{L} = \mathcal{L}(\boldsymbol{R}_i, \dot{\boldsymbol{R}}_i, t') - \mathcal{L}(\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i, t')$$

This second term could then be described by

$$\mathcal{L}(\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i, t) = \frac{1}{2}m\dot{\boldsymbol{r}}_i^2 - U(\boldsymbol{r}_i, t)$$

This will lead us to

$$\delta\mathcal{L} = \frac{1}{2}m((\dot{\boldsymbol{r}}_i + \dot{\boldsymbol{\epsilon}}_i)^2 - \dot{\boldsymbol{r}}_i^2) - (U(\boldsymbol{r}_i + \boldsymbol{\epsilon}_i, t) - U(\boldsymbol{r}_i, t)) = m\dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{\epsilon}}_i - \boldsymbol{\epsilon}_i \cdot \nabla_i U + O(\epsilon^2)$$

To first order, the change in action is then

$$\delta S = \int_0^t \delta\mathcal{L}dt' = \int_0^t (m\dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{\epsilon}}_i - \boldsymbol{\epsilon}_i \cdot \nabla_i U)dt'$$

Integrating by parts, we get

$$\delta S = -\int_0^t \boldsymbol{\epsilon}_i \cdot (m\ddot{\boldsymbol{r}}_i + \nabla_i U)dt'$$

The first term is our total force, which included both constraint forces and nonconstraint forces, and the second term is the negative of our nonconstraint forces. This gives us

$$\delta S = -\int_0^t \boldsymbol{\epsilon}_i \cdot \boldsymbol{F}_{cstr}dt'$$

Constraint forces are perpendicular to the plane of motion and $\boldsymbol{\epsilon}$ lies on the plane of motion, this integral must be zero. This means that the action integral is stationary at the right path.