$\boxed{89 \quad 23\ \text{Jan}\ 19}$

Office hours Th 2:30–4:30pm Birge 359   Quizes Section on syllabus is exact

Final 35%

Midterm 25% 14Mar19 6–9pm

HW 30%

Quizzes 10% – Fri

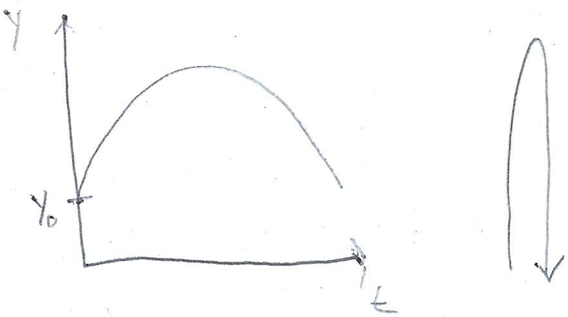
– Linear Algebra

– Differential Equations

Fourier transform, Laplace transform, etc.

– All physical processes can be described with math

Throwing something up in the air



$$\vec{F} = m\vec{a}$$

$$= m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$$

Force ⇒ How momentum changes with time

Kinetic energy $T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2$

$$\frac{dT}{dy} = \frac{d}{dy}\left(\frac{1}{2}mv^2\right) = \frac{d}{dy}\left(\frac{1}{2}m\dot{y}^2\right)$$

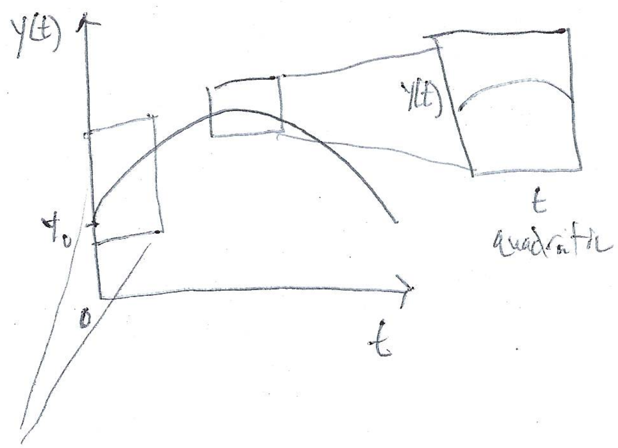
$$= mv\frac{dv}{dy} = m\frac{dy}{dt}\frac{dv}{dy} = m\frac{dv}{dt} = F$$

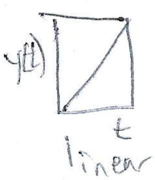
$$F = ma = -mg$$

$$a = \frac{dv}{dt} \qquad v = \int_{t_0}^{t_1} a\, dt = -gt + v_0$$

$$v = \frac{dy}{dt} \qquad y = \int v\, dt = \int -gt + v_0\, dt = -\frac{1}{2}gt^2 + v_0 t + y_0$$

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$$





$$E = T + V = \frac{1}{2}mv^2 + mgh = \frac{1}{2}m v(t)^2 + mg\,y(t)$$

$$v(t) = -gt + v_0, \quad y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$$

$$E = \frac{1}{2}m(-gt + v_0)^2 + mg\left(-\frac{1}{2}gt^2 + v_0 t + y_0\right)$$

$$= \frac{1}{2}m\left(g^2t^2 - 2v_0 g t + v_0^2\right) + \left(-\frac{1}{2}mg^2t^2 + mgv_0 t + mgy_0\right)$$

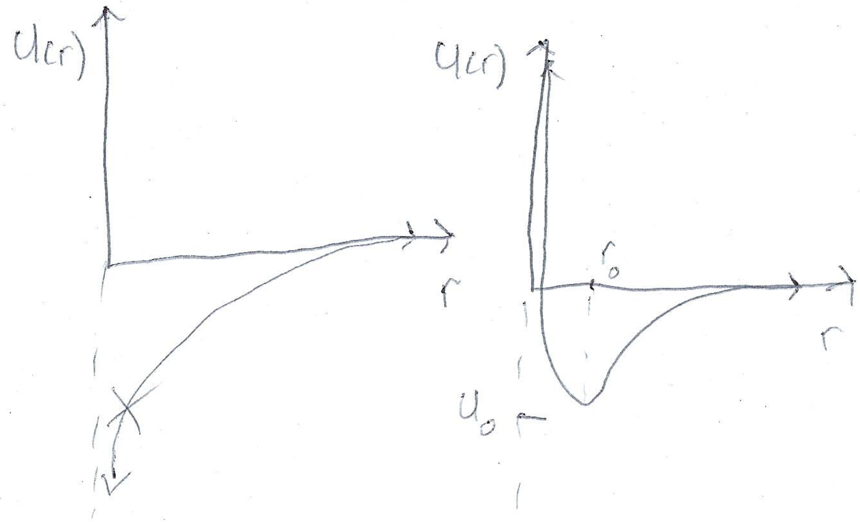
$$= \frac{1}{2}mv_0^2 + mgy_0 \quad \therefore \text{ Energy is conserved}$$

Energy invariant in time implies $\frac{dE}{dt} = 0$

$$\frac{dE}{dt} = \cancel{\frac{1}{2}} m v \frac{dv}{dt} + mg\frac{dy}{dt} = mv\frac{dv}{dt} + mgv = -mgv + mgv = 0$$

Taylor Expansion, Power Series, etc.

$Na^+ \; Cl^- \rightarrow NaCl$

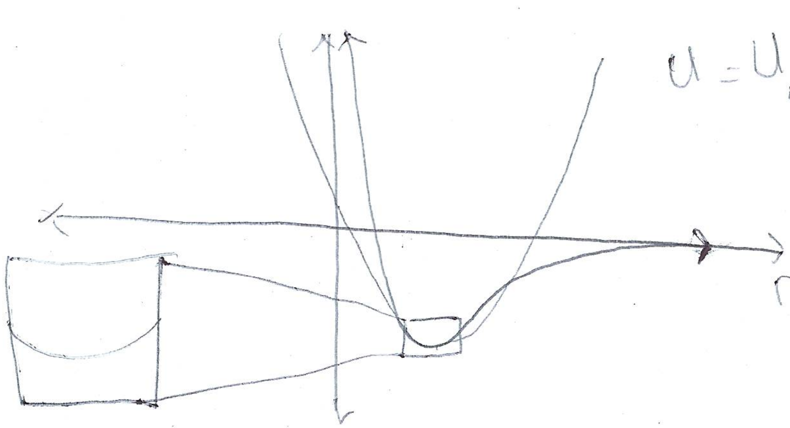


$$\frac{-1}{4\pi\epsilon_0}\frac{q^2}{r} = U(r)$$

$$F = \frac{dT}{dr} = \frac{-dV}{dr}$$

$$U(r) = \frac{-1}{4\pi\epsilon_0}\frac{q^2}{r} + \underbrace{\left(\lambda e^{P/r} - 1\right)}_{\text{phenomenological guess}}$$

$$U = U_0 + a(r-r_0)^2$$



Power series

$$F(x) = \sum_n a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\{a_n\}$ = collection of coefficients specific to this power series

<u>Analytic Function</u>

A function that can always be described as some power series in a region $|x| < R$

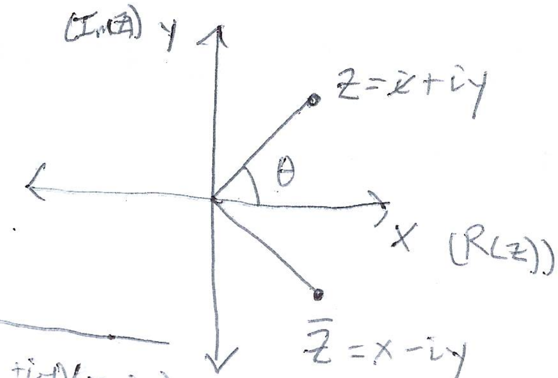
89 28 Jan 19

Complex Numbers and Functions

Kreyszig CH (13.1 - 5)

Bring internet capable device on Wednesday

Complex Numbers



$$Z = x + iy$$

$$i = \sqrt{-1}$$

$$|z| = \sqrt{x^2+y^2} = \sqrt{(x+iy)(x-iy)} = \sqrt{z\bar{z}}$$

$$x = r\cos\theta,\ y = r\sin\theta,\ z = r(\cos\theta + i\sin\theta)$$

Euler relation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$1 = \sqrt{z\bar{z}} = \sqrt{e^{i\theta}e^{-i\theta}} = \sqrt{e^{i0}} = \sqrt{1} = 1$$

$\mathrm{Arg}(z) = \theta$ Modulo 2π

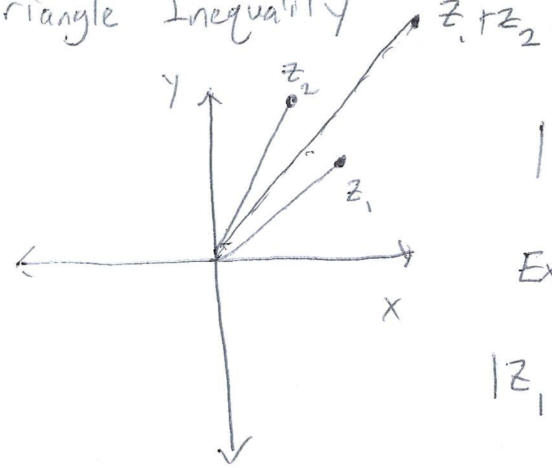
$-\pi < \mathrm{Arg}(z) \leq \pi$

All Possible θ

$$\arg(z) = \mathrm{Arg}(z) + 2\pi k,\ k \in \mathbb{Z}$$

$$re^{i\theta} = re^{i(\theta + k2\pi)} = re^{i\theta}e^{ik2\pi} = re^{i\theta}(\underbrace{\cos k2\pi}_{1} + i\underbrace{\sin k2\pi}_{0}) = re^{i\theta}$$

Triangle Inequality



$$|z_1+z_2| \le |z_1|+|z_2|$$

Extension:

$$|z_1+z_2+z_3+\cdots| \le |z_1|+|z_2|+|z_3|+\cdots$$

Destructive and constructive interference

$$\begin{aligned}|z_1 z_2| &= |z_1||z_2| \\ &= |r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\ &= |r_1 r_2 e^{i(\theta_1+\theta_2)} \\ &= |r_1 r_2| = |r_1||r_2| = |z_1||z_2|\end{aligned}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{x_1^2+y_1^2}}{\sqrt{x_2^2+y_2^2}} =$$

$$w = \frac{z_1}{z_2}, \quad \left|\frac{z_1}{z_2}\right| = \sqrt{(Re(w))^2+(Im(w))^2}$$
$$= \sqrt{\left(Re\left(\frac{z_1}{z_2}\right)\right)^2+\left(Im\left(\frac{z_1}{z_2}\right)\right)^2}$$

$$z_1 = 1-i, \quad z_2 = 1+i$$

$$z_1 = \sqrt{1^2+(-1)^2} = \sqrt{2}$$
$$z_2 = \sqrt{1^2+(1)^2} = \sqrt{2}$$

$$\frac{|z_1|}{|z_2|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

$$\frac{z_1}{z_2} = \frac{1-i}{1+i} = \boxed{z_1\bar{z}_2 = (1-i)^2 = 1-2i-1 = -2i} \text{ X}$$

$$\frac{1-i}{1+i}\cdot\frac{1-i}{1-i} = \frac{-2i}{2} = -i$$

Powers Z

$$Z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$(re^{i\theta})^n = r^n e^{i\theta n}$$

$$Z^{1/n} = r^{1/n}(\cos\theta/n + i\sin\theta/n) = r^{1/n}e^{i\theta/n}$$

$$z = re^{i\theta} = re^{i(\theta+2\pi k)}$$

$$z^{1/n} = \sqrt[n]{z} = r^{1/n}e^{i(\theta+2\pi k)/n}$$

$$= \cancel{r}^{1/n}e^{i\theta/n}e^{i2\pi k/n}$$

$$\sqrt[3]{1} = r^{1/3}e^{i\theta/3}e^{i2\pi k/3} = \sqrt[3]{e^{i\theta}} =$$

$$= e^{i2\pi k/3},\ k \in \mathbb{Z}$$

$k=0$

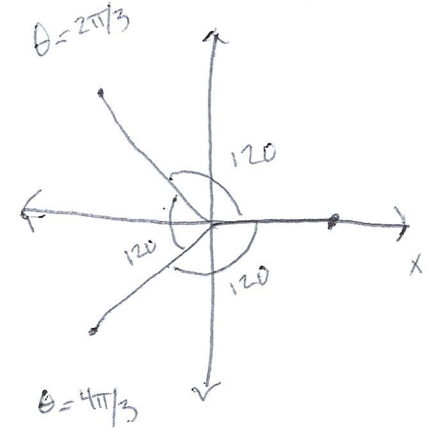
$e^0 = 1$

$k=1$ $\quad$ $k=2$ $\quad$ $k=3$

$e^{i2\pi/3}$ $\quad$ $e^{i4\pi/3}$ $\quad$ $e^{i6\pi/3}$

$(e^{i2\pi/3})^3$

$= e^{i2\pi} = 1$



$\sqrt[n]{z}$ will have n complex roots

Complex functions

$$f(z) = u(x,y) + i\,v(x,y)$$

$z = x + iy \qquad u(x,y) = \mathrm{Re}\{f(z)\}$

$\qquad\qquad\quad v(x,y) = \mathrm{Im}\{f(z)\}$

$\qquad\qquad\quad u(x,y),\ v(x,y)$ both real functions

$$\begin{aligned} f(z) &= z^2 + \bar{z}^2 \\ &= (x+iy)^2 + (x-iy)^2 \\ &= x^2 - y^2 + i2xy + (x^2 - y^2 - i2xy) \\ &= 2(x^2 - y^2) \end{aligned}$$

$\cancel{d,t}$

$u(x,y) = 2(x^2 - y^2),\ v(x,y) = 0$

89 30 Jan 14

Friday Quiz

(Bring internet capable device)

Kreyszig Ch. 13

$z = x + iy \qquad \frac{1}{2}(z + \bar{z}) = x = Re(z)$

$\frac{1}{2i}(z - \bar{z}) = y = Im(z)$

Similarly $\quad \frac{1}{2}(f(z) + \overline{f(z)}) = Re(f(z))$

$\frac{1}{2i}(f(z) - \overline{f(z)}) = Im(f(z))$

where $f(z) = u(x,y) + iv(x,y)$

Complex Functions

$z = x + iy$

$f(z) = u(x,y) + iv(x,y)$; u, v both real functions

$f(z) = z^2 + (\bar{z})^2$

$= (x+iy)^2 + (x-iy)^2$

$= x^2 + 2ixy - y^2 + x^2 - 2ixy - y^2$

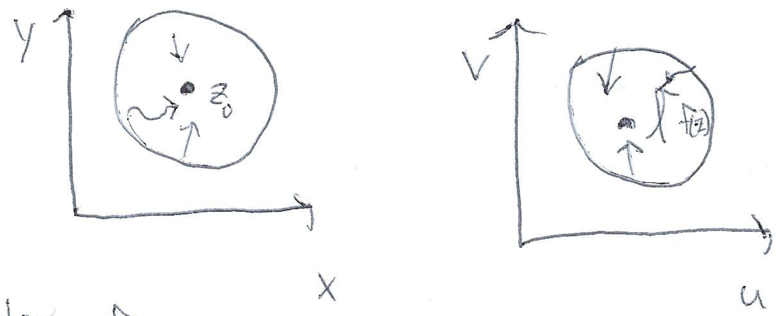
$= 2x^2 - 2y^2$

$u(x,y) = 2x^2 - 2y^2 \quad , v(x,y) = 0$

Differentiability and continuity

$f(z)$ is continuous if the limit exists

$$\lim_{z \to z_0} f(z_0) = l \quad , \; l \text{ is some complex number}$$



A complex function approaches l from all directions

Derivatives

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$ so long as this exist, function is differentiable

$$\frac{dF}{dz} = f'(z)$$

$$\delta_z \to 0$$

$$\delta_x \to 0 \qquad \delta_y \to 0$$

$\bar{z}$ is not differentiable

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \quad \lim_{\delta z \to 0} \frac{\overline{z+\delta z} - \bar{z}}{\delta z} = \lim_{\delta z \to 0} \frac{x - iy + \delta x - i\delta y - (x+iy)}{\delta x + i\delta y}$$

$$= \lim_{\delta z \to 0} \frac{\delta x - i\delta y}{\delta x + i\delta y} \qquad \lim_{\delta x \to 0} \lim_{\delta y \to 0} \frac{\delta x - i\delta y}{\delta x + i\delta y} = +1$$

$$\lim_{\delta y \to 0} \lim_{\delta x \to 0} \frac{\delta x - i\delta y}{\delta x + i\delta y} = -1$$

what this means is that $\frac{d\bar{z}}{dz}$ does not exist though $\frac{d\bar{z}}{d\bar{z}} = 1$ does

If $f(z)$ is differentiable then it is called analytic

(if differentiable in domain D, then it is analytic in D

If a function $f(z)$ is analytic then it satisfies

Cauchy - Riemann equations

$f(z) = u(x,y) + iv(x,y)$

$$\frac{du}{dx} = \frac{dv}{dy}, \quad \frac{du}{dy} = -\frac{dv}{dx}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{Cauchy Riemann Equations}}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{\cancel{f(z)}\, f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) - u(x,y) - iv(x,y)}{\delta x + i\delta y} = G$$

$$\lim_{\delta x \to 0} \lim_{\delta y \to 0} G$$

$$\lim_{\delta y \to 0} \lim_{\delta x \to 0} \quad f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + \frac{i(v(x+\delta x, y) - v(x,y))}{\delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

(Do this algebra)

$$f'(z) = \lim_{\delta y \to 0} -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

so $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

Function is analytic iff Cauchy-Riemann is satisfied

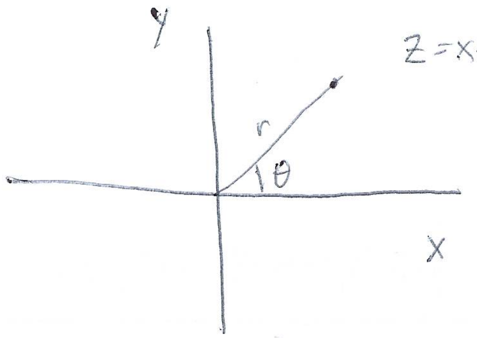
Show z^n is analytic

$$f'(z) = \lim_{\delta z \to 0} \frac{(z+\delta z)^n - z^n}{\delta z} = n z^{n-1}$$

Laplace's Equation

iff $f(z)$ is analytic ~~then~~ and u and v obey Laplace's equation

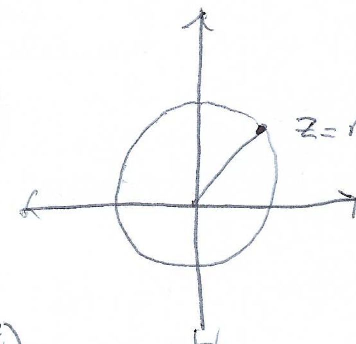
$$\nabla^2 u = 0 \quad , \quad \nabla^2 v = 0$$



$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

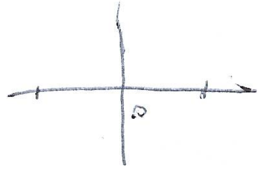
$$z = re^{i\theta}$$

Angular velocity $\omega = \frac{d\theta}{dt}$



$\mathrm{Re}(re^{i\omega t}) = r\cos\omega t$

$F = ma = -kx + f_0\cos\omega t - cv$



$ma = -kx$

$$c\frac{dx}{dt} + m\frac{d^2x}{dt^2} + kx = \cancel{0}\, F_0\cos\omega t \cancel{c}$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0\cos\omega t = F_0\frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$$

Guess Solution $x(t) = Ae^{i\omega t} + Be^{-i\omega t}$

$$\frac{dx}{dt} = Ai\omega e^{i\omega t} - Bi\omega e^{-i\omega t}$$

$$\frac{d^2x}{dt^2} = -\omega^2(Ae^{i\omega t} + Be^{-i\omega t})$$

$\cancel{\frac{dx}{dt} \Rightarrow}$

$$m(-\omega^2(Ae^{i\omega t} + Be^{-i\omega t})) + c\,i\omega(Ae^{i\omega t} - Be^{-i\omega t}) + k(Ae^{i\omega t} + Be^{-i\omega t}) = \frac{F_0}{2}(e^{i\omega t} + e^{-i\omega t})$$

$$(-m\omega^2 + ci\omega + k)A = \frac{F_0}{2} \quad , \quad (-m\omega^2 - ci\omega + k)B = F_0/2$$

$$A = \frac{F_0}{2}\frac{1}{k - m\omega^2 + ic\omega} \quad , \quad B = \frac{F_0}{2}\frac{1}{k - m\omega^2 - ic\omega} \qquad x(t) = \frac{F_0}{2}\left(\frac{e^{i\omega t}}{k - m\omega^2 + ic\omega} + \frac{e^{-i\omega t}}{k - m\omega^2 - ic\omega}\right)$$

$$\frac{1}{k - m\omega^2 + ic\omega} = re^{i\theta}, \quad k - m\omega^2 + ic\omega = re^{i\theta}$$

$$r = |k - m\omega^2 + ic\omega| \qquad \theta = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

$$x(t) = \frac{F_0}{2}\left(\frac{e^{i\omega t}}{re^{i\theta}} + \frac{e^{-i\omega t}}{re^{-i\theta}}\right)$$

$$= \frac{F_0}{2r}\left(e^{i(\omega t - \theta)} + e^{i(\theta - \omega t)}\right)$$

$$= \frac{F_0}{\cancel{2}r}\cos(\omega t - \theta)$$

89 01 Feb 19

Quiz 1

Q1: $z = x+iy$ find $\mathrm{Im}(1/z)$

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \quad \mathrm{Im}\left(\frac{1}{z}\right) = \frac{-y}{x^2+y^2} \quad (d)$$

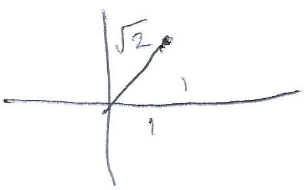
a) $\frac{-x}{x^2+y^2}$

b) $\frac{y^2}{x^2+y^2}$

c) $\frac{x^2}{x^2+y^2}$

d) $\frac{-y}{x^2+y^2}$

Q2: Find the arguments of the roots of $\sqrt[3]{1+i}$



$\tan\left(\frac{y}{x}\right) = +$ $\quad \arctan(1)$

45 $\quad \sqrt{2}\cos$ $\quad$ * Be patient *

~~(B)~~ ~~(A)~~ (d)

a) $\pi/12,\ 9\pi/12,\ \frac{17\pi}{12}$

b) $\pi/4,\ 4\pi/12,\ \frac{19\pi}{12}$

c) $\pi/12,\ \frac{7\pi}{12},\ \frac{13\pi}{12},\ \frac{19\pi}{12}$

d) $\pi/4,\ \frac{3\pi}{4},\ \frac{5\pi}{4}$

Q3: Find $u(x,y)$ $v(x,y)$

$$f(z) = u(x,y) + iv(x,y) = iz\bar{z}$$

$$z\bar{z} = i(u^2+v^2) \quad (c)$$

a) $u=0 \quad v = x^2-y^2$

b) $u = x^2 \quad v = y^2$

c) $u=0 \quad v = x^2+y^2$

d) $u = x^2+y^2 \quad v=0$

Q4: Differentiate $\frac{z-i}{z+i}$ $\quad \frac{-i(z+i) - i(z-i)}{(z+i)^2}$

$-i(z+i) - (z-i)$ $\quad -iz+1 \quad -iz-1$

$-iz+1 \quad -z+i$ $\quad \frac{-2iz}{z^2+2iz-1}$

$-2z+1$

$z+i - z+i$ $\quad$ (b)

$2i$

a) $\frac{2z}{(z+i)^2}$

b) $\frac{2i}{(z+1)^2}$

c) $\frac{z}{(z+i)^2}$

d) $\frac{z-i}{(z+i)^2}$

Vectors and Vector spaces

Kreyszig Ch 4.0, 7.1-7.4

which of these are vectors

[i] $\begin{bmatrix} 1 \\ 0 \\ 10 \end{bmatrix}$ [ii] $\nearrow$ [iii] $3\sin 2x$ [iv] a rib

Linearity $y(x)$ $\quad y(x_1+x_2) = y(x_1) + y(x_2)$

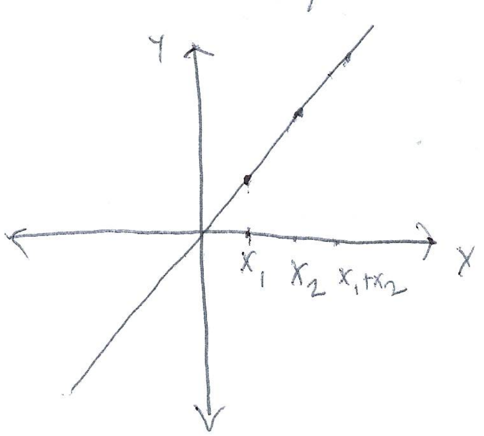
$y(ax)$ $\quad y(ax) = ay(x)$

$y(x_1)$, $\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad \vec{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\vec{y}_1 + \vec{y}_2 = \left.\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right\}$ is also a vector

$\vec{y} = \begin{bmatrix} ax_1 \\ ax_2 \\ ax_3 \end{bmatrix} = a\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Linear $y = mx$ $\quad y(x_1+x_2) = m(x_1+x_2) = m(x_1) + m(x_2)$



$y = mx^2$ not linear

- Any functional operation that obey these properties is called a linear operation
- Linear algebra studies objects that obey these operations

Scalar

be careful with mass

Temperature, energy, speed, mass, charge

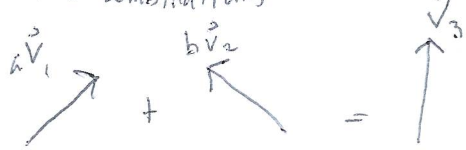
vectors: "magnitude and direction

eg scaling

same direction
twice length

$$2\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\0\\2\end{bmatrix}$$

eg linear combinations



eg $\vec{v}_1 + \vec{v}_2$

$$a\vec{v}_1 + b\vec{v}_2 = \vec{v}_3$$

Vector Space

A vector space $\mathbb{V}$ is a set object equipped with these two operations (scalar multiplication and vector addition)

"closed" vector space is one where if you perform these operations you yield another object in the same vector space

What do we mean by multiplication and addition?

$$a\vec{v} \qquad \vec{V}_1 + \vec{V}_2$$

?   ?

$\vec{V}_1 + \vec{V}_2 = \vec{V}_2 + \vec{V}_1$ (commutativity of addition)

$\vec{V}_1 + (\vec{V}_2 + \vec{V}_3) = (\vec{V}_1 + \vec{V}_2) + \vec{V}_3$ (Associativity)

$\vec{V}_1 + \vec{0} = \vec{V}_1$ (existence of zero)

$\vec{V}_1 + (-\vec{V}_1) = \vec{0}$ (additive inverse exists)

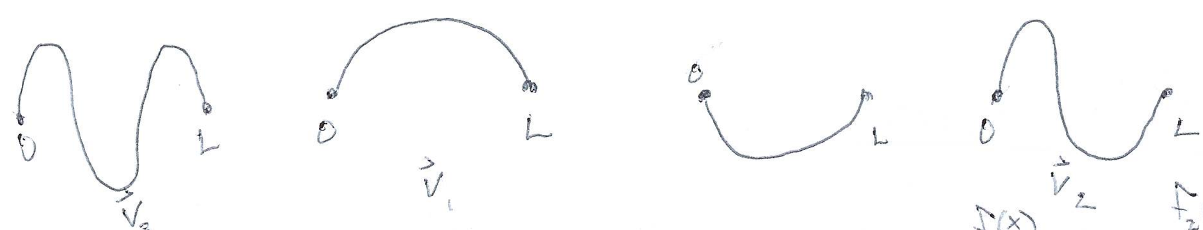
$a(\vec{v}_1 + v_2) = a\vec{v}_1 + a\vec{v}_2$ (distributative)

$0(\vec{v}) = \vec{0}$ (multiplication by zero)

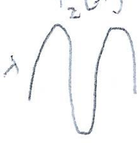
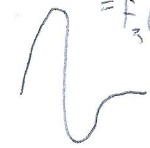
$1\vec{v} = \vec{v}$ (multiplication by one)

$$|\vec{V}_1 + \vec{V}_2| \neq |\vec{V}_1| + |\vec{V}_2|$$

eg waves on a string



$f_1(x)$ $f_2(x)$ $(f_1+f_2)(x) = f_3(x)$

$\sin(42\pi x) = f(x)$ $2 \times$  $+ 3 \times$  $=$ 

Vector addition $f_1(x) + f_2(x) = (f_1 + f_2)(x)$

$2\sin\frac{\pi x}{L} + 3\sin\frac{2\pi x}{L}$ = some other wave

Scalar multiplication ~~af. f(x) =~~ $af(x) = (af)(x)$

BW ~~0 and L~~

0 and L

84 ☐ 4 Feb 2019

Kreyszig Ch 7.9

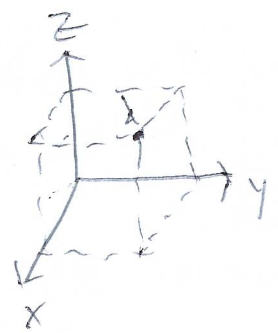
Vector Space

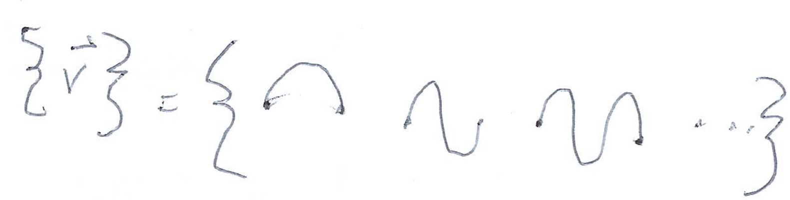
a) a set of object $\{\vec{v}\}$

b) An operation called addition that combines two vectors to yield another vector $\vec{v}_1 + \vec{v}_2 = \vec{v}_3$

c) An operation called scalar multiplication than connect scalar a with vector $\vec{v}$ to give a vector $a\vec{v}$

$\vec{d}$ $\quad\vec{v}\quad$ $\vec{a}$ $\qquad\qquad 0 - L$



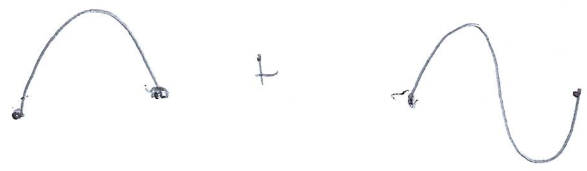


eg. $\vec{r}_1 = f_1(x)$

Addition $f_1(x) + f_2(x) = f_3(x) = (f_1 + f_2)(x)$

Scalar multiplication $a f_1(x) = f_4(x) = (af_1)(x)$

$2\sin\frac{\pi x}{L} + 3\sin\frac{2\pi x}{L} =$



= wave between 0 and L so belongs to the same set of objects in our vector space

3 × 

$3\sin\frac{\pi x}{L}$ $\quad$ $3\sin\frac{\pi x}{L}$

Won't work

$f_1(x) = f(x_1) \qquad f_2(x) = f(x_2)$

Addition as $f(x_1) + f(x_2) = f(x_1 + x_2)$ } This wouldn't work for our object

In order to define a vector space you must define addition and multiplication in a way that works for the objects you are describing

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ bd \end{bmatrix} \qquad \alpha\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha a \\ b^{\alpha} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} + \left[-\begin{bmatrix} a \\ b \end{bmatrix}\right] = \vec{0} = \cancel{\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} -a \\ -1/b \end{bmatrix}} \quad \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} -a \\ 1/b \end{bmatrix}$$

$$\vec{0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Magnitude → Tricky, involves multiplication of object with itself

Direction → Need a reference frame or coordinate system
Need the concept of a basis

A basis is a set of linearly independent vectors $\{\hat{e}_i\}$

Linear independence = no $\hat{e}_i$ for any i can be expressed in terms of a sum of $\hat{e}_j$ $(j \neq i)$

$$\hat{e}_i \neq a_1\hat{e}_1 + a_2\hat{e}_2 + \dots + a_n\hat{e}_n \quad \text{(for any choice of } a_i$$

$$i \neq j$$

$$0 = \sum_i a_i \hat{e}_i$$ (~~Linear indepence~~ if $\{\hat{e}_i\}$ are linearly independent $a_i = 0 \ \forall i$

Span: The span $\text{span}\{\hat{e}_i\}$ is the set of all linearly dependent vectors ie $\vec{V} = \sum_i^n a_i \hat{e}_i$

$\text{span}\{\hat{e}_i\}$ is itself a vector space or we can say $\mathbb{V}$ is spanned by $\{\hat{e}_i\}$

basis

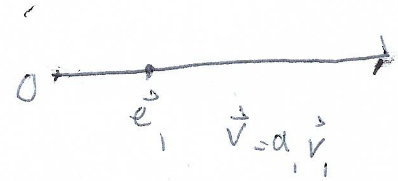
2D space $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\{\vec{e}\} =$ $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Span $\{\vec{e}_1\}$

all $\vec{v} = a_1 \vec{e}_1$

a_1 can be any number

→ is a 1D space

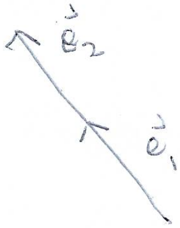


For any choice of $\vec{e}_1$, since there is only one basis vector the span is 1D.

$\{\vec{e}_i\} = \{\vec{e}_1, \vec{e}_2\}$

$\vec{v} = a_1\vec{e}_1 + a_2\vec{e}_2 = a_1\vec{e}_1 + a_2\vec{e}_2$

If $\vec{e}_2 = 2\vec{e}_1$



Because $\vec{e}_1$ and $\vec{e}_2$ are linearly dependent, the span $\{\vec{e}_1, \vec{e}_2\}$ becomes 1D

$0 = \sum_i^n a_i \vec{e}_i = a_1\vec{e}_1 + a_2\vec{e}_2$

$-2\vec{e}_1 + \vec{e}_2 = 0$

Instead of the usual objects (displacement, velocity, acceleration, etc.) consider monomials

$$\{x^i\} = \{1, x, x^2, \ldots\}$$

$$0 = \sum_i a_i x^i, \quad a_i = 0 \ \forall i$$

Set of monomials are linearly independent

$f(x) = \sum_i a_i x^i$ describing function in basis of monomials

$f(x)$ is linearly ~~dep~~ dependent on the span $\{x^n\}$

$f(x)$ can be expanded as a basis of monomials

That allows us to define "direction"

eg. e^x

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$e^x = \begin{bmatrix} 1 \\ 1 \\ 1/2! \\ \vdots \\ \vdots \end{bmatrix}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ 0 \\ \vdots \end{bmatrix}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/3! \\ \vdots \end{bmatrix}$$

Phys 89 08Feb19

Kreyszig 7.9

No Quizz Tode

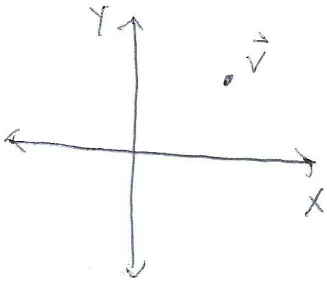
include code in HW

Today

Basis

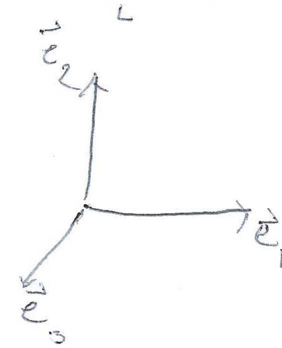
Inner Product Spaces

Orthogonality



$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

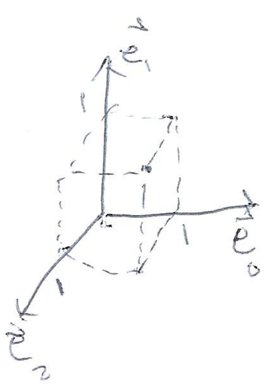
$\sin\frac{\pi x}{L}$ $\quad \sin\frac{2\pi x}{L}$ $\quad \sin\left(\frac{3\pi x}{L}\right)$



$\{x^0, x^1, x^2, x^3, \ldots\}$

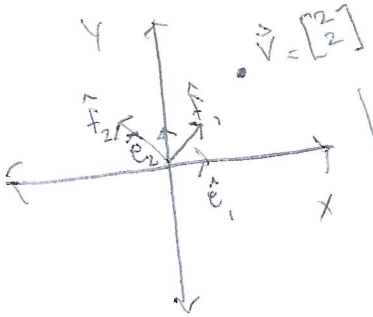
$\vec{e}_0 \quad \vec{e}_1 \quad \vec{e}_2, \ldots$

$F(x) = 1 + x + x^2$



 $= 3\sin\frac{2\pi x}{L} + 2\sin\frac{3\pi x}{L} = 0\cdot\sin\frac{\pi x}{L} + 3\sin\frac{2\pi x}{L} + 2\sin\frac{3\pi x}{L} + 0\cdot\sin\frac{4\pi x}{L} + \ldots$

$$\vec{v} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$



$\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

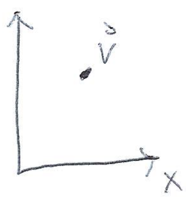
$\longrightarrow \vec{v} = 2\hat{e}_1 + 2\hat{e}_2$

$\hat{f}_1 = 1\hat{e}_1 + 1\hat{e}_2$

$\vec{v}_{[f]} = 2\hat{f}_1 + 0\hat{f}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Magnitude?

↳ concept of length



$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

$|\vec{v}| = \sqrt{a^2 + b^2}$

Dot product = Scalar product = inner product

Doesn't always exist in every $\mathbb{V}$

Inner ~~PR~~ Product Spaces

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 + \dots$

$= \sum_i v_i w_i$

$\vec{v} \cdot \vec{w} = \sum_{ij} v_i w_j \delta_{ij}$

$\boxed{\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \text{Kronecker delta}}$

$= v_1 w_1 \delta_{11} + v_1 w_2 \delta_{12} + v_1 w_3 \delta_{13} + v_2 w_1 \delta_{21} + v_2 w_2 \delta_{22} + \dots$

This is definition for Real $\mathbb{V}$

For Real Vector Spaces

i) Symmetry $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

ii) Linearity $(a\vec{v}_1 + b\vec{v}_2) \cdot \vec{w} = a(\vec{v}_1 \cdot \vec{w}) + b(\vec{v}_2 \cdot \vec{w})$

iii) Positive defintenesss of $\vec{v} \cdot \vec{v} \geq 0$

$$\vec{v} \cdot \vec{v} = \sum_i v_i^2 = v_1^2 + v_2^2 + v_3^2 + \ldots$$

For Complex Vector Spaces

$$\vec{v} \cdot \vec{w} = \sum_i v_i^* w_i$$

i) Symmetry $\vec{v} \cdot \vec{w} = (\vec{w} \cdot \vec{v})^*$

ii) Linearity the second part

$$\vec{v} \cdot (a\vec{w}_1 + b\vec{w}_2) = a(\vec{v} \cdot \vec{w}_1) + b(\vec{v} \cdot \vec{w}_2)$$

$$(a\vec{v}_1 + b\vec{v}_2) \cdot \vec{w} = a^*(\vec{v}_1 \cdot \vec{w}) + b^*(\vec{v}_2 \cdot \vec{w})$$

anti-linear in first part

iii) Positive definiteness

$$\vec{v} \cdot \vec{v} = \sum_i v_i^* v_i = \sum_i |v_i|^2 >> 0$$

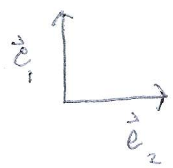
$$\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{v} \cdot \vec{w} = 1 + i$$

$$\vec{w} \cdot \vec{v} = 1 - i$$

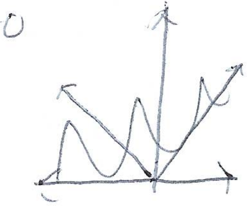
Definition of Perpendicular or orthogonal

Notice $\vec{e}_1 \cdot \vec{e}_2 = 0$



$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$

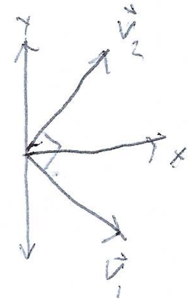
$\overset{\vec{v}_1}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \cdot \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = 1 - 1 = 0$



The orthonormal set $\{\hat{e}_i\}$

hat means unit length

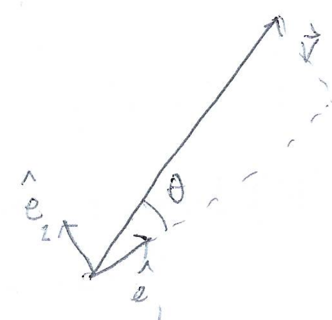
$\{\hat{e}_1, \hat{e}_2, \ldots\}$



$\hat{e}_i \cdot \hat{e}_i = 1$

$\hat{e}_i \cdot \hat{e}_j = 0 \quad i \neq j$

$$\boxed{\hat{e}_i \cdot \hat{e}_j = \delta_{ij}}$$



$\vec{v} = \sum_i v_i \hat{e}_i = v_1 \hat{e}_1 + v_2 \hat{e}_2 + \ldots$

how is $\vec{v}$ represented in this basis

$\vec{V} - v_1 \hat{e} =$ The vector(s) perpendicular to $\hat{e}_1$

$\vec{e}_1 \cdot (\vec{v} - v_1 \hat{e}) = 0$

$\hat{e}_1 \cdot \vec{v} = \hat{e}_1 \cdot \left(\sum_i v_i \hat{e}_i \right)$

$= \hat{e}_1 \cdot \vec{v} = v_1 \hat{e}_1 \cdot \hat{e}_1 + v_2 \hat{e}_1 \cdot \hat{e}_2 + \ldots$

$= v_1 \cancel{\theta_1} = |v| \cos\theta$

$v_1 = |v| \cos\theta$

$\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos\theta$

Gram Schmidt

next week

Phys 89 11 Feb 19

Which of these is a complex Number?

(A) i (B) $3-i$ (C) 2 $\boxed{\text{All 3}}$

Which of these is a complex vector?

(A) $\vec{v} = \begin{bmatrix} i \\ 2 \end{bmatrix}$ (B) $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (C) $\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\boxed{\text{All 3}}$

Real Inner Product Vector Space $\Rightarrow$ Everything is real

Complex Inner Product Vector Space $\Rightarrow$ Everything is complex (which includes real numbers)

Symmetry

Linearity

Positive definiteness $\vec{v} \cdot \vec{v}$

$$\left.\sqrt{\vec{v}\cdot\vec{v}} = \underbrace{\sqrt{\sum_i v_i^* v_i}}_{\text{Euclidian Norm}}\right\} \text{Length}$$

To be a "length" that we are familiar with, basis need to be orthogonal

- Inner Product Definitions
- Gram-Schmidt Procedure

$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} \Big\}$$ Column vector in an n-dimensional space

$$\underset{\sim}{\vec{V}} = \begin{bmatrix} V_1^* & V_2^* & \cdots & V_n^* \end{bmatrix} \Big\}$$ Row vector

$= \vec{V}^\dagger$ Conjugate transpose

$=$ Dual of $\vec{V}$

$$\vec{V}^T = \begin{bmatrix} V_1 & V_2 & V_3 & \cdots & V_n \end{bmatrix} \Big\}$$ Transpose of $\vec{V}$

$$\vec{V} \cdot \vec{W} \equiv \langle \vec{V}, \vec{W} \rangle$$

$$\vec{V} = \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}, \quad \vec{W} = \begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix} \qquad \vec{V} \cdot \vec{W} = \begin{bmatrix} V_1^* & V_2^* & \cdots & V_n^* \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix} = \sum_i V_i^* W_i$$

$$\vec{V} = \begin{bmatrix} i \\ 2 \\ 3i \end{bmatrix} \qquad \vec{W} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix}$$

$$\vec{V} \cdot \vec{W} = \begin{bmatrix} -i & 2 & -3i \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix} = 1(-i) + 2\left(\tfrac{1}{2}\right) - 3i\left(\tfrac{1}{3}\right)$$

$$= -i + 1 - i = 1 - 2i$$

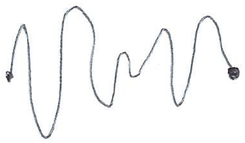
$\mathbb{V}$ waves between 0 and L

$\{\hat{e}_i\} = \{$  $\}$

$\vec{V} =$  $\quad \vec{V} = \sum_i V_i \hat{e}_i = V_1$  $+ \cdots$

Introduce an inner Product

$$\vec{f}\cdot\vec{g} = \int_0^L f(x)g(x)\,dx \quad , 0<x<L$$

$\vec{f}$ 

$\hat{e}_1 =$ 

$\vec{f}(x)e_1(x) =$ 

middle magnitude increases more than the ends

The integral basically adds up the elements of this product

$\hat{e}_1 = c\sin\frac{\pi x}{L}$

$\uparrow$
normalization constant

$$\int_0^L \hat{e}_1(x)\hat{e}_1(x)\,dx = 1 = \int_0^L c^2\sin^2\frac{\pi x}{L}\,dx$$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$= \frac{c^2}{2}\int\left(1-\cos\frac{2\pi x}{L}\right)dx = \frac{c^2}{2}\left(x - \frac{\sin\frac{2\pi x}{L}}{\frac{2\pi}{L}}\right)\Bigg|_0^L$$

$$= \frac{c^2}{2}\left(x - \frac{L\sin\frac{2\pi x}{L}}{2\pi}\right)\Bigg|_0^L = \frac{c^2}{2}(L-0) = \frac{c^2 L}{2} = 1$$

$$c^2 = \frac{2}{L} \Rightarrow c = \frac{\sqrt{2}}{\sqrt{L}} \qquad \hat{e}_1 = \sqrt{\frac{2}{L}}\sin\frac{\pi x}{L} = \sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$$

$\hat{e}_m \cdot \hat{e}_n$, $\hat{e}_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}\right)$

$$= \frac{2}{L}\int_0^L \sin\frac{m\pi x}{L} \cdot \sin\frac{n\pi x}{L}\, dx = \frac{2}{L}\int_0^L \frac{1}{2}\left(\cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}\right)dx$$

$$\frac{1}{L}\int_0^L \cos\frac{\pi(m-n)x}{L} - \cos\frac{(m+n)\pi x}{L}\, dx$$

$$= \frac{1}{L}\left(\frac{\sin\frac{\pi(m-n)x}{L}}{\frac{\pi(m-n)}{L}} - \frac{\sin\frac{(m+n)\pi x}{L}}{\frac{(m+n)\pi}{L}} \right)\Bigg|_0^L = 0$$

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

Orthogonal Functions

Hermite Bessel

Legendre Fourier

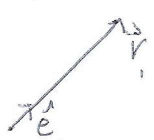
Laguerre

Gram-Schmidt Procedure

Given a set of linearly independent vectors $\{\vec{V}_i\}$ we can generate a set of orthonormal vectors $\{\hat{e}_i\}$ That span the same ~~space~~ vector space

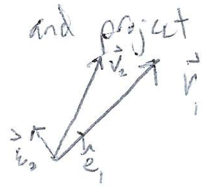
1) Take first element and make it a unit vector

$$\hat{e}_1 = \frac{\vec{V}_1}{|\vec{V}_1|}$$



2) Take the second element and project out the first

$$\vec{\varepsilon}_2 = \vec{V}_2 - (\cancel{\vec{V}_2 \cdot \hat{e}_1})\hat{e}_1$$
$$(\hat{e}_1 \cdot \vec{V}_2)\hat{e}_1$$



3) $$\hat{e}_2 = \frac{\vec{\varepsilon}_2}{|\vec{\varepsilon}_2|} = \frac{\vec{V}_2 - (\hat{e}_1 \cdot \vec{V}_2)\hat{e}_1}{|\vec{V}_2 - (\hat{e}_1 \cdot \vec{V}_2)\hat{e}_1|}$$

4) Take the third element $\vec{V}_3$ and remove the projection from $\hat{e}_1$ and $\hat{e}_2$

$$\vec{\varepsilon}_3 = \vec{V}_3 - (\hat{e}_1 \cdot \vec{V}_3)\hat{e}_1 - (\hat{e}_1 \cdot \vec{V}_3)\hat{e}_2$$

Then $$\hat{e}_3 = \frac{\vec{\varepsilon}_3}{|\vec{\varepsilon}_3|}$$

$$\boxed{\begin{aligned} \vec{\varepsilon}_i &= \vec{v}_i - \sum_j (\hat{e}_j \cdot \vec{v}_i)\hat{e}_j \\ \hat{e}_i &= \frac{\vec{v}_i - \sum_j (\hat{e}_j \cdot \vec{v}_i)\hat{e}_j}{\left|\vec{v}_i - \sum_j (\hat{e}_j \cdot \vec{v}_i)\hat{e}_j\right|} \end{aligned}}$$

$$\frac{\vec{v}_i - \sum_j (\hat{e}_j \cdot \vec{v}_i)\hat{e}_j}{\sqrt{|\vec{v}_i|^2 + \sum_j |\hat{e}_j \cdot \vec{v}_i|^2}}$$

Phys 89 13Feb19

Quiz Friday, end of class

Kreyszig 7.9 1-10, 15-20, 21-25 } Review Problems
7.1 1-16

Today Finish Gram Schmidt
Start Matrices

$$\text{eg } \{\vec{v}\} = \left\{ \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \overset{\vec{v}_3}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \right\}$$

i) $\hat{e}_1 = \dfrac{\vec{v}_1}{|v_1|} = \dfrac{1}{\sqrt{1^2+1^2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

ii) $\vec{\varepsilon}_2 = \vec{v}_2 - (\hat{e}_1 \cdot \vec{v}_2)\hat{e}$ $\qquad e_1 \vec{v}_2 = \left(\dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \dfrac{1}{\sqrt{2}} \left(\dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$ $\qquad = \dfrac{1}{\sqrt{2}}(1) = \dfrac{1}{\sqrt{2}}$

$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \dfrac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$

$\hat{e}_2 = \dfrac{\vec{\varepsilon}_2}{|\vec{\varepsilon}_2|} = \dfrac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1}} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \sqrt{\dfrac{2}{3}} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$

$\hat{e}_1 \cdot \hat{e}_2 = \dfrac{1}{\sqrt{2}} [1\ 1\ 0] \sqrt{\dfrac{2}{3}} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \dfrac{1}{\sqrt{3}} [1/2\ -1/2\ 0] = 0$

iii) $\vec{\varepsilon}_3 = \vec{V}_3 - (\hat{e}_1 \cdot \vec{V}_3)\hat{e}_1 - (\hat{e}_2 \cdot \vec{V}_3)\hat{e}_2$

$$\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \sqrt{\frac{2}{3}}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}, \hat{e}_3\right\}$$

Legendre Polynomials

(Example of orthogonal functions)

$$\{\phi_i\} = \{\underbrace{\tfrac{1}{\sqrt{2}}}_{\phi_0}, \underbrace{\sqrt{\tfrac{3}{2}}x}_{\phi_1}, \underbrace{\sqrt{\tfrac{5}{2}}\left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right)}_{\phi_2}, \dots\}$$

$$\langle \phi_i, \phi_j \rangle = \underbrace{\int_{-1}^{1} \phi_i(x)\phi_j(x)\,dx}_{\text{Definition of inner product for this space}} = \underbrace{\delta_{ij} \begin{array}{l} 1 \Rightarrow i=j \\ 0 \; i \neq j \end{array}}_{\text{normalization condition}}$$

$$\{\mu(x)\} = \{x^0, x^1, x^2, \dots\}$$

set of linearly independent but not orthogonal functions

$\phi_0 = 1/\sqrt{2}$

recall $\qquad x \qquad \sqrt{\int_{-1}^{1} dx} = \sqrt{2}$

$\hat{e}_1 = \dfrac{\vec{V}_1}{|\vec{V}_1| \to \sqrt{IP}}$

$$\phi_1 = \cancel{\tfrac{1}{\sqrt{2}} - (\phi_0 \cdot x)\phi_0}\; x - (\phi_0 \cdot x)\phi_0 = x - \cancel{\tfrac{1}{\sqrt{2}}x} - \int_{-1}^{1} \frac{1}{\sqrt{2}} x\,dx \frac{1}{\sqrt{2}}$$

$$= x - \frac{1}{2\sqrt{2}}x^2\Big|_{-1}^{1}\frac{1}{\sqrt{2}} = x - \frac{1}{4}(1+1) = x$$

$$\frac{\phi_1}{|\phi_1|} = \frac{x}{\sqrt{\int_{-1}^{1} x \cdot x\,dx}} = \frac{\sqrt{3}\,x}{\sqrt{x^3\big|_{-1}^{1}}} = \sqrt{\frac{3}{2}}\,x$$

Matrices

Linear Transformations

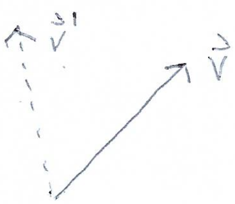
$$a(\vec{v}_1 + \vec{v}_2) = a\vec{v}_1 + a\vec{v}_2$$

a = scalar

$$M(\vec{v}_1 + \vec{v}_2) = M\vec{v}_1 + M\vec{v}_2$$

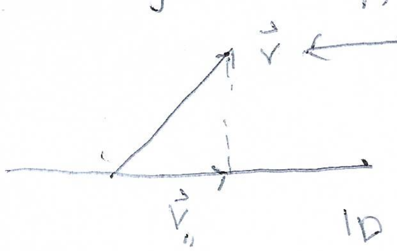
→ Rotations
~~Pojections~~ Projections
Shear

Rotation: Maps a vector $\vec{v}$ to another vector in the same vector space $\vec{v}'$ with the same length



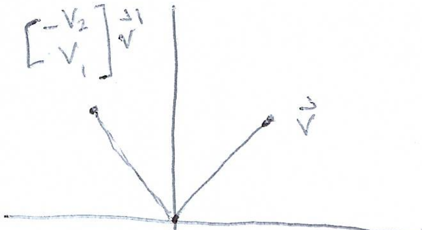
Identity: Maps vector onto itself

Projection: Maps a vector $\vec{v}$ onto a smaller dimension vector space



want to Find $M\vec{v} = \vec{v}'$

$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2, \quad \vec{v}' = -v_2\hat{e}_1 + v_1\hat{e}_2$$



$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad M\vec{v} = v_1 M\hat{e}_1 + v_2 M\hat{e}_2 \text{ (by linearity)}$$

$$= -v_2\hat{e}_1 + v_1\hat{e}_2$$

$$M\hat{e}_1 = 0\hat{e}_1 + 1\hat{e}_2, \quad M\hat{e}_2 = -\hat{e}_1 + 0\hat{e}_2$$

$$M = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}$$

In general we speak of $n \times m$ matrices

$$\begin{array}{c c} & \overbrace{\hphantom{M_{11} \quad M_{12} \quad \cdots \quad M_{1m}}}^{m \text{ columns}} \\ n \text{ rows} \left\{ \vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}} \right. & \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1m} \\ M_{21} & M_{22} & \cdots & \vdots \\ \vdots & & \ddots & \\ M_{n1} & \cdots & & M_{nm} \end{bmatrix} = M \end{array}$$

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = M_{11} = 0, \; M_{12} = -1$$

$$M_{21} = 1, \; M_{22} = 0$$

2×2

$$P = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad P\vec{v} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_2 = \vec{v}'$$

1×2 1D space

$$[1 \times 2] \cdot [2 \times 1] = [1 \times 1]$$

In general a $n \times m$ Matrix takes a vector in an m dimensional space and maps it to an n-dimensional space

$$\underset{(n \times m)}{M} : \mathbb{R}^m \to \mathbb{R}^n$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

3D 2D

$$\mathbb{R}^3 \to \mathbb{R}^2$$

Phys 89 15 Feb 19

Today:
By Popular Demand
<u>Levi-Civita</u>

Quiz #2
Today

- Matrix Multiplication

Monday no lecture

Guest Lecture Wednesday (Please review matrix multiplication)
Kreyszig 7.1

Delta Function

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

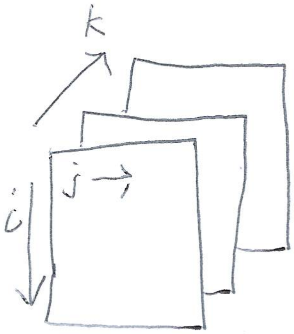
$$\vec{a} \cdot \vec{b} = \sum_i a_i b_i = \sum_i \sum_j a_i b_i \delta_{ij}$$

$$a_i = \sum_j a_j \delta_{ij} = a_1 \delta_{i1} + a_2 \delta_{i2} + a_3 \delta_{i3} + \ldots$$

$$\varepsilon_{ijk} = \begin{cases} 1 & (ijk) = (123)(231)(312) \\ -1 & (ijk) = (132)(213)(321) \\ 0 & \text{Repeated indice} \quad i=j \text{ or } i=k \text{ or } j=k \end{cases}$$

$$\delta_{ij} = \begin{bmatrix} \delta_{11} & \delta_{12} & \cdots \\ \delta_{21} & \delta_{22} & \\ \vdots & & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \\ \vdots & & \ddots \end{bmatrix}$$

$\varepsilon_{ijk} =$



$$(\vec{a} \times \vec{b})_i$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} (\vec{a} \times \vec{b})_1 \\ (\vec{a} \times \vec{b})_2 \\ (\vec{a} \times \vec{b})_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - b_2 a_3 \\ -(a_1 b_3 - b_1 a_3) \\ a_1 b_2 - b_1 a_2 \end{bmatrix}$$

$$(\vec{a} \times \vec{b})_i = \sum_j \sum_k \varepsilon_{ijk} a_j b_k$$ * ith term not involved *

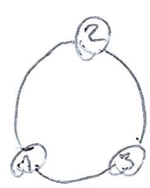
$$= \cancel{\varepsilon_{i11} a_1 b_1} + \varepsilon_{i12} a_1 b_2 + \varepsilon_{i13} a_1 b_3$$
$$+ \varepsilon_{i21} a_2 b_1 + \cancel{\varepsilon_{i22} a_2 b_2} + \varepsilon_{i23} a_2 b_3$$
$$+ \varepsilon_{i31} a_3 b_1 + \varepsilon_{i32} a_3 b_2 + \cancel{\varepsilon_{i33} a_3 b_3}$$

As soon as we choose an $i = 1, 2, 3$
four of the six remaining entries will vanish

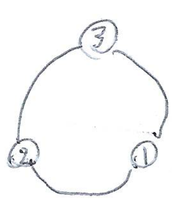
$$(\vec{a} \times \vec{b})_2 = \varepsilon_{213} a_1 b_3 + \varepsilon_{231} a_3 b_1 = -a_1 b_3 + a_3 b_1 = -(a_1 b_3 - a_3 b_1)$$

1 2 3

2 3 1

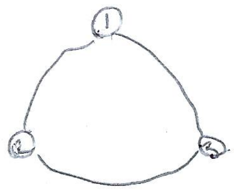


3 1 2



even = cyclic Permutation

1 3 2



odd = ~~non~~ noncyclic Permutation

$$(\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B})$$

$$= \sum_i \sum_j (\vec{A}\times\vec{B})_i (\vec{A}\times\vec{B})_j \delta_{ij}$$

$$\cancel{\sum_l \sum_k \sum_i \sum_j \varepsilon_{ikl} A_k B_l}$$

$$= \sum_i \sum_j \sum_k \sum_l \sum_m \sum_n \varepsilon_{ikl} A_k B_l \varepsilon_{jmn} A_m B_n \delta_{ij}$$

Einstein notation

Repeated indices imply summation

$$\varepsilon_{ikl} A_k B_l \varepsilon_{jmn} A_m B_n \delta_{ij} = \varepsilon_{ikl}\varepsilon_{jmn}\delta_{ij} A_k B_l A_m B_n$$

$$= \underbrace{\varepsilon_{ikl}\varepsilon_{imn}} A_k B_l A_m B_n$$

$$= (\delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}) A_k B_l A_m B_n$$

$$= \delta_{km}\delta_{ln} A_k B_l A_m B_n - \delta_{kn}\delta_{lm} A_k B_l A_m B_n$$

$$= A_k A_k B_l B_l - A_k B_k B_l A_l = A_k A_k B_l B_l - A_k B_k A_l B_l$$

$$= (\vec{A}\cdot\vec{A})(\vec{B}\cdot\vec{B}) - (\vec{A}\cdot\vec{B})(\vec{A}\cdot\vec{B})$$

Quiz #2

1) Which vectors are orthogonal to $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ find 2 linearly independent ones

A) $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}$ B) $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$ C) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$

D) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \cdot [-1\ 0\ 2] = 0$

2) Functions $y(x) = (a+bx)e^{-x}$ define a vector space, what is the basis

A) $(x+1)e^{-x}$

B) xe^{-x}, e^{-x}

C) $\{1, x, x^2, x^3, \dots\}$

D) $\{e^{-x}, e^{x}, x\}$

3) what is the euclidian norm of vector $\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$

A) 1

B) $\sqrt{2}$

C) $\frac{1}{\sqrt{2}}$

D) $1/2$

4) Suppose inner product $\langle f(x), g(x) \rangle = \int_0^\infty f(x)g(x)e^{-x}dx$ Starting from monomial basis $\{1, x, x^2, \dots\}$ what is first normalized function?

$\int_0^\infty e^{-x}dx = -e^{-x}\Big|_0^\infty = -(0-1) = 1$

A) e^{-x}

B) $1+x$

C) 1

D) x

$$(AB)^T = (A_{ij} B_{kl})^T$$

$$(a_{ir} b_{rj})^T = a_{ri} b_{jr} = b_{jr} a_{ri} = B^T A^T$$

$$Tr(AB) = Tr(a_{ir} b_{rj})$$

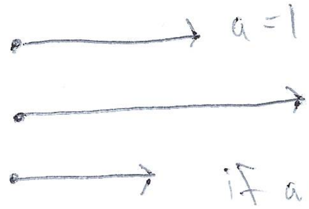
$$Tr(AB) = (ab)_{ii} = a_{ij} b_{ji} = b_{ji} a_{ij} = (ba)_{jj} = Tr\,BA$$

$$Tr((AB)^T) = Tr(B^T A^T) = b_{ji} a_{ij} = a_{ij} b_{ji} = (ab)_{ii} = Tr(AB)$$

Phys 89 20 Feb 19

Mike Zaletel

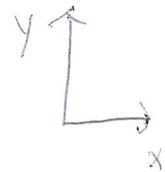
$\vec{V} \to a\vec{V}$ a is $\mathbb{R}$



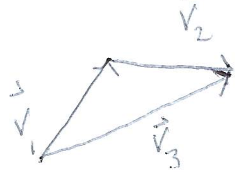
$a=1$

if $a>1$

if $a<1$



$\vec{V}_1 + \vec{V}_2 = \vec{V}_3$



$\vec{V} = x\hat{x} + y\hat{y} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots \\ M_{21} & \ddots & \\ \vdots & & \ddots \end{bmatrix}$$

$\vec{u} = a_1 \vec{V}_1 + a_2 \vec{V}_2$

$$u_i = \sum_{j=1}^{N} M_{ij} V_j$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = M \cdot V = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$f(\vec{v}) = \vec{u}$

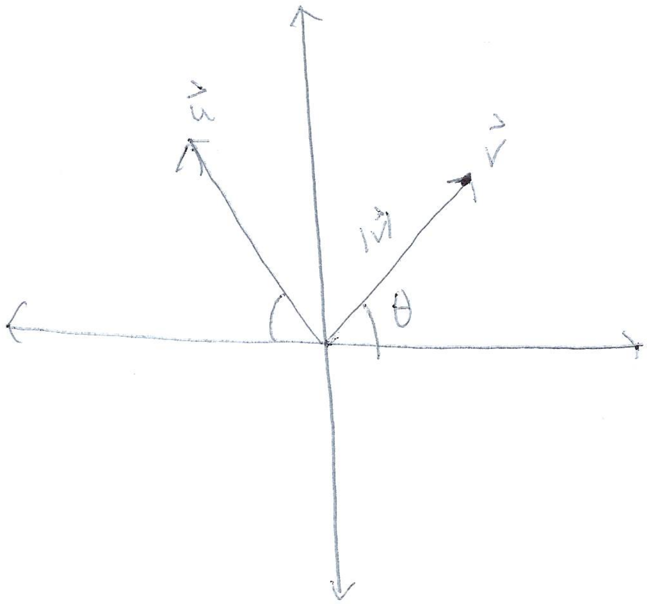
$f(\vec{v}_1 + \vec{v}_2) = a(\vec{v}_1 + \vec{v}_2) = a\vec{v}_1 + a\vec{v}_2 = f(\vec{v}_1) + f(\vec{v}_2)$

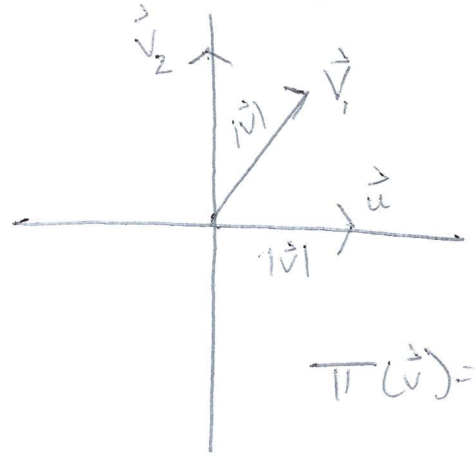
$f(\vec{v}) = a\vec{v}$

$f(c\vec{v}_1) = a(c\vec{v}_1) = c(a\vec{v}_1) = c f(\vec{v}_1)$

$f(a_1\vec{v}_1 + a_2\vec{v}_2) = a_1 f(\vec{v}_1) + a_2 f(\vec{v}_2)$

$$R(\vec{v}) = \vec{u}$$





$$\Pi(\vec{v}) = \vec{u}$$

$$\Pi(\vec{v}_1 + \vec{v}_2) \stackrel{?}{=} \Pi(\vec{v}_1) + \Pi(\vec{v}_2)$$

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} -v_y \\ v_x \end{bmatrix}$$

$$\vec{u} = \sum_{j=1}^{N} M_{ij}(v_j^1 + v_j^2) = \sum_{j=1}^{N} M_{ij} v_j^1 + \sum_{j=1}^{N} M_{ij} v_j^2 = M\vec{v}^1 + M\vec{v}^2$$

$$\vec{u} = \sum_{j=1}^{N} M_{ij}(a v_j) = a \sum_{j=1}^{N} M_{ij} v_j = aM\vec{v}$$

$$\vec{v} = x\hat{x} + y\hat{y} + \ldots$$

$$f(a\vec{v}_1 + \vec{v}_2) = a f(\vec{v}_1) + f(\vec{v}_2)$$

$$\hat{x}' = f(\hat{x}), \quad \hat{y}' = f(\hat{y}), \ldots$$

$$f(x\hat{x} + y\hat{y}) = x f(\hat{x}) + y f(\hat{y}) = x\hat{x}' + y\hat{y}'$$

$$f(\hat{x}) = \hat{x}' = (\ )\hat{x} + (\ )\hat{y}$$

$$= m_{11}\hat{x} + m_{21}\hat{y}$$

$$f(\hat{y}) = \hat{y}' = m_{12}\hat{x} + m_{22}\hat{y}$$

$$f(x\hat{x} + y\hat{y}) = x f(\hat{x}) + y f(\hat{y})$$

$$= x(m_{11}\hat{x} + m_{21}\hat{y}) + y(m_{12}\hat{x} + m_{22}\hat{y})$$

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} m_{11}x + m_{12}y \\ m_{21}x + m_{22}y \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$f: \mathbb{R}^M \to \mathbb{R}^N$$

$$\begin{array}{cc} & M \\ N & \begin{bmatrix} M_{11} & \cdots & M_{1m} \\ \vdots & \ddots & \vdots \\ M_{1m} & \cdots & M_{NM} \end{bmatrix} \end{array} \begin{bmatrix} V_1 \\ \vdots \\ V_m \end{bmatrix} = \begin{array}{cc} \begin{bmatrix} M_{11}V_1 + M_{12}V_2 + \cdots \\ M_{21}V_1 + M_{22}V_2 + \cdots \\ \vdots \end{bmatrix} & N \end{array}$$

Phys 89 22Feb19

Today: Finish Matrix Multiplication
Row Reduction

Matrix $\underset{n\times m}{M}$ connects $\mathbb{R}^m \to \mathbb{R}^n$

$\{\hat{e}_i\}$ Basis for $\mathbb{R}^m$ $i = 1, \ldots, m$

$\{\hat{f}_i\}$ Basis for $\mathbb{R}^n$ $j = 1, \ldots, n$

$$M_{ji} = \hat{f}_j \cdot (M\hat{e}_i)$$

$$\underset{2\times 3}{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \end{bmatrix} \quad \mathbb{R}^3 \to \mathbb{R}^2$$

$$\{\hat{f}\} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}; \quad \{\hat{e}\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$\hat{f}_1 \cdot M\hat{e}_2 = [1\,0] \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = [1\,0] \begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} = M_{12}$$

$$M_{\alpha\gamma} = M_{\alpha\beta} M_{\alpha\gamma}$$

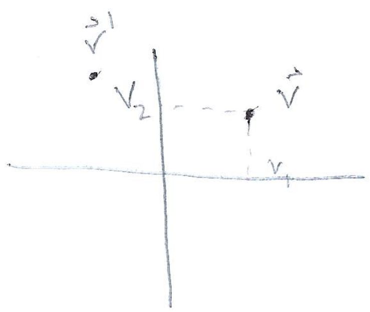
$(m \times p) \quad (m \times n) \quad (n \times p)$

$\mathbb{R}^p \to \mathbb{R}^m \quad \mathbb{R}^n \to \mathbb{R}^m \quad \mathbb{R}^p \to \mathbb{R}^n$

$$\begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \\ 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -3 & -7 \end{bmatrix}$$

1×2 $\quad$ 2×3 $\quad$ 1×3

$\mathbb{R}^2 \to \mathbb{R}^1$ $\quad$ $\mathbb{R}^3 \to \mathbb{R}^2$ $\quad$ $\mathbb{R}^3 \to \mathbb{R}^1$



$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}$$

$$R_{\pi/2} \vec{v} = \vec{v}'$$

$$R_{\pi/2} R_{\pi/2} = R_{\pi}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix} = -\vec{v}$$

Using Matrices to solve Simultaneous Equations

$$ax + by + cz = \gamma \qquad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$dx + ey + fz = \beta$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma \\ \beta \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \end{bmatrix} = \begin{bmatrix} \gamma \\ \beta \end{bmatrix}$$

$$x_1 Cu_2S + x_2 O_2 \rightarrow x_3 Cu + SO_2$$

Want to determine x_1, x_2, x_3

Cu : $x_1 \cdot 2 + x_2 \cdot 0 = x_3 \cdot 1 + 0$

S : $x_1 \cdot 1 + x_2 \cdot 0 = x_3 \cdot 0 + 1$ — move right terms to left

O : $x_1 \cdot 0 + x_2 \cdot 2 = x_3 \cdot 0 + 2$

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad A\vec{x} = \vec{b}$$

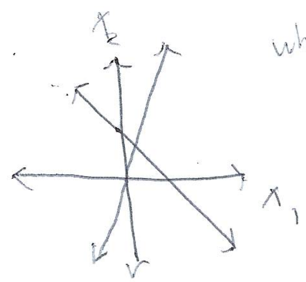
Task is to solve for x

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & -1 & 0 \end{array}\right]$$

① Unique solution

$x_1 + x_2 = 1$

$2x_1 - x_2 = 0$



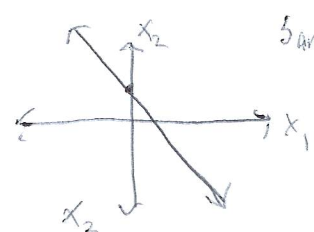
where lines cross is solution

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right]$$

② Infinitely many solutions

$x_1 + x_2 = 1$

$2x_1 + 2x_2 = 2$



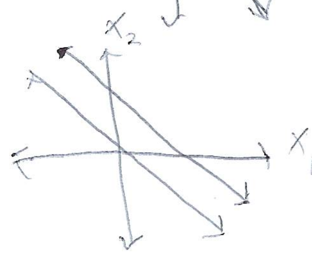
same line

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right]$$

③ No solution

$x_1 + x_2 = 0$

$x_1 + x_2 = 1$



no intercept

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Augmented Matrix

$$\underbrace{\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{array}\right]}_{\substack{3\times 4 \\ \tilde{A}}} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \hline -1 \end{bmatrix}}_{4\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \tilde{A}\,\vec{\tilde{X}} = \vec{0}$$

Row Reduction / Gauss-Jordan

What operations can we do to simultaneous equations on our rows of $\tilde{A}$ and keep the RHS $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ the same

1. Addition of rows keeps the RHS the same
2. Swapping rows keeps the RHS the same
3. Multiply any row by a constant

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{array}\right] \xrightarrow{R_1 \times \frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{array}\right] \xrightarrow[R_2 + R_3]{\text{swap}} \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \times -1} \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 2 & 0 & 2 \\ -1 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 + R_1} \left[\begin{array}{ccc|c} 0 & 0 & -1/2 & -1 \\ 0 & 2 & 0 & 2 \\ -1 & 0 & 0 & -1 \end{array}\right] \xrightarrow[R_1 + R_3]{\text{swap and mult } -1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -1/2 & -1 \end{array}\right] \xrightarrow[R_3 \times -2]{R_2 \frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x_1 - 1 = 0$

$x_2 - 1 = 0$

$x_3 - 2 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$1Cu_2S + 1O_2 \rightarrow 2Cu + SO_2$

Phys 89 01 Mar 19

~~He~~

Hermitian Conjugate

$$M^{\dagger} = (M^*)^T = (M^T)^* \qquad (M^{\dagger})_{ij} = M^*_{ji}$$

Real Matrices

$$M_{ij} = M^*_{ij}$$

Symmetric Matrices

$$M_{ij} = M_{ji} \qquad \begin{bmatrix} a & c \\ c & b \end{bmatrix} \qquad M = M^T$$

Anti-symmetric

$$M_{ij} = -M_{ji} \qquad \begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix} \qquad M = -M^T \text{ , diagonal} = 0$$

Hermitian Matrices

$$M_{ij} = M^*_{ji} \qquad \text{eg} \begin{bmatrix} a & z \\ z^* & b \end{bmatrix} \qquad \text{diagonals are real}$$

Anti-Hermitian (or skew-Hermitian)

$$M_{ij} = -M^*_{ji} \qquad \text{eg} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Can decompose any matrix into symmetric and anti-symmetric

or Hermitian and anti-hermitian

$$M = M_{sym} + M_{anti\,sym} \qquad M = M_{Herm} + M_{anti\,herm}$$

<u>Transpose of a product</u>

$$(ABCD\ldots)^T = \ldots C^T B^T A^T$$

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki} = \sum_k (A^T)_{kj} (B^T)_{ik}$$

$$= \sum_k (B^T)_{ik} (A^T)_{kj} = B^T A^T$$

$$M\vec{v} = \vec{w}$$

$\vec{v} \cdot \vec{w}$ say both real

$$\vec{v} \cdot \vec{w} = \vec{v} \cdot (M\vec{v}) = \vec{v}^T M \vec{v} = (\vec{v}^T M)\vec{v} = (M^T \vec{v})^T \vec{v} = (M^T \vec{v}) \cdot \vec{v}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 3 & -1 \end{bmatrix} \qquad v \cdot M\vec{v} = [1\ 1\ 1] \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 6$$

$$(M^T \vec{v}) \cdot \vec{v} = [1\ 5\ 0] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 6$$

$$M\vec{v} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$M^T \vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

$$(ABC\cdots)^\dagger = \cdots C^\dagger B^\dagger A^\dagger$$

M $\quad M_{sym} = \frac{1}{2}(M + M^T)$ $\qquad M_{sym} = M_{sym}$

$\quad M_{antisym} = \frac{1}{2}(M - M^T)$ $\qquad \frac{1}{2}(M + M^T)^T = \frac{1}{2}(M^T + M) = M_{sym}$

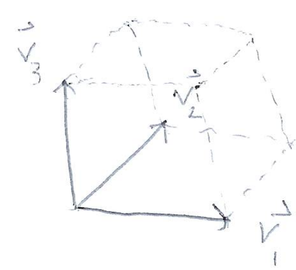
$M_{antisym} = -M_{antisym}$

$M_{Herm} = \frac{1}{2}(M + M^\dagger)$ $\qquad M_{antsym} = \frac{1}{2}(M - M^T)^T = \frac{1}{2}(M^T - M) = -M_{antisym}$

$M_{antiherm} = \frac{1}{2}(M - M^\dagger)$

$$\text{Trace}(M) = \sum_i M_{ii} \qquad M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad Tr(M) = a + e + i$$

<u>Determinant</u> $\quad \det(M) = |M| \qquad M = [\vec{V}_1 \ \vec{V}_2 \ \vec{V}_3]$



If only two are linearly independent
Then this volume collapses to ~~ver~~ zero

"Volume" $= 0$ if vectors in M are linearly dependent
$\updownarrow$
det

$1\times 1 \quad \det(M) = m$

$M = m$

2×2

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det(M) = ad - bc$

(note has units of area)

For Larger Matrices we use a recursive algorithm

$M = \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix} \quad C_{ij} = (-1)^{i+j} \det(\text{leftover terms})$

$C_{ij} = (-1)^{i+j} \det(\cancel{M_{ij}}\, \mathcal{M}_{ij})$ ijth Minor of M

$\det(M) = \sum_i \underbrace{M_{ij}\, C_{ij}}_{\text{fix } j} = b(-1)^{1+2} \det\begin{pmatrix} d & f \\ h & j \end{pmatrix} + e(-1)^{2+2} \det\begin{pmatrix} a & c \\ h & j \end{pmatrix} + i(-1)^{2+3} \det\begin{pmatrix} a & c \\ d & f \end{pmatrix}$

$= -b \det\begin{pmatrix} d & f \\ h & j \end{pmatrix} + e \det\begin{pmatrix} a & c \\ h & j \end{pmatrix} - i \det\begin{bmatrix} a & c \\ d & f \end{bmatrix}$

$= -b(dj - fh) + e(aj - ch) - i(af - dc) = \text{Volume in } \mathbb{R}^3$

1. what is the rank of this matrix

$$\begin{bmatrix} 6 & -4 & 0 \\ -4 & 0 & 2 \\ 0 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 6 & -4 & 0 \\ 0 & -8/3 & 2 \\ 0 & 2 & 6 \end{bmatrix} \to$$

a 1

b 2

c 3

d 4

rank M = 3

2. is the set of vectors lin ind?

[2 0 0 7]

[2 0 1 0]

[2 0 0 9]

[2 0 0 8

a yes

b no

c none of above

d all above

3. Solve sim equation

$-2y - 2z = -8$

$3x + 4y - 5z = 13$

$$\left[\begin{array}{ccc|c} 0 & -2 & -2 & -8 \\ 3 & 4 & -5 & 13 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & -2 & -2 & -8 \\ 3 & 0 & -9 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 1 & 0 & -3 & -1 \end{array}\right]$$

a $[x, y, z] = [-1 + 3t, 4 - t, t]$

b $[x, y, z] = [11 - 3t, t, 4 - t]$

c $[x, y, z] = \frac{1}{3}[3t, 11 - t, 1 + t]$

d all of above

$y + z = 4$

$x - 3z = -1$

$z = 4 + y$

$y = z + 4$

$x = -1 + 3z$

$z = t$

$x = +3z + 1$

$y = z$

4. dot product of

$a = [1+i, 2, 0]^T \quad b = [1, 2-i, 7]^T$

$1 + i \quad + 4 - 2i \quad = 5 - i$

A $5 - 3i$

B $5 - i$

C $5 + i$

d $5 + 3i$

Phys 89 25 Feb 19

Today: Kreyszig 7.4 – 7.5

Rank, Linear independence, kernel image

After Row Reduction, two standard forms:

pivots

$$\begin{bmatrix} \boxed{\cdot} & \cdot & \cdot & \cdot \\ 0 & \boxed{\cdot} & \cdot & \cdot \\ 0 & 0 & \boxed{\cdot} & \cdot \\ 0 & 0 & 0 & \boxed{\cdot} \end{bmatrix}$$

Row Echelon Form

pivots

$$\begin{bmatrix} 1 & \cdot & 0 & \cdot \\ 0 & 1 & \cdot & \cdot \\ 0 & 0 & 1 & \cdot \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reduced Row echelon form

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$x_1 + 2x_2 + 3x_3 = 1$$
$$4x_2 + 2x_3 = 1$$
$$x_3 = 1$$

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & -2/3 \end{bmatrix}$$

Classifying Solutions

$$\tilde{A} = (A|\vec{b}) \xrightarrow[\text{row reduce}]{} (R|\vec{\beta})$$

row reduced forms of $\tilde{A}$

The <u>RANK</u> of a matrix is the number of pivots left after row reduction

Whatever solutions $(R|\vec{\beta})$ has, so does $(A|\vec{b})$

1) <u>No solutions</u>

$$\text{rank}(\tilde{A}) > \text{rank}(A)$$

eg. $\left[\begin{matrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{matrix}\middle| \begin{matrix} 1 \\ 1 \\ 1 \end{matrix}\right]$ $\text{rank}(\tilde{A}) = 3$, $\text{rank}(A) = 2$

2) <u>Infinite solutions</u>

$$\text{rank}(A) < \#\text{ columns in } A$$

eg $\left[\begin{matrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{matrix}\middle| \begin{matrix} 1 \\ 1 \\ 0 \end{matrix}\right]$ $\text{rank}(A) = \text{rank}(\tilde{A})$

$$x_1 + 2x_2 + 3x_3 = 1$$
$$2x_2 = 1 \qquad x_2 = 1/2$$

$$x_1 + 1 + 3x_3 = 1 \qquad x_1 = -3t$$
$$x_1 + 3x_3 = 0 \qquad x_2 = 1/2$$
$$x_1 = -3x_3 \qquad x_3 = t$$

This defines a line

1 dimensional space of solutions

3) Unique Solution

$$\text{rank}(A) = \#\text{ of columns of } A = \text{rank}(\tilde{A})$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$\text{rank}(A) = 3$

$\text{rank}(\tilde{A}) = 3$

$\#$ columns of $A = 4$

$$\dim(\text{solution set}) = \#\text{ columns of } A - \text{rank}(A)$$

Linear Independence

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{w}$$

Some constant

set of $\{\vec{v}_i\} = \{\vec{v}_1, \ldots, \vec{v}_n\}$

$n = 3 \quad \vec{v}_1, \vec{v}_2, \vec{v}_3$

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{w}$$

$$a_1\begin{bmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{bmatrix} + a_2\begin{bmatrix} v_{2x} \\ v_{2y} \\ v_{2z} \end{bmatrix} + a_3\begin{bmatrix} v_{3x} \\ v_{3y} \\ v_{3z} \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$x, y, z = 1, 2, 3$

$$\begin{bmatrix} a_1 v_{11} + a_2 v_{21} + a_3 v_{31} \\ a_1 v_{12} + a_2 v_{22} + a_3 v_{32} \\ a_1 v_{13} + a_2 v_{23} + a_3 v_{33} \end{bmatrix}^T$$

$$\begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 v_{11} + a_2 v_{12} + a_3 v_{13} \\ a_1 v_{21} + \cdots \\ a_1 v_{31} + a_2 v_{32} + a_3 v_{33} \end{bmatrix}$$

For arbitrary # of $\{\vec{v}_i\}$

$$V\vec{a} = \vec{w}$$

$$V = \begin{bmatrix} V_{11} & \cdots & V_{1m} \\ \vdots & \ddots & \vdots \\ V_{n1} & \cdots & V_{nm} \end{bmatrix}$$

If the set $\{\vec{V}_1, \ldots, V_m\}$ is linearly independent, then

$\mathrm{rank}(V) = m$ rank of matrix made of column vectors is equal to the number of vectors

$$V \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

So if $\mathrm{rank}(A) = m$, $\{\vec{v}_i\}$ are linearly independent

Phys 89 27Feb2019

Suppose $\{\vec{v}_1, \ldots, \vec{v}_m\}$

vector space

$\text{span}\{\mathbb{V}\}$ where $\mathbb{V} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix}$

$n \times m$

Now consider $\mathbb{V}\vec{a} = \vec{w}$ $\quad \vec{w} \in \text{span}\{\vec{v}_i\}$

If this has no solution then

$$\text{rank}(\tilde{\mathbb{V}}) > \text{rank}(\mathbb{V})$$

$$\tilde{\mathbb{V}} = [\mathbb{V} | \vec{w}]$$

What is the dimension of this space $\text{span}\{\vec{v}_i\}$

$$\text{rank}(V) = \dim(\text{span}(V))$$

dimension

$$\{\vec{v}_i\} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(V) = 2$$

if V row reduces such that it has as many pivots as ~~the~~ dimension of $\mathbb{V}$ (the vector space to which $\vec{v}_i$ belong)

eg $\vec{v}_i = \begin{bmatrix} v_{1i} \\ v_{2i} \\ \vdots \\ v_{ni} \end{bmatrix}$ $\quad \mathbb{V} = \mathbb{R}^n$ (n-dimensional space)

then $\text{span}\{\vec{v}_i\}$ ~~describes~~ spans all of $\mathbb{V}$

ie $\{\vec{v}_i\}$ can be used as a basis for $\mathbb{V}$

$\dim(\mathbb{V}) = n$

There are at most n linearly independent vectors to describe $\mathbb{V}$

$\mathbb{V}$ needs at least n vectors to span it

$$\{\vec{v}_i\} \to V = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\dim(\mathbb{V}) = 3$

But since there are 4 we know at least one is linearly dependent on the others

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= -\frac{3}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

eg

$$V = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$\dim(\mathbb{V}) = 4$

so span $\{\vec{v}_i\}$ cannot span $\mathbb{V}$

Consider $M: \mathbb{V} \to \mathbb{W}$ $\quad \dim(\mathbb{V}) = m$

$\dim(\mathbb{W}) = n$

M $\quad \cancel{m \times n}$ $\quad n \times m$

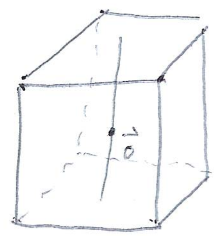
$$\vec{v}_i = \begin{bmatrix} v_{1i} \\ v_{2i} \\ \vdots \\ v_{ni} \end{bmatrix} \quad \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} w_i \\ \vdots \\ w_n \end{bmatrix}$$

Kernel, ker(M), Null space

Image, Im(M), Column Space of M

Kernel

$\ker(M) \in \mathbb{V}$ All vectors $\vec{v}$ in $\mathbb{V}$ such that $M\vec{v} = \vec{0}$



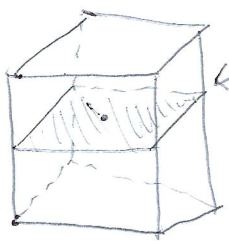
$\mathbb{V} = \mathbb{R}^3$

$\ker(M) = \{\vec{0}\}$ M is one to one

"injective"

Image

$\mathrm{Im}(M) \in \mathbb{W}$ All vectors $\vec{w}$ $M\vec{v} = \vec{w}$



$M\vec{0} = \vec{0}$

$\mathrm{Im}(M) = \mathbb{W}$

M "onto" or surjective

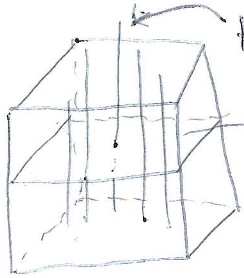
Rank-Nullity Theorem

Given $M \; \mathbb{V} \to \mathbb{W}$ $= \dim(\mathbb{V}) = m$

$\dim(\ker(M)) + \dim(\mathrm{Im}(M)) = \dim(\mathbb{V}) = m$

$$\vec{V} = \underbrace{\vec{V}_k}_{\in \ker(M)} + \underbrace{\vec{V}_{\vec{w}}}_{\text{maps to } \vec{w}}$$

$$M\vec{V} = M\vec{V}_k + M\vec{V}_{\vec{w}} = \vec{0} + \vec{w} = \vec{w}$$



Projection

$$M\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

$$\ker(M) = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

$$\mathrm{Im}(M) = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\dim(\mathrm{Im}(M)) = 2$

$\dim(\ker(M)) = 1$

$\dim(\mathbb{V}) = 3$

If M is one to one and onto it is called an isomorphism

Every distinct $\vec{v}$ will map to a distinct $\vec{w}$ and fill both spaces

If $\exists$ an M that is an isomorphism then $\underbrace{\mathbb{V} \text{ and } \mathbb{W}}_{\text{equivalent spaces}}$ are isomorphic

Definition

$\mathrm{null}(M) = \dim(\ker(M))$

$\mathrm{rank}(M) = \dim(\mathrm{Im}(M))$

$$\text{rank}(M) + \text{null}(M) = \dim(\mathbb{V}) = m$$

Manipulation of Matrices

$$M_{ij} = \begin{array}{cc} & j \\ i & \begin{bmatrix} M & - & - & - \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & - & - & \ddots \end{bmatrix} \end{array}$$

~~Ad~~

Addition $\quad P = M + N$

$$P_{ij} = M_{ij} + N_{ij}$$

multiply by constant

$$cM = N \qquad cM_{ij} = N_{ij}$$

Matrix Multiplication

$$MN = P \Rightarrow P_{ij} = \sum_k M_{ik} N_{kj}$$

Complex Conjugation

$$(M)^*_{ij} = M^*_{ij} \qquad M = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad M^* = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

Transpose

$$(M^T)_{ij} = M_{ji} \qquad M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad M^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\text{rank}(M^T) = \text{rank}(M)$$

Phys 89 04Mar19

Properties of Square Matrices Determinants

Kreyszig 7.6 and 7.7

$$M = \begin{bmatrix} \mu_{11} & \cdots & \mu_{1n} \\ \vdots & & \\ \mu_{n1} & \cdots & \mu_{nn} \end{bmatrix}$$

$n \times n$

$$m_{ij} = \det \begin{bmatrix} \mu_{11} & \cdots & \mu_{1n} \\ & \mu_{ij} & \\ \mu_{n1} & \cdots & \mu_{nn} \end{bmatrix} = m_{ij}$$

$$W = \begin{vmatrix} f_1(0) & f_2(0) & f_3(0) & \cdots \\ \frac{df_1(0)}{dx} & \frac{df_2(0)}{dx} & \cdots & \\ \frac{d^2f}{dx^2} & & & \\ \vdots & & & \end{vmatrix} \neq 0 \quad f_1, \ldots \text{ are linearly independent}$$

$$\det(M) = \sum_i \mu_{ij} C_{ij}$$

Choice of j arbitrary

C_{ij} = (ij th) co-factor $= (-1)^{i+j} m_{ij}$

m_{ij} = (ij th) minor of M

Consider $M = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & \frac{1}{2} & -1 \end{bmatrix}$

choose $j = 3$

$$m_{13} = \det\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = 0 \quad m_{23} = \det\begin{bmatrix} 1 & 2 \\ 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \quad m_{33} = \det\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = 0$$

$$\det(M) = \sum_i \mu_{ij} C_{ij} = -2\left(\frac{1}{2}\right) = -1$$

$j = 1$

$$\sum_i \mu_{i1} C_{i1}$$

$$1\begin{vmatrix} 0 & 2 \\ \frac{1}{2} & -1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{vmatrix}$$

$= -1$

$i = 2$

$$\sum_j \mu_{2j} C_{2j}$$

$$\cancel{0}\begin{vmatrix} 2 & 0 \\ \frac{1}{2} & -1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 0 \ldots - 2\begin{vmatrix} 1 & 2 \\ 0 & \frac{1}{2} \end{vmatrix}$$

$= -1$

$$\det\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = a(df)$$

upper and lower triangular matrices have determinants equal to the product of their diagonal entries

Wronskian $f_1(x) = f_1(0) + \left.\frac{df_1}{dx}\right|_0 x + \left.\frac{d^2 f_1}{dx^2}\right|_0 \frac{x^2}{2!} + \cdots$

$$f_1(x) \doteq \begin{bmatrix} f_1(0) \\ \left.\frac{df_1}{dx}\right|_0 \\ \left.\frac{d^2f_1}{dx^2}\right|_0 \\ \vdots \end{bmatrix} \leftarrow \text{coefficients to } n\text{th power of } x$$

$$f_2(x) = \begin{bmatrix} f_2(0) \\ \left.\frac{df_2}{dx}\right|_0 \\ \vdots \end{bmatrix}$$

If these vectors are linearly independent, then so is $f_1(x), f_2(x), f_3(x), \ldots$

Defining Properties of determinants

① Antisymmetry

$$M = [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_a \cdots \vec{v}_b \cdots \vec{v}_n] \qquad M' = [\vec{v}_1 \cdots \vec{v}_b \cdots \vec{v}_a \cdots \vec{v}_n]$$

$$\det M = -\det M' \qquad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb \qquad \det\begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - ad$$

if $a = b$

$$\det(M) = -\det(M) = 0$$

When you have linearly dependent vectors in your matrix

$$\det M = 0$$

② Linearity in columns

$$\vec{v}_a = c_1\vec{w}_1 + c_2\vec{w}_2 \qquad \det(\vec{v}_1 \cdots \vec{v}_a \cdots \vec{v}_n) = c_1\det(\vec{v}_1 \cdots \vec{w}_1 \cdots \vec{v}_n) + c_2\det(\vec{v}_1 \cdots \vec{w}_2 \cdots \vec{v}_n)$$

$$cM = (c\vec{v}_1\ c\vec{v}_2 \cdots c\vec{v}_n)$$

$$\det(cM) = c^n \det M$$

Linear independence

$\det(M) = 0$ some columns are linearly dependent

$\det(M) \neq 0$ columns must be linearly independent

$\underset{n\times n}{M}$ $\text{rank}(M) = n$

③ Identity Matrix

$$\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \qquad \det(\mathbb{I}) = 1$$

Other Cute Properties

$$\det(A^T) = \det(A) \qquad \det(AB) = \det(A)\det(B)$$

Note $\det(A+B) \neq \det(A) + \det(B)$

Cramer's Rule $\quad A\vec{x} = \vec{b} \qquad \tilde{A} = [A \mid \vec{b}] \qquad B_i = [\vec{a}_1 \cdots \overset{\text{ith column}}{\vec{b}} \cdots \vec{a}_n]$

$$A = [\vec{a}_1 \cdots \vec{a}_n]$$

$$x_i = \frac{\det(B_i)}{\det(A)}$$

P23 7.7

$$3y - 4z = 16$$
$$2x - 5y + 7z = -27$$
$$-x \qquad - 9z = 9$$

$$\left[\begin{array}{ccc|c} 0 & 3 & -4 & 16 \\ 2 & -5 & 7 & -27 \\ -1 & 0 & -9 & 9 \end{array}\right] \qquad B_1 = \begin{bmatrix} 16 & 3 & -4 \\ -27 & -5 & 7 \\ 9 & 0 & -9 \end{bmatrix}$$

$$\det A = -3(-11) - 4(-5) = 33 + 20 = 53$$

$$x = \frac{\det B_1}{\det A},\ y = \frac{\det B_2}{\det A},\ z = \frac{\det B_3}{\det A}$$

$$\det B_3 = -3(\cancel{-53} + 18 - 27) + 16(-5) = 3(9) - 80 = -53$$

$$\frac{\det B_1}{\det A} = 0 = x$$

$$\frac{\det B_2}{\det A} = 4 = y$$

$$\frac{\det B_3}{\det A} = -1 = z$$

$$\{x^k\} = \{1, x, x^2, \ldots\}$$

$$f_1(x) = 1 + x,\ f_2(x) = 2 + 2x,\ f_3(x) = 1 + x^2,\ f_4(x) = 2 + x + x^2$$

Phys 89 08 Mar 19

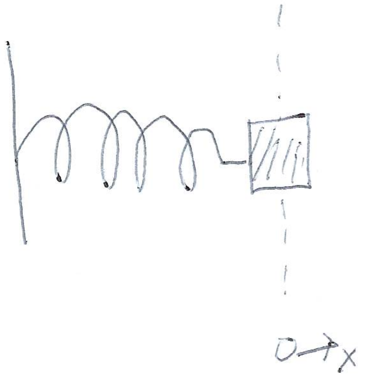
Eigen values and Eigen vectors

Ch 8 Kreyszig

Midterm: 3 Hrs

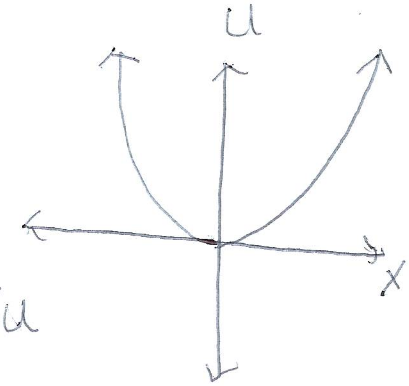
Topics: Everything until today

Hint: heavily based on lectures



$U = \frac{1}{2} k x^2$

$F = -kx = -\nabla U$



$F = ma = -kx = m\ddot{x}$

$\ddot{x} = -\frac{k}{m} x$

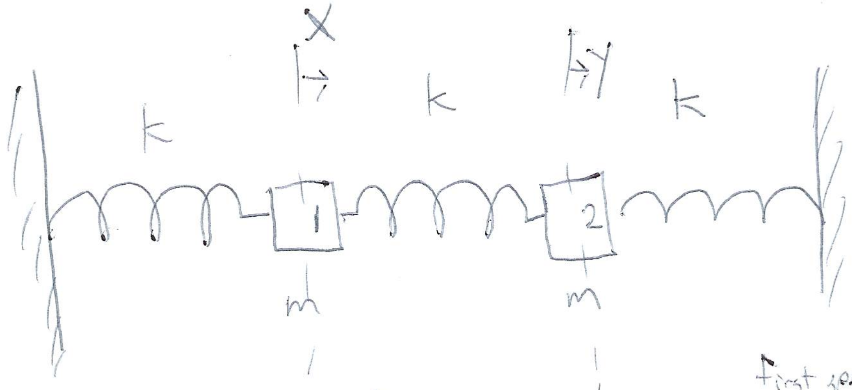
$x = x_0 e^{i\omega t}$ } Re(x) = Physically meaningful solution

$\ddot{x} = x_0 \frac{d}{dt}\left(i\omega e^{i\omega t}\right) = (i\omega)^2 x_0 e^{i\omega t}$

$= -x_0 \omega^2 e^{i\omega t}$

$= -\omega^2 x$

$\omega = \sqrt{\frac{k}{m}}$



$$\vec{X} = \begin{bmatrix} x \\ y \end{bmatrix}$$
$x \to$ position of mass 1
$y \to$ position of mass 2

first spring, last spring, middle spring

$$U = \frac{1}{2}kx^2 + \frac{1}{2}ky^2 + \frac{k}{2}(x-y)^2$$

→ This equation is in a "quadratic form"

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad U(x,y) = \vec{x}^T A \vec{x} \qquad A = \text{symmetric} = A^T$$

$$U(x,y) = [x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[x \ y] \begin{bmatrix} ax+by \\ bx+cy \end{bmatrix} = ax^2 + bxy + bxy + cy^2$$
$$= ax^2 + cy^2 + 2bxy$$

$$\cancel{A = \frac{1}{2}\begin{bmatrix} k & 2k \\ 2k & k \end{bmatrix}} \qquad \cancel{A = \frac{1}{2}\begin{bmatrix} k & k/2 \\ k/2 & k \end{bmatrix}}$$

$$\cancel{A = \frac{1}{2}\begin{bmatrix} k & 2k \\ 2k & k \end{bmatrix}}$$

$$U(x,y) = \frac{1}{2}kx^2 + \frac{k}{2}(x^2 - 2xy + y^2) + \frac{k}{2}y^2$$
$$= kx^2 - kxy + ky^2$$

$$A = \begin{bmatrix} k & -k/2 \\ -k/2 & k \end{bmatrix}$$

$$K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} = 2A$$

$$U(x,y) = \vec{x}^T A \vec{x} = \frac{1}{2}\vec{x}^T K \vec{x}$$

Recall $U = \frac{1}{2}kx^2 = \frac{1}{2}xkx$

$$u(x,y) = \vec{x}^T Q \vec{x} = \frac{1}{2}\vec{x}^T K \vec{x}$$

$$K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}, \quad Q = \begin{bmatrix} k & -k/2 \\ -k/2 & k \end{bmatrix}$$

$$F = -\nabla u = \frac{1}{2}\begin{bmatrix} -\frac{\partial}{\partial x}(\vec{x}^T K \vec{x}) \\ -\frac{\partial}{\partial y}(\vec{x}^T K \vec{x}) \end{bmatrix} = -\begin{bmatrix} 2kx - ky \\ 2ky - kx \end{bmatrix}$$

$$u(x,y) = kx^2 - kxy + ky^2 \qquad = -\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -K\vec{x}$$

$$\frac{\partial u}{\partial y} = 2ky - kx$$

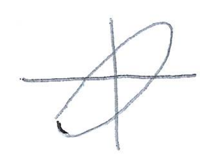
$$\ddot{x} = -\frac{2k}{m}x + \frac{k}{m}y, \quad \ddot{y} = \frac{kx}{m} - \frac{2ky}{m}$$

Aside

$r^2 = x^2 + y^2$

$r^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

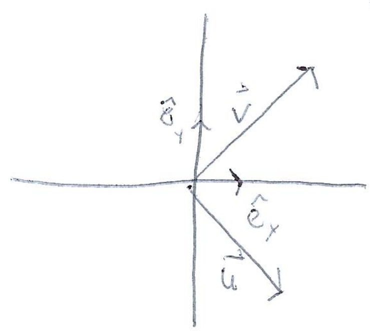
$r^2 = \frac{x^2}{a^2} + \frac{xy}{c} + \frac{y^2}{b^2}$



$$\vec{x}_e = x\hat{e}_x + y\hat{e}_y$$

use different basis

$$\hat{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \hat{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Lets define two variables

$$u = \frac{x-y}{\sqrt{2}}, \quad v = \frac{x+y}{\sqrt{2}}$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = u\hat{u} + v\hat{v} \qquad \frac{1}{\sqrt{2}}\frac{x-y}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{\sqrt{2}}\frac{x+y}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_{(u,v)} = \begin{bmatrix} u \\ v \end{bmatrix} \text{ (in } \hat{u}\text{-}\hat{v} \text{ basis)}$$

$$u = \vec{x}\cdot\hat{u} \quad v = \vec{x}\cdot\hat{v}$$

$$\ddot{u} = \frac{1}{\sqrt{2}}(\ddot{x} - \ddot{y}), \quad \ddot{v} = \frac{1}{\sqrt{2}}(\ddot{x} + \ddot{y})$$

$$\ddot{u} = \frac{1}{\sqrt{2}}\left(-\frac{2k}{m}x + \frac{k}{m}y - \frac{k}{m}x + \frac{2k}{m}y\right)$$

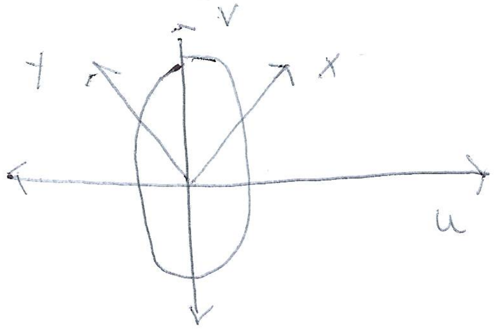
$$= \frac{1}{\sqrt{2}}\left(-\frac{3k}{m}x + \frac{3k}{m}y\right)$$

$$u = \frac{1}{\sqrt{2}} \frac{3k}{m}(-x+y) = \frac{3k}{m}(-1)\frac{1}{\sqrt{2}}(x-y) = \frac{-3k}{m} u$$

$$\ddot{u} = \frac{-3k}{m} u$$

$$\ddot{v} = \frac{-k}{m} v \qquad u = u_0 e^{i\omega_1 t} \qquad v = v_0 e^{i\omega_2 t}$$

$$U_{(u,v)} = \overbrace{\ldots}^{\frac{1}{2}kx^2 + \frac{1}{2}k(x-y)^2 + \frac{1}{2}ky^2} = 3ku^2 + kv^2 = [u\ v]\begin{bmatrix} 3k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$



$$\ddot{u} = -\frac{3k}{m} u, \quad \ddot{v} = -\frac{k}{m} v$$

$$\begin{bmatrix} \ddot{u} \\ \ddot{v} \end{bmatrix} = \frac{-1}{m}\begin{bmatrix} 3k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \rightarrow D = \begin{bmatrix} 3k & 0 \\ 0 & k \end{bmatrix}$$

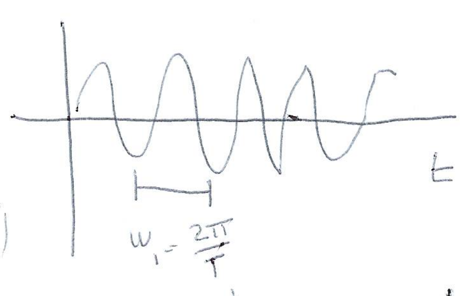
$$\omega_1 = \sqrt{\frac{3k}{m}}, \quad \omega_2 = \sqrt{\frac{k}{m}}$$

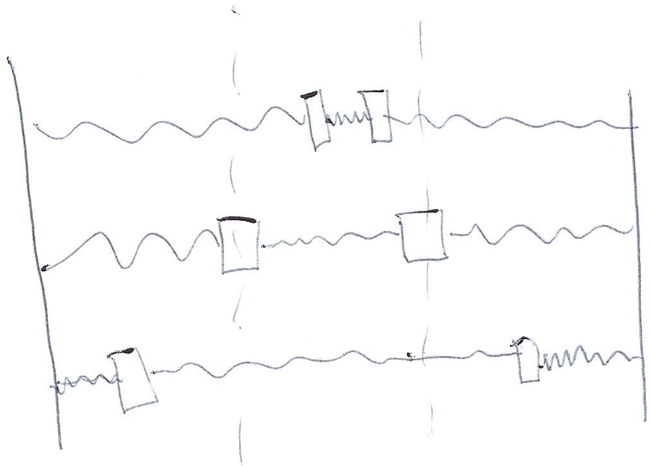
v = constant $= 0$

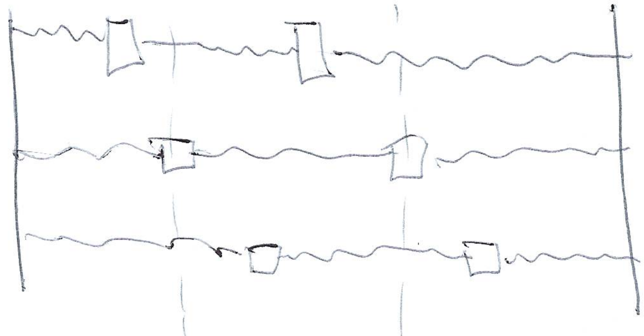
$$\frac{x+y}{\sqrt{2}} = 0, \quad x = -y$$

u = constant $= 0$

$$\frac{x-y}{\sqrt{2}} = 0, \quad x = +y$$







Phys 89 11 Mar 19

Today Eigen Values and Determinants

Eigen vectors and the Kernel

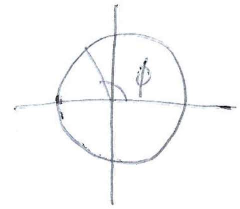
Some Trick

$$\det M^* = (\det M)^*$$

$$\det(U^\dagger) = \det((U^T)^*) = (\det(U^T))^* = (\det(U))^*$$

$$\det(UU^\dagger) = \det(U^\dagger U) = \det U (\det U)^* = 1$$

$$\det(U) = e^{i\phi}$$



$$K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

$$\hat{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \hat{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \frac{1}{2}\vec{x}^T K \vec{x} = U(x,y)$$

$$K\hat{u} = 3k\hat{u} \qquad K\hat{v} = k\hat{v} \qquad \vec{a}^T K a$$

$$\hat{u}^T K \hat{u} = 3k \qquad \hat{v}^T K \hat{v} = k$$

$$A\vec{v} = \lambda \vec{v}$$

$$A\vec{v} = \lambda\vec{v}$$

eigen-vector

eigenvalue

As many λ as distinct eigenvector

Set of $\{\lambda\}$ is called the spectrum

If several λ have the same value the spectrum is called degenerate

→ if two λ's same, λ has degeneracy 2

$$A\vec{v} = \lambda\vec{v} \qquad \mathbb{I}\vec{v} = \vec{v}$$

$$= \lambda\mathbb{I}\vec{v}$$

$$A\vec{v} - \lambda\mathbb{I}\vec{v} = \vec{0}$$

$$(A - \lambda\mathbb{I})\vec{v} = \vec{0} \qquad M\vec{v} = \vec{0} \qquad \det(M) = \det(A - \lambda\mathbb{I}) = 0$$

$$\lambda\mathbb{I} = \begin{bmatrix} \lambda & 0 & \cdots \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \\ \vdots & & & \ddots \end{bmatrix} \qquad A - \lambda\mathbb{I} = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn}-\lambda \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A - \lambda\mathbb{I} = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \qquad \det(A - \lambda\mathbb{I}) = (a-\lambda)(d-\lambda) - bc$$

expect n solutions

$$K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \qquad K - \lambda \mathbb{I} = \begin{bmatrix} 2k-\lambda & -k \\ -k & 2k-\lambda \end{bmatrix}$$

$$\det(K - \lambda\mathbb{I}) = (2k-\lambda)^2 - k^2 = \lambda^2 - 4k\lambda + 3k^2 = (3k-\lambda)(k-\lambda)$$

$\cancel{4k \pm \sqrt{16k^2 -}}$

$\lambda = 3k, k$

The roots are the spectrum

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\det(R_\theta - \lambda\mathbb{I}) = (\cos\theta - \lambda)^2 + \sin^2\theta$$

$$\lambda^2 - 2\lambda\cos\theta + 1 \quad \ldots \rightarrow \quad \lambda = e^{\pm i\theta}$$

only $\theta = \pi n$ $n \in \mathbb{Z}$
give real solutions

Set of eigen vectors

$(A - \lambda\mathbb{I})\vec{v} = 0$

Kernel $(A - \lambda\mathbb{I})$ is the set of eigenvectors

$\lambda = 3k, k \qquad (K - 3k\mathbb{I})\vec{v} = \vec{0}$

$$\begin{bmatrix} 2k-3k & -k \\ -k & 2k-3k \end{bmatrix} \vec{v} = \vec{0} \qquad \left[\begin{array}{cc|c} -k & -k & 0 \\ -k & -k & 0 \end{array}\right]$$

$R_2 - R_1$

$$\left[\begin{array}{cc|c} -k & -k & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1/-k} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$v_1 = -v_2 \qquad \vec{v} = \begin{bmatrix} +1 \\ -1 \end{bmatrix} t$$

$$\hat{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} t$$

$\lambda = k$

$$(A - k\mathbb{I}) = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{cc|c} k & -k & 0 \\ -k & k & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Degeneracy of λ given by $\text{null}(A - \lambda\mathbb{I})$
(dimension of kernel)

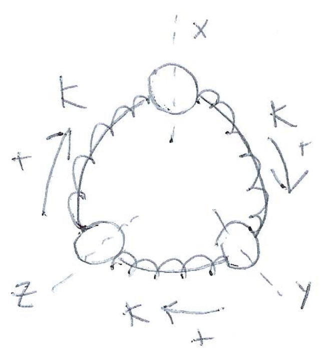
Defective Matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$$

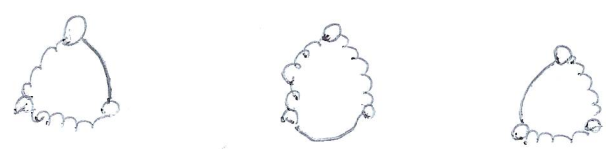
$\lambda = 1$ degeneracy 2

$\text{null}(A - \lambda\mathbb{I}) = 1$

~~Degeneracy~~ Degeneracies tell about the symmetry of the system



$$U(x,y,z) = \frac{1}{2}k(x-y)^2 + \frac{1}{2}k(y-z)^2 + \frac{1}{2}k(z-x)^2$$



$$U(x,y,z) = \frac{1}{2}k(x^2 - 2yx + y^2 + y^2 - 2yz + z^2 + z^2 - 2xz + \cancel{x}^2)$$

$$= \frac{1}{2}k(2x^2 - 2yx + 2y^2 - 2yz + 2z^2 - 2xz)$$

$$= k(\cancel{x^2} - xy + y^2 - yz + z^2 - xz)$$

$U = \frac{1}{2}\vec{x}^T K \vec{x}$

$$K = \begin{bmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{bmatrix}$$

$$U(x,y,z) = \frac{1}{2}\left(2kx^2 + 2ky^2 + 2kz^2 - 2kxy - 2kxz - 2kyz\right)$$

$$K = \begin{bmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{bmatrix}$$

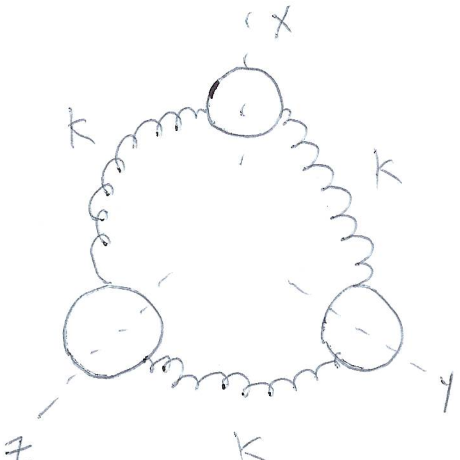
Eigenvalues

$$\det(K - \lambda I) = \begin{vmatrix} 2k-\lambda & -k & -k \\ -k & 2k-\lambda & -k \\ -k & -k & 2k-\lambda \end{vmatrix}$$

Every row sums to same number

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ must be an eigenvector

Phys 89 13Mar19



$$U(x,y,z) = \frac{1}{2}\vec{x}^T K \vec{x}$$

$$= \frac{1}{2}[x\ y\ z]\begin{bmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \frac{1}{2}k(x-y)^2 + \frac{k}{2}(y-z)^2 + \frac{k}{2}(x-z)^2$$

First Eigenvector

(Noting all rows sum to the same value)

$$K = \begin{bmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t \text{ or normalize } \hat{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \lambda_1 = 0$$

$$\det(\underbrace{K - \lambda I\!I}_{M \text{ is singular}}) = 0 \qquad \begin{bmatrix} 2k-\lambda & -k & -k \\ -k & 2k-\lambda & -k \\ -k & -k & 2k-\lambda \end{bmatrix} \qquad \lambda = 3k$$

$$K - \lambda I\!I = \begin{bmatrix} -k & -k & -k \\ -k & -k & -k \\ -k & -k & -k \end{bmatrix}$$

Find eigenvector for $\lambda = 3k$

$$-k\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\vec{v}_2 = \vec{0} \qquad \left[\begin{array}{ccc|c} -k & -k & -k & 0 \\ -k & -k & -k & 0 \\ -k & -k & -k & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \begin{bmatrix} r \\ s \\ -r-s \end{bmatrix} = r\underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{\vec{v}_2} + s\underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{\vec{v}_3}$$

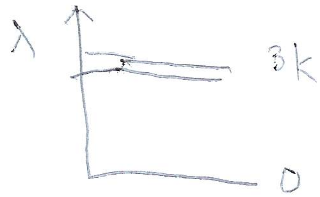
$x + y + z = 0$

$x = r \qquad z = -r-s$

$y = s$

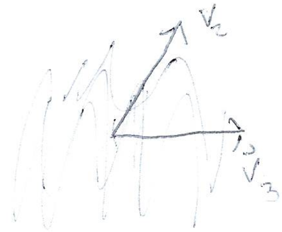
$$\begin{bmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3k \\ -3k \end{bmatrix} = 3k\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 3k\hat{v}_3$$

$\{\lambda\} = 3k, 3k, 0$



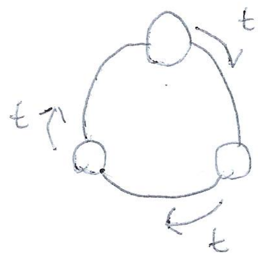
$\lambda = 3k$ has degeneracy 2

$\text{null}(K - 3k\mathbb{I}) = 2$



Any linear superposition of $\vec{v}_2$ and $\vec{v}_3$ is an eigenvector of K eigenvalue $3k$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t$$

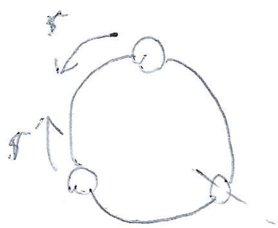


$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

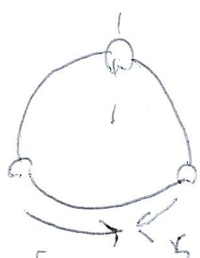
not eigenvector (distinct)

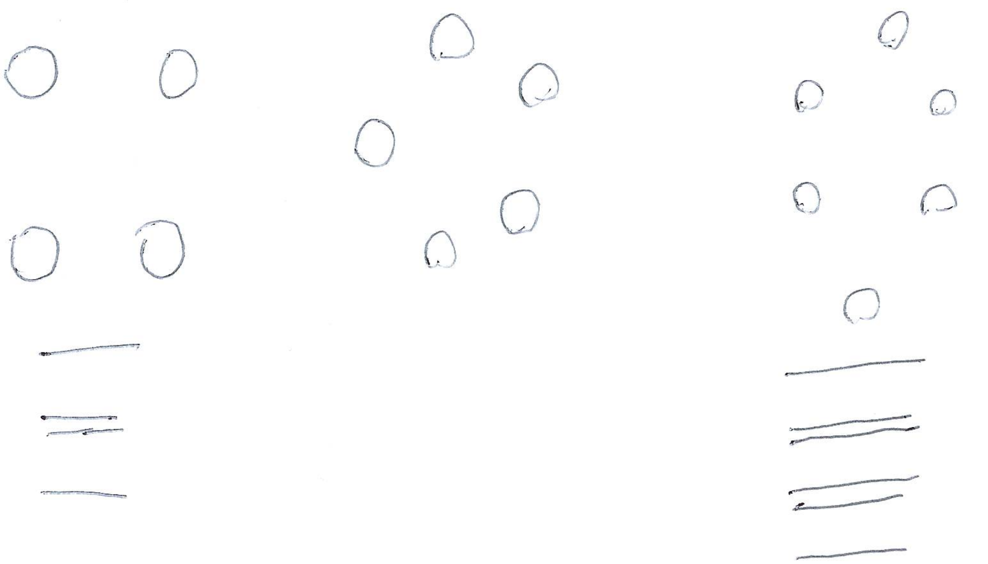
$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{V}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

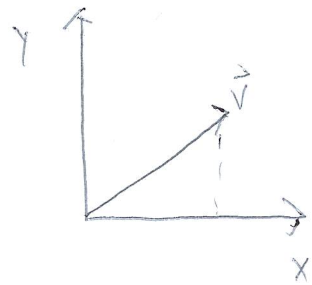


$$\vec{V}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$





Projections



$$P_x \vec{X} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}X\\ Y\end{bmatrix} = \begin{bmatrix}X\\ 0\end{bmatrix}$$

$$\vec{X} = \begin{bmatrix}x\\ y\end{bmatrix}$$

$$\vec{X}^T\vec{X}$$

$$\vec{X}\vec{X}^T = \begin{bmatrix}X\\ Y\end{bmatrix}[x\ y] = \begin{bmatrix}x^2 & xy\\ xy & y^2\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} = \vec{w}\vec{w}^T = \begin{bmatrix}1\\ 0\end{bmatrix}[1\ 0] = \hat{X}\hat{X}^T = \hat{e}_1\hat{e}_1^T$$

$$P_x$$

$$\cancel{\vec{x}}\ \vec{d} = \begin{bmatrix}x\\ y\end{bmatrix} = x\hat{e}_1 + y\hat{e}_2 \qquad \hat{e}_1\hat{e}_1^T\vec{d} = \hat{e}_1\hat{e}_1^T x\hat{e}_1 + \hat{e}_1\hat{e}_1^T y\hat{e}_2$$

$$= x\hat{e}_1(\underbrace{\hat{e}_1^T\hat{e}_1}_{1}) + y\hat{e}_1(\underbrace{\hat{e}_1^T\hat{e}_2}_{0})$$

$$= x\hat{e}_1$$

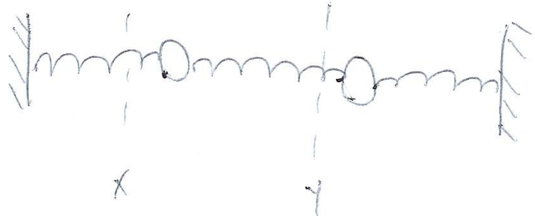
$P_{\hat w} = \hat w \hat w^T$ } Projection onto $\vec w$

$P_{\hat y} = \hat y \hat y^T = \hat e_2 \hat e_2^T$, $P_{\hat z} = \hat z \hat z^T = \hat e_3 \hat e_3^T$

$P_{\hat w}^2 = P_{\hat w}$ $\quad P_{\hat w} \vec v = \lambda \vec v = P_{\hat w}^2 \vec v = P_w (P_w \vec v) = \lambda P_w \vec v = \lambda^2 \vec v$

$\lambda = \lambda^2 \quad \lambda = \cancel{\pm 1}, 1, 0$

$\vec d = \begin{bmatrix} x \\ y \end{bmatrix}$



$\vec d = x \hat e_1 + y \hat e_2 \quad \cancel{\frac{1}{2} \vec d \vec d^T}$

$\frac{1}{2} \vec d^T K \vec d = \cancel{2kx^2} + \frac{1}{2} kx^2 + \frac{1}{2} ky^2 - \frac{1}{2} k (x-y)^2$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} K_{11} & 0 \\ 0 & 0 \end{bmatrix}$

$K = K_{11} \hat e_1 \hat e_1^T + K_{22} \hat e_2 \hat e_2^T$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} 0 & K_{12} \\ K_{21} & 0 \end{bmatrix} \hat e_1 \quad \hat e_1 \hat e^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\frac{1}{2}\vec{d}^T K \vec{d} = \frac{1}{2}kx^2 + \frac{1}{2}ky^2 - \frac{1}{2}k(x-y)^2$$

$$\hat{w}^T \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \hat{w} = k_{11}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\hat{e}_1} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\hat{e}_1} \begin{bmatrix} k_{11} \\ k_{21} \end{bmatrix} = k_{11}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\hat{e}_1} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\hat{e}_2} \begin{bmatrix} k_{12} \\ k_{22} \end{bmatrix} = k_{12}$$

$$K = \begin{bmatrix} \hat{e}_1^T K \hat{e}_1 & \hat{e}_1^T K \hat{e}_2 \\ \hat{e}_2^T K \hat{e}_1 & \hat{e}_2^T K \hat{e}_2 \end{bmatrix} \qquad K = K_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + K_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + K_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + K_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \hat{e}_1\hat{e}_1^T \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \hat{e}_1\hat{e}_2^T \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \hat{e}_2\hat{e}_1^T \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \hat{e}_2\hat{e}_2^T$$

$$K = K_{11}\hat{e}_1\hat{e}_1^T + K_{12}\hat{e}_1\hat{e}_2^T + K_{21}\hat{e}_2\hat{e}_1^T + K_{22}\hat{e}_2\hat{e}_2^T$$

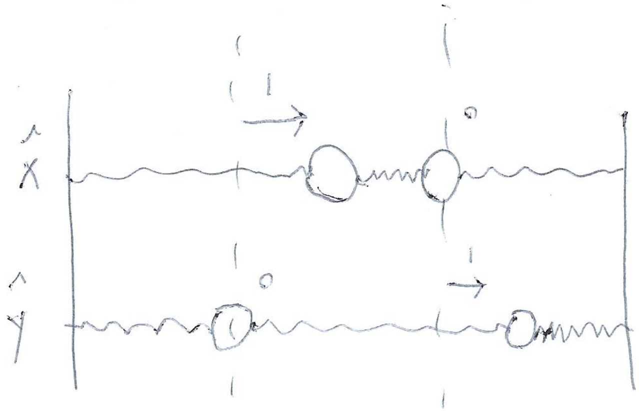
$$\hat{x} = \hat{e}_1, \quad \hat{y} = \hat{e}_2$$

$$K = k_{11}\hat{x}\hat{x}^T + k_{12}\hat{x}\hat{y}^T + K_{21}\hat{y}\hat{x}^T + K_{22}\hat{y}\hat{y}^T$$

$$kx^2 + ky^2 - \frac{1}{2}k(yx) - \frac{1}{2}kxy$$

Phys 89 15 Mar 19

$$\vec{d}_{(x,y)} = \begin{bmatrix} x \\ y \end{bmatrix}$$



Now $\vec{d}^T \hat{x}\hat{x}^T \vec{d} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = x^2$

$$\vec{d}^T \hat{x}\hat{y}^T \vec{d} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} = xy$$

$$\vec{d}^T \hat{y}\hat{x}^T \vec{d} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} = yx$$

$$\vec{d}^T \hat{y}\hat{y}^T \vec{d} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = y^2$$

$$U(x,y) = \frac{1}{2}(2kx^2 - kxy - kyx - 2ky^2)$$

$$= \frac{1}{2}\vec{d}^T(2k\hat{x}\hat{x}^T - k\hat{x}\hat{y}^T - k\hat{y}\hat{x}^T - 2k\hat{y}\hat{y}^T)\vec{d}$$

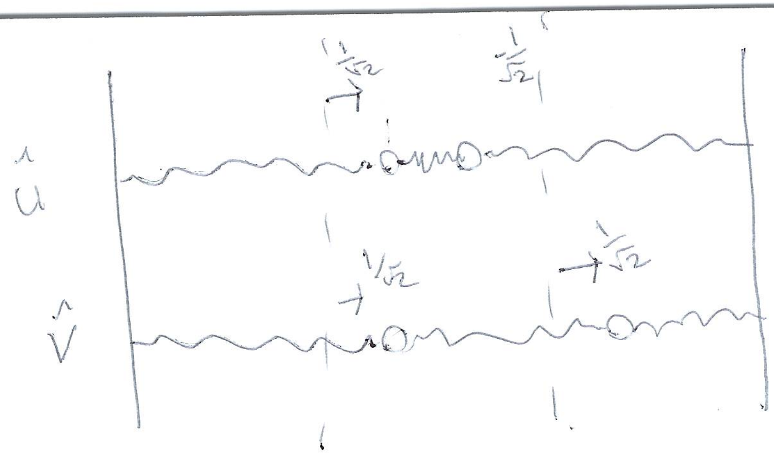
$K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$ $\quad \hat{x}^T K \hat{x} = k_{11} \quad \hat{x}^T K \hat{y} = k_{12}$

$$\hat{x}^T K \hat{y} = \hat{x}^T(2k\hat{x}\hat{x}^T - k\hat{x}\hat{y}^T - k\hat{y}\hat{x}^T + 2k\hat{y}\hat{y}^T)\hat{y}$$

$$= 2k(\hat{x}^T\hat{x})(\hat{x}^T\hat{y}) - k(\hat{x}^T\hat{x})(\hat{y}^T\hat{y}) - k(\hat{x}^T\hat{y})(\hat{x}^T\hat{y}) + 2k(\hat{x}^T\hat{y})(\hat{y}^T\hat{y})$$

$$= 0 \quad -k + 0 \quad + 0$$

$$= -k$$



$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\vec{d} = u\hat{u} + v\hat{v} = \begin{bmatrix}u\\v\end{bmatrix}$$

$$K_{[uv]} = K_{11(uv)}\hat{u}\hat{u}^T + K_{12(uv)}\hat{u}\hat{v}^T + K_{21(uv)}\hat{v}\hat{u}^T + K_{22(uv)}\hat{v}\hat{v}^T$$

$$= \begin{bmatrix}K_{uu} & K_{uv}\\ K_{vu} & K_{vv}\end{bmatrix}$$

$$K_{uu} = \hat{u}^T K\hat{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}2k & -k\\ -k & 2k\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}3k\\-3k\end{bmatrix} = \frac{1}{2}6k = 3k$$

$$K_{uv} = \hat{u}^T K\hat{v} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}2k & -k\\ -k & 2k\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}k\\k\end{bmatrix} = 0$$

This always works in converting basis as long as you are consistent on the right side basis

$$K_{vu} = \hat{v}^T K\hat{u} = \dots = 0$$

$$K_{vv} = \hat{v}^T K\hat{v} = \dots = K$$

$$K_{[uv]} = \begin{bmatrix}3k & 0\\ 0 & k\end{bmatrix}$$

$$K_{uu} = \hat{u}^T K \hat{u} \qquad \hat{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \hat{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K_{uu} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 3k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3k$$

$$\vec{d}_{(x,y)} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \vec{d}_{(u,v)} = \begin{bmatrix} u \\ v \end{bmatrix} \qquad u = \vec{d} \cdot \hat{u} \quad v = \vec{d} \cdot \hat{v}$$

$$u = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} \qquad v = \frac{1}{\sqrt{2}} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{d}_{(u,v)} = \frac{1}{\sqrt{2}} \begin{bmatrix} x-y \\ x+y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \qquad u = \frac{1}{\sqrt{2}}(x-y) \qquad v = \frac{1}{\sqrt{2}}(x+y)$$

$$K_{(uv)} = \begin{bmatrix} \hat{u} \cdot (K\hat{u}) & \hat{u} \cdot (K\hat{v}) \\ \hat{v} \cdot (K\hat{u}) & \hat{v} \cdot (K\hat{v}) \end{bmatrix}$$

$$\vec{d}_{(u,v)} = \begin{bmatrix} & \\ & \end{bmatrix} \vec{d}_{(x,y)} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x-y \\ x+y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$C^T = C^{-1}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{I}$$

$$\vec{d}_{(xy)} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{d}_{(u,v)}$$

$$\vec{d}_{(xy)} = \begin{bmatrix} \hat{u}_{(x,y)} & \hat{v}_{(x,y)} \end{bmatrix} \vec{d}_{(uv)}$$

$$F = -\frac{du}{dx} \qquad U_{(x,y)} = \frac{1}{2}(2kx^2 - 2kxy + 2ky^2)$$

$$\frac{du}{dx} = 2kx - ky$$

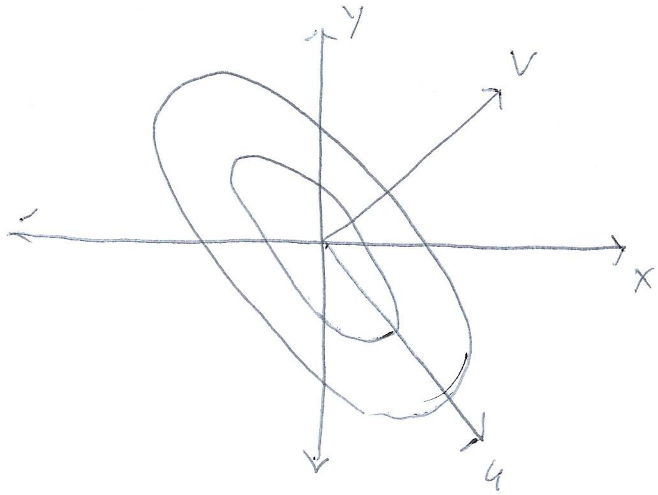
$$\frac{du}{dy} = 2ky - kx$$

$$F = \begin{bmatrix} ky - 2kx \\ kx - 2ky \end{bmatrix}$$

$$U(u,v) = \frac{1}{2} \vec{d}^{\,T}_{[u,v]} \begin{bmatrix} 3k & 0 \\ 0 & k \end{bmatrix} \vec{d}_{[u,v]} = \frac{1}{2}(3ku^2 + kv^2)$$

$$\frac{du}{du} = \frac{3}{2}k \qquad \frac{du}{dv} = \frac{k}{2}$$

$$\vec{F} = \begin{bmatrix} -\frac{3}{2}k \\ -\frac{k}{2} \end{bmatrix}$$



"Generalized Force"

$\lambda, \vec{v} \qquad M$

M^T has same λ

$$\det(M - \lambda I) = \det(M^T - \lambda I)$$

$$\det(A^T) = \det(A)$$

$$\det((M - \lambda I)^T) = \det(M - \lambda I)$$

$$= \det(M^T - \lambda I^T) = \det(M^T - \lambda I)$$

M^{-1} has eigenvalues $\frac{1}{\lambda}$ for same $\vec{v}$

$$\vec{v} = M^{-1}M\vec{v} = \lambda M^{-1}\vec{v}$$

$$M^{-1}v = \frac{1}{\lambda}\vec{v}$$

M^n has eigen values $\lambda^n, \vec{v}$

$$M^n \vec{v} = M^{n-1} \lambda \vec{v} = M^{n-2} \lambda^2 \vec{v} = \dots = \lambda^n \vec{v}$$

A hermitian matrix always has real eigenvalues

A hermitian matrix has orthogonal eigenvectors if those eigenvectors correspond to a distinct eigen value

Phys 89 01 Apr 19

Exact Differentials
Implicit Solutions
Ch 1.4 Kreyszig

$$M(y,t)\,\underbrace{\dot{y}}_{\frac{dy}{dt}} + N(y,t) = 0$$

Explicit

$y(t) = f(t)$

Implicit

$f(y,t) = C$

eg $\underbrace{\sin(y+t) + y^2 t}_{u} = C$

$$M(y,t)\,dy + N(y,t)\,dt = 0$$

$u(y,t)$

$$du = \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial t}dt \qquad \frac{du}{dt} = \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial t}$$

when $\int du$ = Constant or $du = 0 = \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial t}dt$ or $u(y,t)$ = constant

Task: To find implicit solution for y u is implicit solution for y ie $u(y,t)$ = constant

By definition

$$M(y,t) = \frac{\partial u}{\partial y} \qquad N(y,t) = \frac{\partial u}{\partial t}$$

$$f(t) + \int M(y,t)\,dy = u(y,t) \qquad u(y,t) = \int N(y,t)\,dt + g(y)$$

Find u that satisfies both is first task eg $\underbrace{\left(3y^2+2y+\cos(y+t)\right)}_{M(y,t)}\dot{y} + \underbrace{\cos(y+t)}_{N(y,t)} = 0$

$M = \frac{\partial u}{\partial y} \quad N = \frac{\partial u}{\partial t}$

$$\frac{\partial M}{\partial t} = \frac{\partial^2 u}{\partial t \partial y} = \frac{\partial^2 u}{\partial y \partial t} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial N}{\partial y}$$

If u exists $\boxed{\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y}}$

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\left(3y^2+2y+\cos(y+t)\right) = \frac{\partial N}{\partial y} = \frac{\partial}{\partial y}\left(\cos(y+t)\right)$$

$$= -\sin(y+t) = -\sin(y+t) \checkmark$$

$u(y,t)$. $\quad N = \frac{\partial u}{\partial t} = \cos(y+t)$

$$u(y,t) = \int \cos(y+t)\,dt = \sin(y+t) + f(y)$$

$\frac{\partial u}{\partial y} = M(y,t) \quad \frac{\partial}{\partial y}(\sin(y+t)) + \frac{\partial}{\partial y}(f(y)) = 3y^2+2y+\cos(y+t)$

$$3y^2+2y = \frac{\partial f}{\partial y} \qquad = \cos(y+t) + \frac{\partial f}{\partial y}$$

$$f(y) = y^3 + y^2 + C$$

so $u(y,t) = \sin(y+t) + y^3 + y^2 + C = \text{constant}$

$$\boxed{u(y,t) = \sin(y+t) + y^3 + y^2 = \text{constant}}$$

Implicit Solution

Sometimes M and N do not ~~tra~~ satisfy our required relationship $\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y}$

Try multiplying by a fuction of t which is called the <u>integrating factor</u>.

$$\alpha(t) M(y,t)\,dy + \alpha(t) N(y,t)\,dt = 0\,\alpha(t) = 0$$

$\alpha(t)$ = integrating factor

$$\tilde{M} = \alpha(t) M(y,t) \qquad \tilde{N} = \alpha(t) N(y,t)$$

$$\boxed{\frac{\partial \tilde{M}}{\partial t} = \frac{\partial \tilde{N}}{\partial y}}$$

Example $\quad t\dot{y} - y = 0$

$$M = t \quad , N = -y$$

$$\frac{\partial M}{\partial t} = 1 \neq \frac{\partial N}{\partial y} = -1$$

Try $\underbrace{\alpha(t) t\dot{y}}_{\tilde{M}} - \underbrace{\alpha(t) y}_{\tilde{N}} = 0$

$$\frac{\partial \tilde{M}}{\partial t} = \alpha(t) + \dot{\alpha} t \qquad \frac{\partial \tilde{N}}{\partial y} = -\alpha(t)$$

$$\dot{\alpha} t + \alpha = -\alpha$$

$$\dot{\alpha} t = -2\alpha \qquad \frac{d\alpha}{\alpha} = -2\frac{dt}{t}$$

$$\ln \alpha = -2 \ln t + c$$

$$\alpha = C t^{-2}$$

Any constant will satisfy this condition

Let's choose $C = 1$ $\qquad \alpha = \frac{1}{t^2}$

$$\alpha M\dot{y} + \alpha N = 0$$

$$\frac{1}{t^2} t\dot{y} + \frac{1}{t^2} y = 0 \qquad \frac{\dot{y}}{t} + \frac{y}{t^2} = 0$$

$$\tilde{M} = \frac{1}{t}, \quad \tilde{N} = \frac{-y}{t^2}$$

$$\frac{\partial \tilde{M}}{\partial t} = \frac{-1}{t^2} \qquad \frac{\partial \tilde{N}}{\partial y} = \frac{-1}{t^2} = \frac{\partial \tilde{N}}{\partial y}$$

$$\tilde{M} = \frac{1}{t} = \frac{\partial u}{\partial y}$$

$$\boxed{u = \frac{y}{t} + f(t)}$$

$$\frac{\partial u}{\partial t} = \frac{-y}{t^2} + \frac{\partial f}{\partial t} = \frac{-y}{t^2}$$

$$\frac{\partial f}{\partial t} = 0 \Rightarrow f(t) = \text{constant}$$

$$u(y,t) = \frac{y}{t} + C = \text{constant}$$

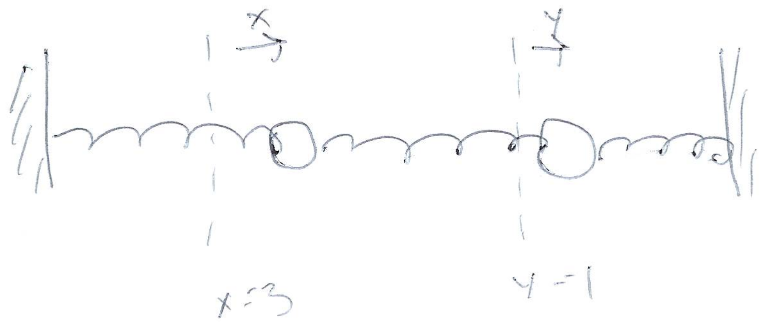
$$\boxed{u(y,t) = \frac{y}{t} = \text{constant}}$$

$$\frac{y}{t} = \text{constant} \Rightarrow y = Ct$$

Phys 89 18 Mar 19

Today: Change of basis

Similarity transformations



$d_{[xy]} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{d} = 3\hat{x} + 1\hat{y}$

What about in $[uv]$ basis $\hat{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\hat{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

basis $\hat{u}, \hat{v}$ represented in $\hat{x}, \hat{y}$

$\vec{d} = \alpha\hat{u} + \beta\hat{v}$

$\vec{d}_{[xy]} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \alpha\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\alpha = \sqrt{2}$, $\beta = 2\sqrt{2}$

$\vec{d}_{[uv]} = \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$ one basis $\{\hat{e}_i\}$ $\{\hat{f}_i\}$ span the same vector space

Suppose C exists such that $C\hat{e}_i = \hat{f}_i$

$C_{ij} = \hat{e}_i \cdot C\hat{e}_j = \hat{e}_i^{\dagger} C\hat{e}_j = \hat{e}_i \cdot \hat{f}_j = \hat{e}_i^{T}\hat{f}_j$

$\hat{f}_i = \sum_i [f_j]_i \hat{e}_i$

This is the ith component of $\hat{f}_j$ in the $\{\hat{e}_i\}$ basis

$\hat{e}_i\hat{f}_j = \hat{e}_i^{T}\sum_i [f_j]_{i[e]}\hat{e}_i = [\hat{f}_j]_{i[e]}\hat{e}_i^{T}\hat{e}_i$

$= [\hat{f}_j]_{i[e]}$

projection of $\hat{f}_j$ on $\hat{e}_i$ (ie in $\hat{e}_i$ basis)

$$C = \begin{bmatrix} \hat{e}_1 \cdot \hat{f}_1 & \hat{e}_1 \cdot \hat{f}_2 & \hat{e}_1 \cdot \hat{f}_3 & \cdots \\ \hat{e}_2 \cdot \hat{f}_1 & \hat{e}_2 \cdot \hat{f}_2 & \cdots & \\ \hat{e}_3 \cdot \hat{f}_1 & \vdots & \ddots & \\ \vdots & & & \hat{e}_n \cdot \hat{f}_n \end{bmatrix}$$

has to be square since they span the same vector space

$$C = \begin{bmatrix} f_{11[e]} & f_{12[e]} & \cdots \\ f_{21[e]} & f_{22[e]} & \\ f_{31[e]} & & \ddots \\ \vdots & & \end{bmatrix} = \begin{bmatrix} \hat{f}_{1[e]} & \hat{f}_{2[e]} & \cdots & \hat{f}_{n[e]} \end{bmatrix}$$

$$\hat{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \hat{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad C\hat{e}_i = \hat{f}_i \qquad \hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \hat{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad C\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\{\hat{e}_i\}$ and $\{\hat{f}_i\}$ are orthonormal basis vectors

C is by definition unitary

$$C^{-1} = C^{\dagger} \qquad CC^{\dagger} = C^{\dagger}C = \mathbb{I}$$

$$\cancel{CC^\dagger = \begin{bmatrix} \hat{f}_{1(x)} & \hat{f}_{2(x)} & \cdots \end{bmatrix}}$$

$$\cancel{CC^\dagger}$$

$$C^\dagger C = \begin{bmatrix} \hat{f}^\dagger_{1(x)} \\ \hat{f}^\dagger_{2(x)} \\ \vdots \end{bmatrix} \begin{bmatrix} \hat{f}_{1(x)} & \hat{f}_{2(x)} & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \hat{f}^\dagger_1 \hat{f}_1 & \hat{f}^\dagger_1 \hat{f}_2 & \cdots \\ \hat{f}^\dagger_2 \hat{f}_1 & \ddots & \\ \vdots & & \ddots \end{bmatrix} = \mathbb{I}$$

eg $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = C \qquad C^{-1} = C^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

As an aside $\mathbb{I} = \sum_i \hat{e}_i \hat{e}_i^\dagger = \hat{e}_1\hat{e}_1^\dagger + \hat{e}_2\hat{e}_2^\dagger + \cdots = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cdots$

eg $\hat{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \hat{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\sum_i \hat{f}_i \hat{f}_i^\dagger = \hat{u}\hat{u}^\dagger + \hat{v}\hat{v}^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\vec{d} = x\hat{x} + y\hat{y} = \alpha\hat{u} + \beta\hat{v}$$

$$= x\hat{e}_1 + y\hat{e}_2 = \alpha\hat{f}_1 + \beta\hat{f}_2 = \alpha C\hat{e}_1 + \beta C\hat{e}_2$$

want α, β

$$\alpha C\hat{e}_1 + \beta C\hat{e}_2$$

To get $\alpha = \hat{e}_1^{\dagger}\left(C^{-1}(\alpha C\hat{e}_1 + \beta C\hat{e}_2)\right)$

$$= \hat{e}^{\dagger}(\alpha\hat{e}_1 + \beta\hat{e}_2)$$

$$= \alpha\hat{e}_1^{\dagger}e_1 + \beta\hat{e}_1^{\dagger}\hat{e}_2 = \alpha + 0 = \alpha$$

$\vec{d}_{(uv)} = C^{-1}\vec{d}_{(xy)}$ $\qquad \hat{f}_i = C\hat{e}_i$

$\vec{d}_{(xy)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\qquad C = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\qquad C^{-1} = C^{\dagger} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\vec{d}_{(uv)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix} = d_{(uv)}$$

$\vec{d}_{[f]} = C^{-1} d_{[e]}$ $\qquad C = \begin{bmatrix} \hat{f}_{1(e)} & \hat{f}_{2(e)} & \cdots \end{bmatrix}$

What about Matrices?

$M\vec{v}$ $\qquad \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad M = R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$M_{[e]}$ $M_{[f]}$

perform same transformation on some vector $\vec{v}$

$$M_{[e]}\vec{v}_{[e]} = \vec{w}_{[e]} \qquad M_{[f]}\vec{v}_{[f]} = \vec{w}_{[f]}$$

$$\vec{w}_{[f]} = C^{-1}\vec{w}_{[e]} \qquad \vec{v}_{[f]} = C^{-1}\vec{v}_{[e]}$$

$$C^{-1}\underbrace{M_{[e]}\vec{v}_{[e]}}_{\vec{w}_{[e]}} = M_{[f]}\vec{v}_{[f]} = \vec{w}_{[f]}$$

$$C^{-1}M_{(e)}\mathbb{I}\vec{v}_{(e)} = C^{-1}M_{(e)}CC^{-1}\vec{v}_{(e)}$$

$$C^{-1}M_{(e)}C\underbrace{C^{-1}\vec{v}_{(e)}}_{\vec{v}_{(f)}} = M_{(f)}\vec{v}_{(f)} \qquad \boxed{C^{-1}M_{(e)}C = M_{(f)}}$$

Similarity transformation

$\tilde{A} = L^{-1}AL$ $\quad\tilde{A}$ and A are called "similar"

$$\det(\tilde{A}) = \cancel{\det(A)}\det(L^{-1})\det(A)\det(L) = \det A$$

$\det(\tilde{A} - \lambda\mathbb{I}) = \det(A - \lambda\mathbb{I}) \Rightarrow \tilde{A}$ and A have same set of eigenvalues

$Tr(ABC) = Tr(ACB) = Tr(BAC)$ any combination of A, B, C

$$Tr(A) = \sum_i A_{ii} \qquad Tr(AB) = \sum_i\sum_k A_{ik}B_{ki} = \sum_k\sum_i A_{ki}B_{ik} = Tr(BA)$$

So $Tr(\tilde{A}) = Tr(L^{-1}AL) = Tr(L^{-1}LA) = Tr(A)$

Phys 89 20 Mar 19

Finish up change of basis

introduction to differential equations Kreyszig ch 1-3

$$\tilde{M} = C^{-1} M C$$

if C is orthogonal or unitary

$$\tilde{M} = C^{\dagger} M C$$

eg $\cancel{M} = K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$ $\quad \hat{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \hat{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$C = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\tilde{M} = \frac{1}{2} \underbrace{\begin{bmatrix} 1 & -1 \\ +1 & 1 \end{bmatrix}}_{C^{\dagger}} \underbrace{\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{C}$$

$$= \dots \begin{bmatrix} 3k & 0 \\ 0 & k \end{bmatrix} \leftarrow \text{eigenvalues}$$

way of diagonalizing K

$$f(D) = \sum_n a_n D^n = a_0 + a_1 D + a_2 D^2 + \dots$$

$$\begin{bmatrix} 3k & 0 \\ 0 & k \end{bmatrix}^n = \begin{bmatrix} (3k)^n & 0 \\ 0 & k^n \end{bmatrix}$$

$AB = \sum_k A_{ik} B_{kj}$ $\quad (A^2)_{ij} = \sum_k A_{ik} A_{ki}$ $\quad k \neq i$ all entries 0

$$(A^2)_{ij} = \cancel{\sum_i} A_{ij}^2 \, \delta_{ij}$$

$(A^2)_{ii} = A_{ii}^2$ only diagonals survive

$\tilde{M}$ eigenvectors are not the same

$D = \begin{bmatrix} 3k & 0 \\ 0 & k \end{bmatrix}$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$M\vec{v} = \lambda\vec{v} = \lambda C C^{-1}\vec{v}$

$\tilde{M} = C^{-1}MC$ $\quad M = C\tilde{M}C^{-1}$

$C\tilde{M} = MC$ $\quad M\vec{v} = \lambda\vec{v}$

$C\tilde{M}C^{-1} = M$ $\quad C\tilde{M}C^{-1}\vec{v} = \lambda\vec{v}$

$C^{-1}C\tilde{M}C^{-1}\vec{v} = \lambda C^{-1}\vec{v}$

$\tilde{M}\underbrace{C^{-1}\vec{v}}_{\vec{w}} = \lambda \underbrace{C^{-1}\vec{v}}_{\vec{w}}$

$\vec{w}$ is eigenvector of $\tilde{M}$ with eigenvalue λ

$\vec{w} = C^{-1}\vec{v}$

$C^{-1}\hat{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}} = \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ eigenvector for a diagonal matrix

$D = \begin{bmatrix} 3k & 0 \\ 0 & k \end{bmatrix}$

$D\vec{w} = \begin{bmatrix} 3k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = k\begin{bmatrix} 0 \\ 1 \end{bmatrix} = k\vec{w}$

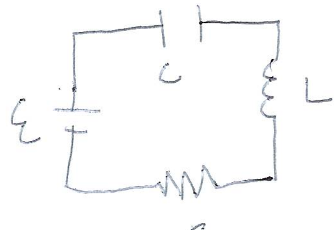
Normal Matrices can always be diagonalized by a unitary transformation

$$D = U^{\dagger} A U \quad \text{or if all real} \quad D = C^{T} A C$$

↑ matrix constructed by eigenvectors

Since A is diagonalizable by U and U is a matrix with orthonormal vectors those vectors must be the eigenvectors and span the full space and are orthonormal

Differential Equations



$$V_0 = \frac{Q}{C} + \frac{L\,dI}{dt} + IR$$

$$= \frac{Q}{C} + \frac{L\,d^2Q}{dt^2} + R\frac{dQ}{dt} \Big\} \text{ Second order DE since it has highest order of 2 in derivative}$$

Looking for $Q(t)$ $\quad \frac{dQ}{dt} = \dot{Q} \qquad \frac{dQ}{dx} = Q'$

$$\frac{\partial^2 Q}{\partial x^2} = \gamma \frac{\partial Q}{\partial t}$$ Second order in x, first order in t

Is whether the DE is linear or not

$\frac{d}{dx} \sim D$ (differential operator)

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

$D = \frac{d}{dt}$

$\left(LD^2 + RD + \frac{1}{C}\right)Q = V$

D is like an operator

$A = LD^2 + RD + \frac{1}{C}$

$AQ = V$ $\qquad$ $A\vec{x} = \vec{b}$

Q becomes $\vec{x}$ $\qquad$ V becomes $\vec{b}$

Now the "vector space" describes a function
eg $Q(t)$

$\dot{y}^2 + y = 3$ $\qquad$ $\left(\frac{dy}{dt}\right)^2 + y = 3$

Can we construct $Ay = 3$ $\qquad$ Nope!

$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$

y_1 and y_2 consider $y = y_1 + y_2$

$\left(\frac{d}{dt}(y_1 + y_2)\right)^2 + y_1 + y_2 = 3 \neq \left(\frac{dy_1}{dt}\right)^2 + y_1 + \left(\frac{dy_2}{dt}\right)^2 + y_2$

$\cos(y(t))$, yy', $y^2 + y'$ are all non-linear

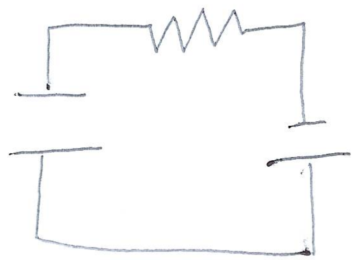
Techniques for Solving Linear DE

① Separation of variables

Can you convert your DE ~~into~~ so that $f(y)dy = g(t)dt$

$$y(t) = \int_0^t f(\tilde{t})\,d\tilde{t} \qquad \frac{dy}{dt} = f(t)$$

$$\int_{y_0}^{y} d\tilde{y} = \int_0^t f(\tilde{t})\,d\tilde{t}$$



charging capacitor

$$\frac{Q}{C} + R\frac{dQ}{dt} = V$$

$$R\frac{dQ}{dt} = V - Q/L \qquad \frac{dQ}{dt} = \frac{V}{R} - \frac{Q}{RL} = \frac{VC - Q}{RC}$$

$$\frac{dQ}{VL - Q} = \frac{dt}{RL} \qquad \int_{Q_0}^{Q} \frac{d\tilde{Q}}{\tilde{Q} - VL} = \int_0^t \frac{-d\tilde{t}}{RL}$$

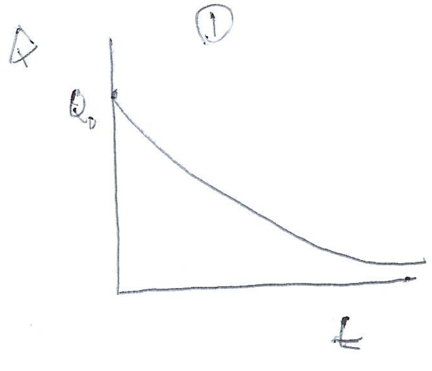
$$\ln|\tilde{Q} - VL|\Big|_{Q_0}^{Q} = \frac{-t}{RL} = \ln(Q - VL) - \ln(Q_0 - VL) = \ln\left(\frac{Q - VL}{Q_0 - VL}\right)$$

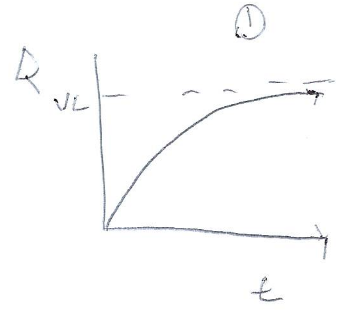
$$\ln\left(\frac{Q - VL}{Q_0 - VL}\right) = -\frac{t}{RL} \qquad \frac{Q - VL}{Q_0 - VL} = e^{-t/RC}$$

$$Q - VL = (Q_0 - VL)e^{-t/RC} \qquad Q = (Q_0 - VL)e^{-t/RL} + VL$$

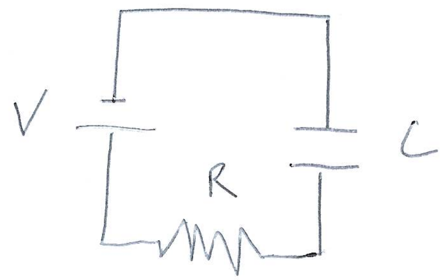
$$= \underset{①}{Q_0 e^{-t/RL}} + \underset{②}{VL(1 - e^{-t/RC})}$$

①





Phys 89 22Mar19



$$\frac{Q}{C} + R\frac{dQ}{dt} = V$$

$$\frac{q}{C} + R\frac{dq}{dt} = V$$

Today: Integration factors

Homogeneous and inhomogeneous Equations

First order, linear ODE's

Ordinary Differential Equations

no partial derivatives

Standard Form

$$\dot{y} + P(t)y = Q(t)$$

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V}{R} \qquad P(t) = \frac{1}{RC}, \quad Q(t) = \frac{V}{R}$$

$$\left(\frac{d}{dt} + P(t)\right)y = Q(t)$$

$$Ay = Q(t) \quad : \quad A\vec{y} = \vec{b}, \quad \vec{b} = Q(t), \quad \vec{y} = y(t)$$

y_p particular solution

y_h $\quad Ay_h = 0$ homogeneous

$$\frac{dy_h}{dt} + P(t)y_h = 0$$

$$y = y_p + y_h$$

$$\frac{dy}{dt} + P(t)y = \frac{dy_p}{dt} + P(t)y_p + \frac{dy_h}{dt} + P(t)y_h = Q(t)$$

$$\frac{dy_h}{dt} + P(t)\,y_h = 0 \qquad \text{[Homogeneous]}$$

$$\cancel{\frac{dy_h}{dt} = -P(t)\,dt} \qquad \frac{dy_h}{y_h} = -P(t)\,dt$$

$$\ln\left(\frac{y_h}{y_0}\right) = -\int_0^t P(t')\,dt' \qquad y_h = y_0\, e^{-\boxed{\int_0^t P(t')\,dt'}} = y_0\, e^{-I(t)}$$

$I(t)$

integrating factor

For RC - circuit $\rightarrow I(t) = \frac{t}{RC}$

Inhomogeneous Equation $A_y = Q(t)$

will give us the particular solution y_p

$y = y_p + y_h$ is always a solution by linearity

Find $y(t) = Z(t)\, e^{-I(t)}$ Assume this is true Guess ANSATZ and see if it works

$$\dot{y}(t) = \dot{Z}(t)\, e^{-I(t)} + -Z(t) \underbrace{\boxed{\frac{dI}{dt}}}_{P(t)} e^{-I(t)}$$

$$\dot{y}(t) = \dot{Z}(t)\, e^{-I(t)} - P(t)\, Z(t)\, e^{-I(t)}$$

$$\dot{y}(t) + P(t)\, y(t) = Q(t)$$

$$\dot{z}e^{-I} - Pze^{-I} + Pze^{-I} = Q(t)$$

$$\dot{z}e^{-I} = Q(t)$$

$$\dot{z} = Q(t)\, e^{I}$$

$$\boxed{Z = \int_0^t Q(t')\, e^{I}\, dt'}$$

$$y_p(t) = e^{-I(t)} \int_0^t Q(t') e^{I(t')} dt'$$

$$I = \frac{t}{RL} \quad y_p(t) = e^{-t/RL} \int_0^t \frac{V}{R} dt' e^{t'/RL}$$

$$q_p = \frac{V}{R} e^{-t/RL} RL \left(e^{t'/RL}\right)\Big|_0^t = VC\left(1 - e^{-t/RL}\right)$$

$$q_h = q_0 e^{-t/RL}$$

$$q = q_p + q_h = q_0 e^{-t/RL} + CV\left(1 - e^{-t/RL}\right)$$

Sometimes you can solve nonlinear equations using this method

eg) Bernoulli Equation

$$\dot{y} + P(t) y = Q(t) y^n$$

Let $u(t) = y^{1-n}$ $\quad \dot{u} = (1-n) y^{-n} \dot{y}$

$$\dot{y} = \frac{\dot{u}}{(1-n) y^{-n}} = \frac{\dot{u} y^n}{1-n}$$

$$\frac{\dot{u} y^n}{1-n} + P(t) y = Q(t) y^n$$ Put into standard form

$$\dot{u} + P(t) \cdot \frac{y}{y^n} (1-n) = Q(t) \frac{y^n}{y^n} (1-n)$$

$$\dot{u} + P(t) y^{1-n} (1-n) = Q(t)(1-n)$$

$$\boxed{\dot{u} + P(t) u (1-n) = Q(t)(1-n)}$$

Quiz

Q1 Find eigenvalues

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \qquad \begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} \qquad (a-\lambda)^2 + b^2 = 0$$

$$(a-\lambda)^2 = -b^2$$

$$\lambda = a \pm ib$$

(C)

Q2 $\begin{bmatrix} i & 1+i \\ -1+i & 0 \end{bmatrix}$ (B) skew hermitian

$$\begin{matrix} -9 & 2 & 3 \\ 0 & -9 & -1 \\ 0 & 0 & -9 \end{matrix} \qquad \begin{matrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}$$

Q3 A^{-1} $\vec{x}^T A \vec{x}$

$-2 \; -4+6i$

$-2 \; -4+6i = -6+6i$

$4i-6$

$4i-6i = -2i$

$12i+12$

$+8i-12i$

$12+8i$

$8i-12$

$$A = \begin{bmatrix} i & -2+3i \\ 2+3i & 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 2i \\ 2 \end{bmatrix}$$

$$[-2i \;\; 2] \begin{bmatrix} i & -2+3i \\ 2+3i & 0 \end{bmatrix} \begin{bmatrix} 2i \\ 2 \end{bmatrix} = [-2i \;\; 2] \begin{bmatrix} -6+6i \\ -2 \end{bmatrix}$$

$$= +12 + 12 - 4 = \cancel{28} \; 20$$

(D) $20i$ C

Q4 $\begin{bmatrix} 4 & 3 & 3 \\ 3 & 6 & 1 \\ 3 & 1 & 6 \end{bmatrix}$, $\lambda_1 = 10$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 5 & -5 \\ 3 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & -5 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 3 \\ 3 & 6 & 1 \\ 3 & 1 & 6 \end{bmatrix}$$

(A)

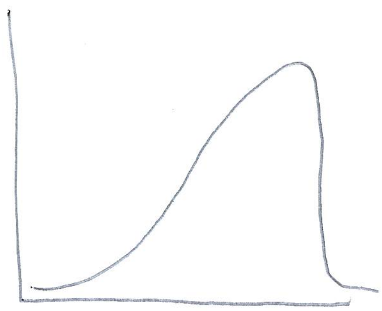
$$\begin{matrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & -1 & 3 \end{matrix} \qquad \begin{matrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 10x$$

Phys 89 05 Apr 2019

Ch 4.1 Kreyszig System of ODEs

Ch 2.1 Kreyszig Reduction of order



$$m\ddot{y} + \gamma\dot{y} + \omega^2 y = 0 \qquad m=1$$

$$y_1 = y \qquad y_2 = \dot{y} \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

$$\dot{\vec{y}} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = M\vec{y}$$

$$\lambda_\pm = \frac{-\gamma}{2} \pm \sqrt{\gamma^2/4 - \omega^2} = \frac{-\gamma}{2} \pm i\sqrt{\omega^2 - \gamma^2/4}$$

$$e^{\lambda t} \qquad e^{\frac{-\gamma}{2} \pm i\sqrt{\omega^2-\gamma^2/4}}$$

$$\vec{y}(t) = \sum_j c_j \vec{v}_j e^{\lambda_j t} = c_+ \begin{bmatrix} 1 \\ \lambda_+ \end{bmatrix} e^{\lambda_+ t} + c_- \begin{bmatrix} 1 \\ \lambda_- \end{bmatrix} e^{\lambda_- t} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

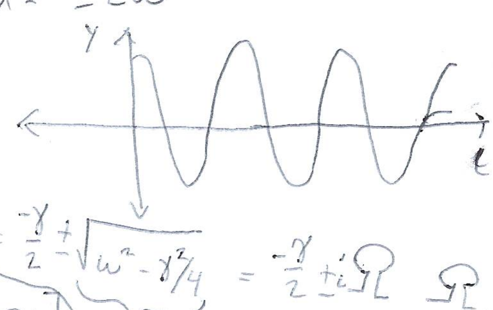
$$y(t) = c_+ e^{\lambda_+ t} + c_- e^{\lambda_- t}$$

$$\dot{y}(t) = c_+\lambda_+ e^{\lambda_+ t} + c_-\lambda_- e^{\lambda_- t}$$

$c_\pm$ determined by initial conditions ie $y(0)$ and $\dot{y}(0)$

"0" frictionless $\gamma = 0$ $\lambda = \pm i\omega$

$$\vec{y}(t) = y(0)\begin{bmatrix} \cos\omega t \\ -\omega\sin\omega t \end{bmatrix}$$



Underdamped $\left(\frac{\gamma}{2}\right)^2 < \omega^2$ $\quad \lambda_\pm = \frac{-\gamma}{2} \pm \underbrace{\sqrt{\omega^2 - \gamma^2/4}}_{\Omega} = \frac{-\gamma}{2} \pm i\Omega$ $\quad \Omega$, real and positive

$$\vec{y}(t) = y(0)e^{-\gamma t/2}\begin{bmatrix} \cos\Omega t + \frac{\gamma}{\Omega}\sin\Omega t \\ -\Omega\left(1+\frac{\gamma^2}{4\Omega^2}\right)\sin\Omega t \end{bmatrix}$$

damping gives a phase

Overdamped case

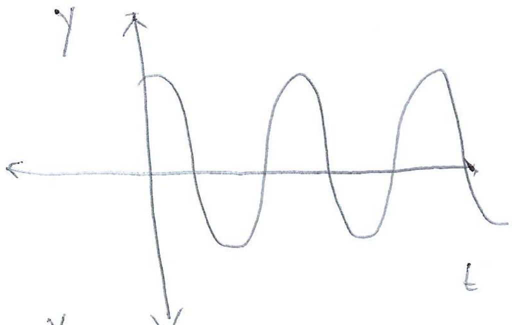
$\vec{X}(t)$ $\quad \left(\frac{\gamma}{2}\right)^2 > \omega^2 \qquad \lambda_{+-} = \frac{-\gamma}{2} \pm i\sqrt{\omega^2 - \gamma^2/4}$

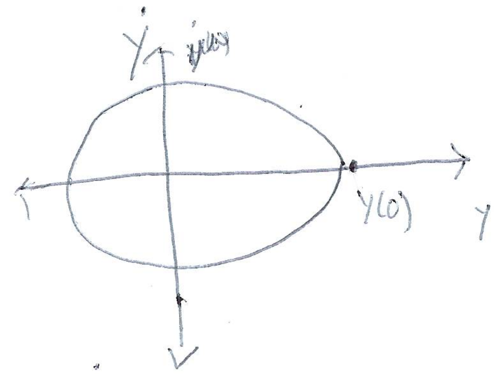
$$= \frac{-\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega^2} = \frac{-\gamma}{2} \pm \Gamma \qquad \Gamma > 0\text{, real}$$

Add exponentials with purely real terms $\quad \frac{e^{-\lambda t} + e^{\lambda t}}{2} = \cosh t$

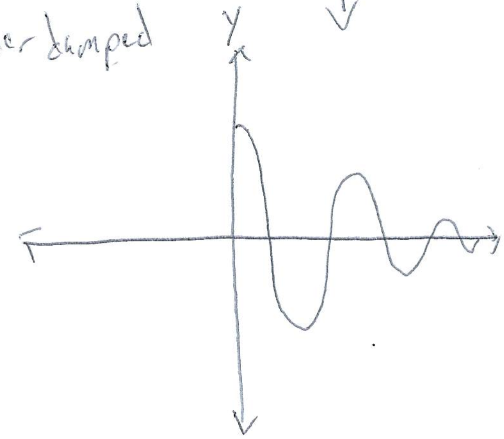
$$\vec{y}(t) = y(0)\, e^{-\frac{\gamma}{2}t} \begin{bmatrix} \cosh(\Gamma t) + \frac{\gamma}{\Gamma}\sinh(\Gamma t) \\ \Gamma\left(1 - \frac{\gamma^2}{\Gamma^2}\right)\sinh(\Gamma t) \end{bmatrix}$$

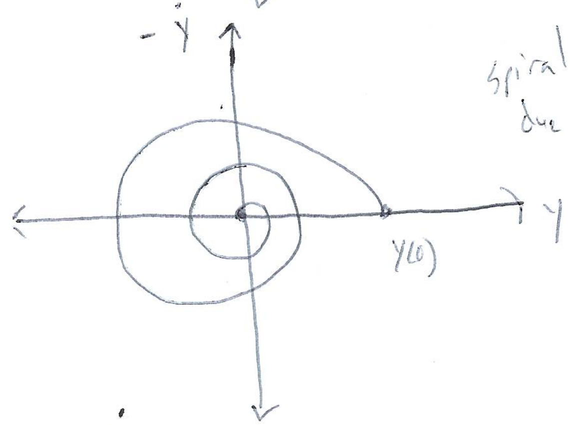
Frictionless





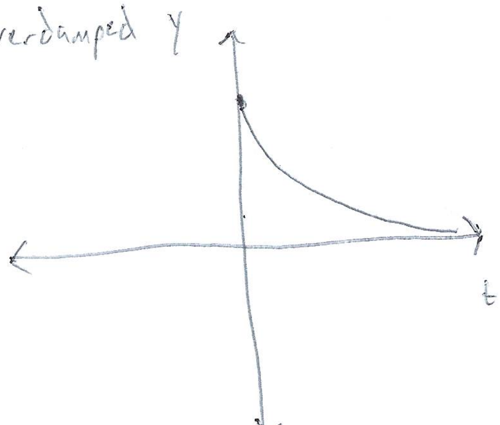
Underdamped

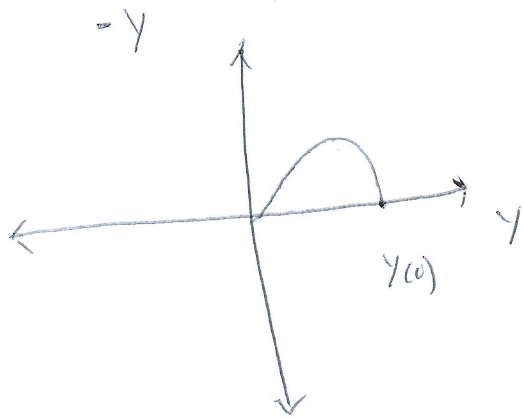




spiral backwards due to negative initial velocity

Overdamped





Critical Damping $\frac{\gamma}{2} = \omega$

$\lambda_{\pm} = -\frac{\gamma}{2} \pm 0$

$\boxed{\text{Two degenerate solutions}}$

Expect 2 solutions for a 2nd order ODE

So we need to generate a second solution

→ we use "Reduction of order"

Applicable if you know one solution already

<u>Theory</u> (in general)

$$\ddot{y} + p(t)\dot{y} + q(t)y = 0$$

And y_1 is known

Ansatz

$$y_2(t) = u(t)y_1(t) \qquad e^{2t} \quad t^2 e^{2t}$$

generating a linearly indepent function

$$\dot{y}_2 = \dot{u}y_1 + u\dot{y}_1 \qquad \ddot{y}_2 = \ddot{u}y_1 + 2\dot{u}\dot{y}_1 + u\ddot{y}_1$$

$$\ddot{u}y_1 + 2\dot{u}\dot{y}_1 + u\ddot{y}_1 + p(t)(\dot{u}y_1 + u\dot{y}_1) + q(t)uy_1$$

$$\ddot{u}y_1 + \dot{u}(2\dot{y}_1 + p(t)y_1) + u\underbrace{(\ddot{y}_1 + p(t)\dot{y}_1 + q(t)y_1)}_{0 \text{ solution to ODE}} = 0$$

$$\ddot{u}y_1 + \dot{u}(2\dot{y}_1 + p(t)y_1) = 0$$

$\dot{u} = v$

$$\dot{v}y_1 + v(2\dot{y}_1 + p(t)y_1) = \dot{v}f(t) + vg(t) = 0$$

1st order ODE

$$\dot{v} + v\frac{g(t)}{f(t)} = 0$$

$$\dot{v} + v\,\tilde{p}(t) = 0$$

$$v(t) = Ae^{I(t)}$$

$$I(t) = \int_0^t \tilde{p}(t')\,dt'$$

$$= \int_0^t \frac{2\dot{y}_1 + p(t)y_1}{y_1}\,dt$$

$$= \int_0^t \frac{2\dot{y}_1}{y_1}\,dt' + \int_0^t p(t)\,dt'$$

$$* \; 2\int_0^t \frac{1}{y_1}\frac{dy_1}{dt}\,dt = 2\int_0^t \frac{dy_1}{y_1} = 2\ln y_1\Big|_0^t$$

Phys 89 08Apr19

Reduction of Order Kreyszig 2.1

Particular Solutions

Method of Undetermined Coefficients 2.8, 2.9

Variation of Parameters Kreyszig 2.10

~~$y'' + p(t)$~~

$$\ddot{y} + p(t)\dot{y} + q(t)y = 0$$

$$Y_1 = y_1 \qquad y_2 = u(t)\,y_1$$

$$v = \dot{u} \qquad y_2(t) = y_1(t)\int_0^t \frac{y_1^2(0)}{y_1^2(t')}\, e^{-\int_0^{t'} p(t'')\,dt''}\,dt'$$

critically damped $\frac{\gamma}{2} = \omega$

$$p(t) = \gamma = 2\omega \quad y_1(t) = e^{-\omega t}$$

$$\dot{v} + \left(\frac{2\dot{y}_1}{y_1} + p(t)\right)v = 0$$

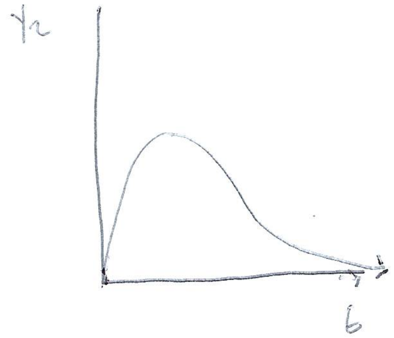
$$\dot{v} + \left(\frac{2(-\omega)e^{-\omega t}}{e^{-\omega t}} + 2\omega\right)v = 0$$

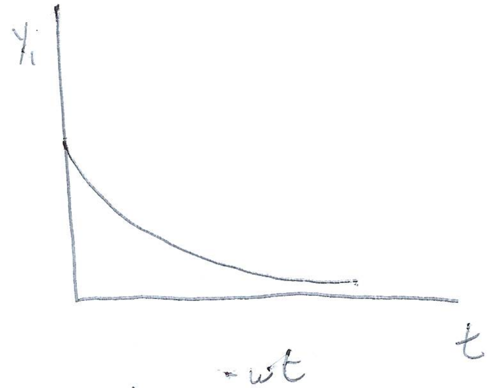
$$\dot{v} + 0\cdot v = 0 \qquad \dot{v} = 0 \qquad v = \text{constant}$$

$$\dot{v} = u = \int_0^t C\,dt' = Ct$$

$$y_2 = u(t)\,y_1 = Cte^{-\omega t} = te^{-\omega t}$$

~~Any function~~ $C = 1$





$y_2 = te^{-wt}$ $\qquad$ $y_1 = e^{-wt}$

y_1, y_2 are basis for solutions

$f(t) = 1 + a_1 t + a_2 t^2 + \dots$

Way of checking each term to generate solutions

Particular solution

"Method of undetermined coefficients"

$\ddot{y} + p(t)\dot{y} + q(t)y = r(t)$

y looks like $r(t)$

$r(t)$	Ansatz y_p	
$ke^{\gamma t}$	$ce^{\gamma t}$	$\gamma \in \mathbb{C}$ only have c to determine
kt^n	$a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$	~~all~~ all a_n need to be determined $(n+1)$ unknowns
$k\cos wt$ $k\sin wt$	$c_1 e^{iwt} + c_2 e^{-iwt}$ $A\cos wt + B\sin wt$ $A = c_1 + c_2$ $B = \frac{c_2 - c_1}{i}$	The first line of this table twice 2 unknowns
$ke^{\alpha t}\cos wt$ $ke^{\alpha t}\sin wt$	$Ae^{\alpha t}\cos wt + Be^{\alpha t}\sin wt$	2 unknowns
n^{th} order polynomial $k_0 + k_1 t + k_2 t^2 + \dots + k_n t^n$	$a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$	n unknowns

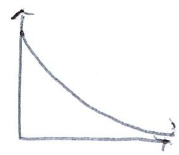
Frictionless Driven SHO

$$m\ddot{y} = -ky + F(t)$$

$m\ddot{y} + ky = 0 \qquad \ddot{y} + \frac{k}{m}y = 0 \qquad Y_1 = \cos\omega t = \cos\sqrt{\frac{k}{m}}\,t$

$Y_2 = \sin\omega t = \sin\sqrt{\frac{k}{m}}\,t$

$F(t) = \alpha e^{-\beta t}$



$Y_p = Ce^{-\beta t} \qquad \dot{Y}_p = -C\beta e^{-\beta t}$

$\ddot{Y}_p = C\beta^2 e^{-\beta t}$

$mC\beta^2 e^{-\beta t} + k(Ce^{-\beta t}) = \alpha e^{-\beta t}$

$C(m\beta^2 + k) = \alpha$

$C = \frac{\alpha}{m\beta^2 + k} \qquad Y_p(t) = \frac{\alpha}{m\beta^2+k} e^{-\beta t}, \quad F(t) = \alpha e^{-\beta t}$

$m\ddot{y} + ky = \gamma t^2$

$Y_p = a_0 + a_1 t + a_2 t^2$

$\dot{Y}_p = a_1 + 2a_2 t$

$\ddot{Y}_p = 2a_2$

$Y_p = -\frac{2m\gamma}{k^2} + \frac{\gamma}{k} t^2$

$m(2a_2) + k(a_0 + a_1 t + a_2 t^2) = \gamma t^2$

$(2ma_2 + ka_0) + ka_1 t + ka_2 t^2$

$2ma_2 + ka_0 = 0 \qquad ka_1 = 0 \qquad ka_2 = \gamma$

$2m\frac{\gamma}{k} = -ka_0 \qquad a_1 = 0 \qquad a_2 = \frac{\gamma}{k}$

$-\frac{2m\gamma}{k^2} = a_0$

What if r(t) resonates with the homogeneous solution

$$F(t) = F_0 \cos wt$$

$$m\ddot{y} + ky = F_0 \cos wt$$

$$y_p = A\cos wt + B\sin wt$$

$$\dot{y}_p = \cancel{\quad} - Aw\sin wt + Bw\cos wt$$

$$\ddot{y}_p = -Aw^2\cos wt - Bw^2\sin wt$$

$$\cancel{\to} \quad A = \frac{F_0}{k - mw^2} \qquad B = 0 \qquad y_p(t) = \frac{F_0}{k - mw^2}\cos wt$$

$\boxed{\text{linearly dependent on } y_{h1}}$

What if $w = \sqrt{\frac{k}{m}}$ (diverges) Even if $w \neq \sqrt{\frac{k}{m}}$

So to get around this we use something like Lagrange's idea. Keep ~~multiplying~~ trying Ansatzes that have higher and higher orders of t multiplying them

$\boxed{\text{eg Try } y_p = t\, y_h}$

Phys 89 10Apr19

Ch 2.10 "Variation of parameters"

ch 4 "Power series techniques"

"Method of Frobenius"

$\hookrightarrow$ Fourier techniques

$m\ddot{y} + ky = at^2 \qquad y = a_0 + a_1 t + a_2 t^2$

$m\ddot{y} + ky = F_0 \cos(\sqrt{\tfrac{k}{m}}\, t) \leftarrow$ Forcing fun

$\omega = \sqrt{\tfrac{k}{m}}$ Driving Force Resonant with homogeneous solution

$y_h(t) = C_1 \cos\omega t + C_2 \sin\omega t$

But we want the particular solution

The strategy is to create some linearly independent function to y_h

$$\boxed{\underset{\text{Ansatz}}{y_p = t y_h}} \qquad \dot{y}_p = y_h + t\dot{y}_h$$

$$\ddot{y}_p = \dot{y}_h + \dot{y}_h + t\ddot{y}_h = 2\dot{y}_h + t\ddot{y}_h$$

$$m\ddot{y} + ky = m(2\dot{y}_h + t\ddot{y}_h) + kty_h$$

$$= \underbrace{t(m\ddot{y}_h + ky_h)}_{\text{homogeneous part} \to 0} + 2m\dot{y}_h = 2m\dot{y}_h = F_0 \cos\omega t$$

$\dot{y}_h = -C_1\omega \sin\omega t + C_2 \omega \cos\omega t$

$$\underbrace{-2m\omega C_1 \sin\omega t}_{C_1 = 0} + \underbrace{C_2 \omega 2m \cos\omega t}_{} = F_0 \cos\omega t$$

$2C_2\omega m = F_0$

$C_2 = \dfrac{F_0}{2\omega m}$

$\cancel{y_p = \frac{F_0}{2\omega m} t\cos\omega t}$

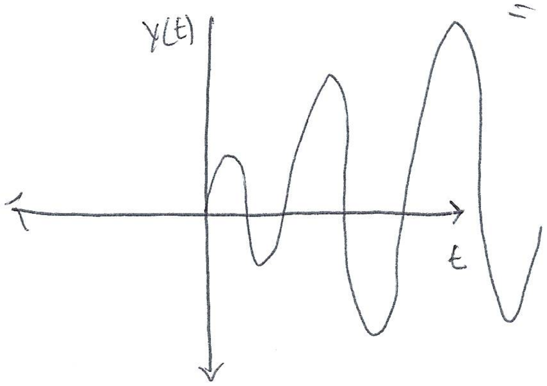
$y_p = \dfrac{F_0}{2\omega m} t \sin\omega t$

for some initial condition

y_h $a_1 = 0$ $a_2 \neq 0$

$$y = y_h + y_p = a_2 \sin \omega t + \frac{f_0 t}{2m\omega} \sin t\omega$$

$$= \left(a_2 + \frac{t f_0}{2m\omega}\right) \sin \omega t$$



$$m\ddot{y} + ky = F_0 \cos \omega t \qquad y_p = at^2 \quad \dot{y}_p = 2at \quad \ddot{y}_p = 2a$$

$$m\,2a + kat^2 = F_0 \cos \omega t$$

can't equate coefficients

Variation of Parameters

$$\ddot{y}(t) + p(t)\dot{y}(t) + q(t)y(t) = r(t)$$

$$\ddot{y}_h(t) + p(t)\dot{y}_h(t) + q(t)y_h(t) = 0$$

Assume you have y_{h1} and y_{h2}

$y_p(t) = t y_h(t)$ extend idea $y_p(t) = c_1(t) y_{h1}(t) + c_2(t) y_{h2}(t)$

parameter

some function of t

Assume: $\dot{c}_1 y_{1h} + \dot{c}_2 y_{2h} = 0$

$$\dot{y}_p(t) = \dot{c}_1 y_{h1} + c_1 \dot{y}_{h1} + \dot{c}_2 y_{h2} + c_2 \dot{y}_{h2} = c_1 \dot{y}_{h1} + c_2 \dot{y}_{h2}$$

$$\ddot{y}_p(t) = \ddot{c}_1 y_{h1} + \dot{c}_1 \dot{y}_{h1} + \dot{c}_1 \dot{y}_{h1} + c_1 \ddot{y}_{h1} + \ddot{c}_2 y_{h2} + \dot{c}_2 \dot{y}_{h2} + \dot{c}_2 \dot{y}_{h2} + c_2 \ddot{y}_{h2}$$

$$= \ddot{c}_1 y_{h1} + 2\dot{c}_1 \dot{y}_{h1} + c_1 \ddot{y}_{h1} + \ddot{c}_2 y_{h2} + 2\dot{c}_2 \dot{y}_{h2} + c_2 \ddot{y}_{h2}$$

$$= c_1 \ddot{y}_{h1} + \dot{c}_1 \dot{y}_{h1} + \dot{c}_2 \dot{y}_{h2} + c_2 \ddot{y}_{h2}$$

$$c_1(\ddot{y}_{h1} + p(t)\dot{y}_{h1} + q(t) y_{h1}) + c_2(\ddot{y}_{h2} + p(t)\dot{y}_{h2} + q(t) y_{h2}) + \dot{c}_1 \dot{y}_{h1} + \dot{c}_2 \dot{y}_{h2} = r(t)$$

$$\dot{c}_1 \dot{y}_1 + \dot{c}_2 \dot{y}_2 = 0$$

$$\dot{c}_1 \dot{y}_1 + c_2 \dot{y}_2 = r$$

$$\begin{bmatrix} c_1 & c_2 \\ \dot{c}_1 & \dot{c}_2 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

$$\dot{c}_1 y_1 + \dot{c}_2 y_2 = 0$$

$$\dot{c}_1 \dot{y}_1 + \dot{c}_2 \dot{y}_2 = r$$

$$\begin{bmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{bmatrix} \begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

$$A\vec{y} = \vec{b}$$

Cramer's Rule

$$A\vec{y} = \vec{b} \qquad \vec{y} = \begin{bmatrix} y' \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} |B_1|/|A| \\ |B_2|/|A| \\ \vdots \\ |B_n|/|A| \end{bmatrix}$$

$$B_2 = \begin{bmatrix} \ldots & \vec{b} & \ldots \end{bmatrix}$$

matrix A with ith column replaced

$$\dot{c}_1 = \frac{\det \begin{bmatrix} 0 & y_{2h} \\ r & \dot{y}_{2h} \end{bmatrix}}{\det \begin{bmatrix} y_{1h} & y_{2h} \\ \dot{y}_{1h} & \dot{y}_{2h} \end{bmatrix}} = \frac{-r y_{2h}}{W}$$

$$\dot{c}_2 = \frac{\det \begin{bmatrix} y_{1h} & 0 \\ \dot{y}_{1h} & r \end{bmatrix}}{\det \begin{bmatrix} y_{1h} & y_{2h} \\ \dot{y}_{1h} & \dot{y}_{2h} \end{bmatrix} \to W} = \frac{r y_{1h}}{W}$$

W = Wronskian

Phys 89 15 Apr 19

Power Series Solutions Kreyszig Ch 5.1 - 5.3

The Monomials 'Basis' $\{t^n\}$ $\doteq$ represents

$$\vec{e}_n \doteq \{t^n\}$$

$$y(t) = \vec{y} = \sum_n^{\infty} a_n \vec{e}_n = \sum_n^{\infty} a_n t^n \Big\} \text{ Power Series}$$

$$m\ddot{y} + ky = 0 \qquad \left(m\frac{d^2}{dt^2} + k\right)y(t) = 0$$

$\to$ $A\vec{y} = 0$

$$A\vec{y} = 0$$

$A \sum a_n \vec{e}_n$ $\to$ Look for set of a_n that will solve the problem

$A\vec{y} = \vec{b}$

$$\dot{y} = \sum_{n=1} a_n n t^{n-1} \qquad \ddot{y} = \sum_{n=2} a_n (n)(n-1) t^{n-2}$$

$$m \sum_2^{\infty} a_n n(n-1) t^{n-2} + k \sum_{n=0}^{\infty} a_n t^n = 0$$

$\tilde{n} = n-2$

$$m \sum_0^{\infty} a_{\tilde{n}+2} (\tilde{n}+2)(\tilde{n}+1) t^{\tilde{n}} \Leftrightarrow \sum_0^{\infty} a_{n+2} (n+2)(n+1) t^n \qquad \tilde{n} \text{ is dummy}$$

$$\sum_0^{\infty} m a_{n+2} (n+2)(n+1) t^n + k a_n t^n = 0$$

$$= \sum_0^{\infty} \left(a_{n+2} m (n+2)(n+1) + k a_n\right) t^n = 0 \qquad \{t^n\} \text{ are linearly independent}$$

$$a_{n+2} m (n+2)(n+1) + k a_n = 0$$

$$\boxed{a_{n+2} = \frac{-k}{m} \frac{1}{(n+2)(n+1)} a_n}$$ Recursion relation

$a_2 \to a_0 \qquad a_4 \to a_2$ Evens to evens related to a_0

$a_5 \to a_3 \to a_1$ Odds to odds related to a_1

$$a_3 = \frac{-k}{m} \frac{a_1}{(1+2)(1+1)} = \frac{-k}{m} \frac{a_1}{3 \cdot 2}$$

$$a_5 = \left(-\frac{k}{m}\right) \frac{a_3}{(3+2)(3+1)} = \left(-\frac{k}{m}\right)^2 \frac{a_1}{5432}$$

$$a_{2n+1} = \left(-\frac{k}{m}\right)^n \frac{1}{(2n+1)!} = \frac{(-1)^n \omega^{2n+1}}{(2n+1)!} \frac{a_1}{\omega}$$

$$a_2 = \left(\frac{-k}{m}\right) \frac{a_0}{2}$$

$$a_4 = \left(\frac{-k}{m}\right) \frac{a_2}{4 \cdot 3} = \left(\frac{-k}{m}\right)^2 \frac{a_0}{4 \cdot 3 \cdot 2}$$

$$a_{2n} = \left(\frac{-k}{m}\right)^n \frac{a_0}{(2n)!} = \frac{(-1)^n \omega^{2n}}{(2n)!} a_0 \qquad \omega = \sqrt{\frac{k}{m}}$$

$$Y = \sum_0^\infty a_n t^n = \underbrace{\sum_s \frac{(-1)^s \omega^{2s} t^{2s} a_0}{(2s)!}}_{\text{even}} + \underbrace{\sum_r \frac{(-1)^r \omega^{2r+1} a_1 t^{2r+1}}{(2r+1)!\, \omega}}_{\text{odd}}$$

$$y = a_0 \sum_s \frac{(-1)^s (\omega t)^{2s}}{(2s)!} + \frac{a_1}{\omega} \sum_r \frac{(-1)^r (\omega t)^{2r+1}}{(2r+1)!}$$

$$y(t) = a_0 \cos \omega t + \frac{a_1}{\omega} \sin \omega t = a_0 y_{1h} + \frac{a_1}{\omega} y_{2h}$$ 2 linearly independent solutions

Legendre Polynomials

$$\boxed{\int_{-1}^{1} P_\ell(t)\, P_{\ell'}(t)\, dt = \delta_{\ell\ell'} \frac{2}{2\ell+1}}$$

$$(1+t^2)\ddot{y} - 2t\dot{y} + \ell(\ell+1)y = 0$$

Legendre Equation   Legendre Polynomials solve this equation

$Ay = \lambda y$ ("Eigenvectors" of the Legendre Diff Eq)
"Eigenfunctions"

$\ell = 2$ $P_2(t) = \frac{1}{2}(3t^2 - 1)$

$\dot{P}_2 = 3t$

$\ddot{P}_2 = 3$

$(1+t^2)3 - 2t(3t) + 6\left(\frac{1}{2}(3t^2-1)\right) = 0$

$3 + 3t^2 - 6t^2 + 9t^2 - 3$

$P_\ell(t) = \sum_n a_n t^n$

$(1+t^2)\sum_2^\infty a_n (n-1) n t^{n-2} - 2t \sum_1^\infty a_n n t^{n-1} + \ell(\ell+1)\sum_0^\infty a_n t^n$

$\sum_2^\infty a_n (n-1) n t^{n-2} - \sum_2^\infty a_n (n-1) n t^n - 2\sum_1^\infty a_n t^n n + \ell(\ell+1)\sum_0^\infty a_n t^n$

$\tilde{n} \to n-2$

$\sum_0^\infty a_{n+2}(n+1)(n+2)t^n - \sum_2^\infty a_n (n-1) n t^n - 2\sum_1^\infty a_n t^n n + \ell(\ell+1)\sum_0^\infty a_n t^n$

Kreyszig 5.2

...

Again you will end up with two recursive relations

$$a_{n+2} = \frac{-(\ell-n)(\ell+n+1)}{(n+2)(n+1)} a_n$$

$a_0 = a_0$

$a_2(1 \cdot 2) t^0 + l(l+1) a_0 t^0 = 0$

$$a_2 = -a_0 \frac{l(l+1)}{2}$$

$n = 1$

$$a_3 = \frac{2 - l(l+1)}{3 \cdot 2 \cdot 1} a_1 \qquad a_4 = \frac{(l-2)l(l+1)(l+3)}{4!} a_0$$

$$a_3 = \frac{-(l-1)(l+2)}{3!} a_1 \qquad a_5 = \frac{(l-3)(l-1)(l+2)(l+4)}{5!} a_1$$

$$y(t) = \overset{\text{even}}{a_0 Y_{1h}} + \overset{\text{odd}}{a_1 Y_{2h}}$$

$$Y_{1h} = a_0 - \frac{l(l+1)}{2!} a_0 t^2 + \frac{(l-2)l(l+1)(l+3)}{4!} a_0 t^4 + \dots$$

$l = 2$ First 2 terms survive

$$Y_{1h}(l=2) = a_0(1 - 3t^2)$$

$$c_n = \hat{e}_n \cdot \hat{f} = \cancel{\frac{1}{2\pi}\int}$$

$$= \hat{e}_n \cdot \sum_m^{\infty} c_m \hat{e}_m = \sum_m^{\infty} c_m \hat{e}_n \cdot \hat{e}_m = \sum_{-\infty}^{\infty} c_m \delta_{mn} = c_n$$

$f(x)$ real $\{a_0, a_n, b_n\}$ real

$$c_{-n} = c_n^*$$

$f(-x) = f(x)$ even function

$$\{b_n\} = 0 \qquad c_n = c_{-n}$$

$\cancel{c_n = c_{-n}}$ $f(x) = -f(-x)$ odd function

$$\{a_n\} = 0 \qquad c_n = -c_{-n}$$

$$f(x) = \sum_n c_n e^{inx} \qquad \frac{df}{dx} = \sum_n c_n in\, e^{inx}$$

$$\begin{bmatrix} -in & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & in \end{bmatrix}$$

$$f(x) = e^{i2x} + e^{-i2x} = 2\cos 2x$$

$$c_2 = 1 \qquad c_{-2} = 1 \qquad a_2 = 2 = c_2 + c_{-2}$$

$$\frac{df}{dx} = i2\left(e^{i2x} - e^{-i2x}\right)$$

$$\vec{f} = \begin{array}{l} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{array}{l} \\ \\ n=-2 \\ \\ n=2 \end{array} \\ \;\; 0 \\ \;\; 0 \\ \;\; 0 \end{array} \qquad \frac{df}{dx} = \begin{bmatrix} -2i & & & \\ & -i & & \\ & & \ddots & \\ & & & i \\ & & & & 2i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 2i \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{f}\cdot\vec{f} = |f|^2 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2$$

Parseval's Theorem

$$\langle |f|^2 \rangle = \frac{1}{2}\left(\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 \right)$$

(Average energy of light composed of n frequencies)

A Complex Basis is even better

$$\{\hat{e}_n\} = \{e^{inx}\} \quad n \in \mathbb{Z} \quad \text{All integers}$$

$$\vec{f}\cdot\vec{g} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(x)g(x)\,dx$$

$$\hat{e}_n\cdot\hat{e}_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}e^{imx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}\,dx$$

for $m=n$ $\quad \hat{e}_n\cdot\hat{e}_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{0}\,dx = \frac{1}{2\pi}(2\pi) = 1$

$m \neq n$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ilx}\,dx = \frac{1}{2\pi il} e^{ilx}\Big|_{-\pi}^{\pi} = \frac{1}{2\pi il}\left(e^{il\pi} - e^{-il\pi}\right) = 0$$

$$= \frac{2i\sin \pi l}{2\pi il} = 0$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\hat{e}_n = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

$n \to -\infty$ to ∞ integers

Phys 89 17 Apr 19

Fourier Techniques Ch 11 Kreyszig

(But watch definition of Fourier Transforms)

$$y(x) = t^r \sum_0^\infty a_n t^n$$

~~r~~ r can be non integer or integer

Bessel Functions

$$y(x) = 1 + 2x + x^2 = \sum_0^\infty a_n x^n$$

$a_0 = 1$ $a_3 = 0$

$a_1 = 2$ $a_4 = 0$

$a_2 = 1$ $\vdots$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\{\vec{e}_n\} - \{x^n\}$$

$$\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

dropped $n \geq 3$

$$\frac{dy}{dx} = 2 + 2x \qquad \frac{d\vec{y}}{dx} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$= \sum_0^\infty a_n n x^{n-1}$$

$$y' = \begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix} \quad y = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix}$$

What about

general for even longer vector

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} \qquad \frac{d}{dx} \doteq \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4 & 0 & \cdots \\ \vdots \end{bmatrix}$$

Fourier Series ((Ch. 11 Kreyszig)

$$\{\hat{e}_n\} = \{\overset{\hat{e}_0}{1/2},\ \overset{\hat{e}_{c,n}}{\cos(nx)},\ \overset{\hat{e}_{s,n}}{\sin(nx)}\}$$ (infinite dimensional space) $n = 1$ to ∞

This defines a vector space for functions $n \in \mathbb{Z}^+$

with inner product

$$\vec{f}\cdot\vec{g} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx$$

for real functions

$$f(x) = a_0\hat{e}_0 + \sum_{n=1}^{\infty} a_n\hat{e}_{c,n} + \sum_{n=1}^{\infty} b_n\hat{e}_{s,n}$$

$$= \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx}_{\text{Fourier coefficients}}$$

Use dot product

$$a_0 = 2\vec{e}_0\cdot\vec{f} = \frac{2}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$a_{c,n} = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx \qquad b_{s,n} = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

Check $\hat{e}_n\cdot\hat{e}_m = \delta_{mn}$ $\quad g(x) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n\cos nx + \sum_{n=1}^{\infty} d_n\sin nx$

$$\vec{f}\cdot\vec{g} = \frac{1}{2}a_0c_0 + \sum_{n=1}^{\infty} a_nc_n + \sum_{n=1}^{\infty} b_nd_n$$

a_n doesn't mix with b_n or any c_m (with $m \neq n$)

Phys 89 19Apr19

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\hat{e}_m \cdot \hat{e}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-imx} e^{inx}\, dx$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(x) g(x)\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}\, dx = \delta_{m,n}$$

$$\frac{df}{dx} = \sum_{-\infty}^{\infty} in\, e^{inx}$$

$$\lim_{n\to\infty} \begin{bmatrix} -in & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & in \end{bmatrix} \begin{bmatrix} c_{-n} \\ \vdots \\ c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} -inc_n \\ \vdots \\ 0 \\ \vdots \\ inc_n \end{bmatrix}$$

$$\begin{bmatrix} c_{-n} \\ \vdots \\ 0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_k \\ 0 \\ 0 \\ \vdots \end{bmatrix} \rightleftharpoons f(x) = c_k e^{ikx}$$

$$p = \frac{h}{\lambda} = \hbar k$$

$$\hat{p} \to -i\frac{d}{dx}$$

$$\frac{-d^2\psi}{2m\, dx^2} = E\psi \qquad \frac{-1}{2m}\left(\frac{d}{dx}\left(\frac{d}{dx}(\psi)\right)\right) = E\psi$$

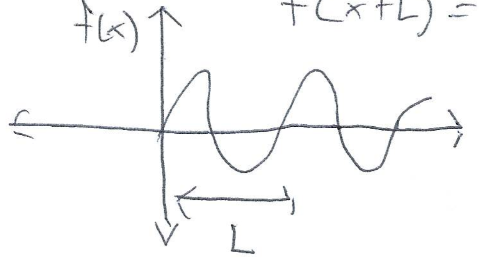
$$\frac{\hat{p}^2\psi}{2m} = E\psi$$

Fourier series is good at describing periodic functions.

what if period is L and not 2π

$$k_0 = \frac{2\pi}{L} \qquad u = k_0 x$$

$f(x+L) = f(x)$



$$f(x) = f\left(\frac{u}{k_0}\right) = \tilde{f}(u)$$

$$\cos\left(\frac{2\pi x}{L}\right) = \cos(u)$$

$$\cancel{f(x+L) = f\left(\frac{u}{k_0} + \frac{L}{k_0}\right) = f\left(\frac{u}{k_0} + \frac{L\,2\pi}{L}\right) = \tilde{f}(u+2\pi)}$$

$$\tilde{f}(u+2\pi) = f\left(\frac{u}{k_0} + \frac{2\pi}{k_0}\right) = f(x+L) = f(x) = \tilde{f}(u)$$

so $\tilde{f}(u) = \tilde{f}(u+2\pi)$

so $\tilde{f}(u) = \sum_{-\infty}^{\infty} c_n e^{inu}$ } Fourier expansion should work

$$= \sum_{-\infty}^{\infty} c_n e^{ink_0x} \qquad c_n = \hat{e}_n \cdot \vec{f} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inu}\,\tilde{f}(u)\,du$$

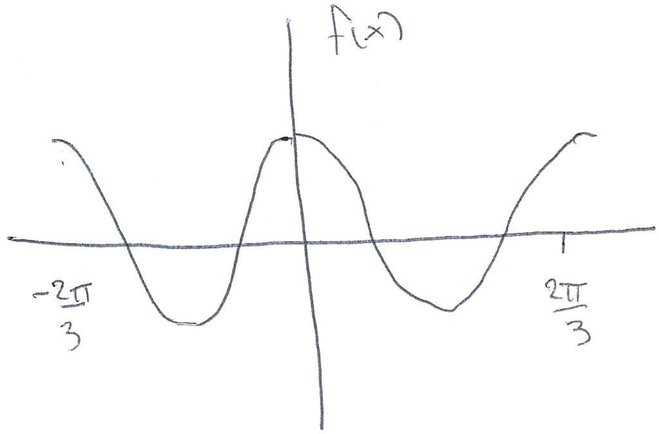
$$u = k_0 x \qquad du = k_0\,dx \qquad c_n = \frac{k_0}{2\pi}\int_{-\pi}^{\pi} e^{-ink_0x} f(k_0x)\,dx$$

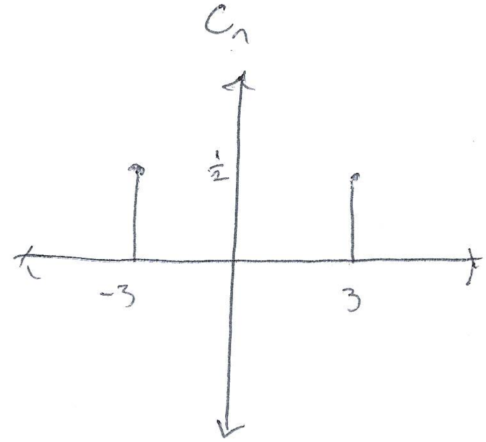
$$= \frac{1}{L}\int_{-L/2}^{L/2} e^{-ink_0x} f(x)\,dx$$

$$\cos 3x = \sum_{-\infty}^{\infty} c_n e^{inx}$$

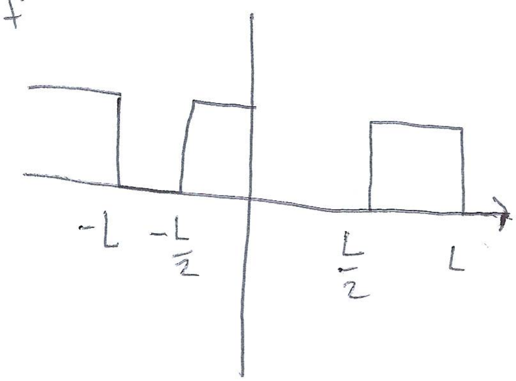
$$= \frac{1}{2}\left(e^{i3x} + e^{-i3x}\right) \qquad c_{-3} = 1/2 \qquad c_3 = 1/2$$

$\cancel{\cos}$ $\cos 3x = \frac{1}{2}\left(e^{i3x} + e^{-i3x}\right)$





C/f



$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inxk_0}$$

$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} e^{-inxk_0} f(x)\,dx$$

$$= \frac{1}{L}\left(\int_{-L/2}^{0} e^{-inxk_0}\cdot 1\,dx + \int_{0}^{L/2} e^{-inxk_0}\cdot 0\,dx\right)$$

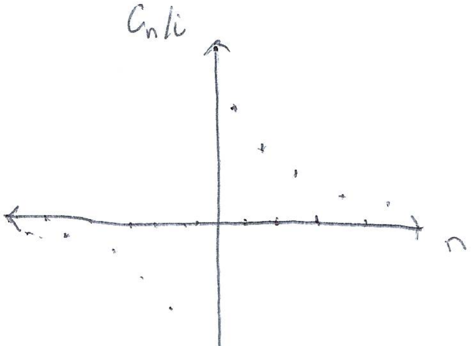
$$= \frac{1}{L}\int_{-L/2}^{0} e^{-inxk_0}\,dx = \frac{1}{L}\,\frac{1}{-ink_0}\,e^{-ink_0x}\Big|_{-L/2}^{0}$$

$$= \frac{+1}{-in2\pi}\left(1 - e^{+in\frac{2\pi}{L}\frac{L}{2}}\right)$$

$$= \frac{1}{-in2\pi}\left(1 - e^{in\pi}\right)$$

$$= \frac{-1}{in2\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{-1}{in\pi} & \text{if } n \text{ odd} \end{cases}$$

$$f(x) = \begin{cases} 0 : & 0 < x < L/2 \\ 1 : & \frac{L}{2} < x < L \\ \cancel{-\frac{L}{2} < x < 0} & \end{cases}$$



C_n diverges as $n \to 0$ (Get $\frac{0}{0}$)

So treat C_0 separately

$$\frac{1}{2\pi}\int_{-L/2}^{L/2} C_0\, dx$$

Quiz

1. $\frac{dT}{dt} = k(T - T_A)$

A) $T(t) = T_A e^{kt}$

B) $T(t) = T_A + Ce^{-kt}$

C) $T(t) = T_A + Ce^{kt}$

D) $T(t) = T_A e^{-kt}$

2. $y' + y\tan x = \sin 2x$
what is integrating factor

A) $\int \tan x\, dx$

B) $\int \sin x\, dx$

c) $\int \cot x\, dx$

D) $\int \cos x\, dx$

3. What is anatz?

$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x$

A) $C_1 \cos x + C_2 \sin x$

B) $K_1 + K_2 x$

c) A and B

D) none of the above

4. $\vec{y}' = A\vec{y}$ write as first order
find A

$y'' + 2y' - 24y = 0$

A) $\begin{bmatrix} 0 & 1 \\ 24 & -1 \end{bmatrix}$

B) $\begin{bmatrix} 0 & 1 \\ 24 & -2 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 \\ -24 & 2 \end{bmatrix}$

D) $\begin{bmatrix} 1 & 0 \\ 2 & -24 \end{bmatrix}$

Phys 89 22Apr19

- Briefly Back to Solving ODEs
- Fourier Transforms

$$m\ddot{y} + ky = F_0 \cos\omega t$$

$$y = \sum_{-\infty}^{\infty} c_n e^{in\omega t} \quad \Big\} \text{ Assume This}$$

$$\dot{y} = \sum_{-\infty}^{\infty} c_n e^{in\omega t} \, i\omega n \qquad \ddot{y} = \sum_{-\infty}^{\infty} c_n e^{in\omega t} (i\omega n)^2$$

$$F_0 \cos\omega t = F_0 \tfrac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) = \frac{F_0}{2} \sum_{-\infty}^{\infty} e^{in\omega t}\left(\delta_{n,1} + \delta_{n,-1}\right)$$

$$-m \sum_{-\infty}^{\infty} c_n e^{in\omega t} \omega^2 n^2 + k \sum_{-\infty}^{\infty} c_n e^{in\omega t} = \frac{F_0}{2} \sum_{-\infty}^{\infty} e^{in\omega t}\left(\delta_{n,1} + \delta_{n,-1}\right)$$

$$\sum_{-\infty}^{\infty} e^{in\omega t}\left[(-m\omega^2 n^2 + k) c_n - \frac{F_0}{2}\left(\delta_{n,1} + \delta_{n,-1}\right)\right] = 0$$

$$\sum_{-\infty}^{\infty} a_n \hat{e}_n = 0 \text{ iff } a_n = 0$$

$$(-m\omega^2 n^2 + k) c_n - \frac{F_0}{2}\left(\delta_{n,1} + \delta_{n,-1}\right) = 0$$

$$c_n = \frac{\frac{F_0}{2}\left(\delta_{n,1} + \delta_{n,-1}\right)}{k - m\omega^2 n^2}$$

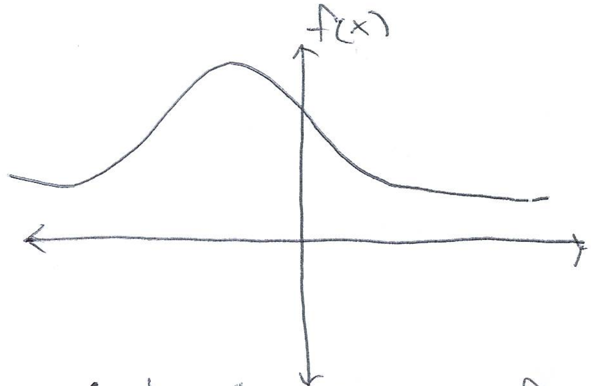
$$y = \sum_{-\infty}^{\infty} c_n e^{in\omega t} = \sum_{-\infty}^{\infty} \frac{F_0\left(\delta_{n,1} + \delta_{n,-1}\right)}{2(k - m\omega^2 n^2)} e^{in\omega t}$$

$$= \frac{F_0}{2}\left(\frac{e^{i\omega t}}{k - m\omega^2}\right) + \frac{F_0}{2}\left(\frac{e^{-i\omega t}}{k - m\omega^2}\right) = \frac{F_0}{k - m\omega^2} \cos\omega t$$

Fourier Transform

so far $\{e^{inx}\}$ $\{x^n\}$

use continuous basis $\{e^{ikx}\}$

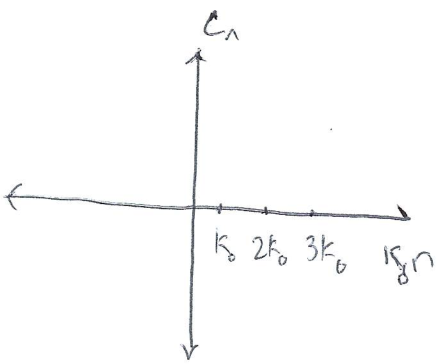


Can't really express in Fourier

so we allow $L \to \infty$

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{ink_0 x} \qquad k_0 = \frac{2\pi}{L} \to 0 \text{ as } L \to \infty$$

so $\{e^{ink_0 x}\}$



So if $k_0 \to 0$ as $L \to \infty$
These numbers squish together
making $\{e^{ink_0 x}\}$ continuous

$k_0 n \leftrightarrow k_n$

$$f(x) = \boxed{\sum_{-\infty}^{\infty}} c_n e^{ik_n x}$$

k_n is continuous variable

$$\rightarrow \boxed{f(x) = \int_{-\infty}^{\infty} c(k_n)\, e^{ik_n x}\, dk_n \quad \text{inverse Fourier Transform}}$$

$f(x)$ expanded in a continuous basis $\{e^{ikx}\}$

Let's explore this idea

→ We want to find $\boxed{c(k)}$ → This is the Fourier Transform

$$\vec{f}\cdot\vec{g} = \frac{1}{2\pi}\int f^*(x)g(x)dx$$ use dot product

$$\boxed{\hat{e}\cdot\vec{f} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx}$$

$$\hat{e}_k\cdot\vec{f} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\int_{-\infty}^{\infty} c(k')e^{ik'x}dk'dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k')\int_{-\infty}^{\infty} e^{-ikx}e^{ik'x}dx\,dk'$$

$$\hat{e}_k\cdot\vec{f} = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k')\boxed{\int_{-\infty}^{\infty} e^{i(k'-k)x}dx}dk'$$ → behaves like delta function

expect to behave like a delta function

$$f(u) = \int_{-\infty}^{\infty} f(u')\underline{\quad\quad}du' = \int_{-\infty}^{\infty} f(u')\delta(u'-u)du'$$

$$a_n = \sum_m \delta_{nm}a_m$$ Defines the delta function

$$\int_{-\infty}^{\infty} f(u')\delta(au')du'$$ sub $v = au'$

→ $a > 0$ $\quad \frac{dv}{a} = du'$

$$= \int_{-\infty}^{\infty} f\left(\frac{v}{a}\right)\delta(v)dv$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} f\left(\frac{v}{a}\right)\delta(v)dv = \frac{1}{a}f(0)$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} f\left(\frac{v}{a}\right)\delta(v)dv$$

$a < 0$

$$= \frac{1}{a}\int_{\infty}^{-\infty} f\left(\frac{v}{a}\right)\delta(v)dv$$

$$= \frac{-1}{a}\int_{-\infty}^{\infty} f\left(\frac{v}{a}\right)\delta(v)dv$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} f\left(\frac{v}{a}\right)\delta(v)dv$$

$$\int_{-\infty}^{\infty} f(u')\delta(au')du' = \frac{1}{|a|}f(0) = \frac{1}{|a|}\int_{-\infty}^{\infty} f(u')\delta(u')du'$$

$$\delta(au) = \frac{1}{|a|}\delta(u)$$

$k' - k = a$

$$\int_{-\infty}^{\infty} e^{iax}\,dx = \frac{1}{ia} e^{iax}\Big|_{-\infty}^{\infty}$$ not defined

$$\cancel{\lim} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iax}\,dx$$ add in gauss

$$\lim_{b\to 0} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iax} e^{-bx^2}\,dx$$

$$= \lim_{b\to 0} \frac{1}{2\pi}\sqrt{\frac{\pi}{b}}\, e^{-a^2/b}$$ $= 0$ except if $a = 0$

$$\delta(a) = \begin{cases} \infty & a = 0 \\ 0 & a \neq 0 \end{cases}$$

$$\hat{e}\cdot\vec{f} = \int_{-\infty}^{\infty} C(k')\,\delta(k'-k)\,dk' = C(k)$$

$$\boxed{C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx}$$ Fourier Transform

$$\boxed{f(x) = \int_{-\infty}^{\infty} C(k)e^{ikx}\,dk}$$ Inverse Fourier Transform

Properties of Fourier Transforms <u>Read</u>

FT of Dirac Delta Function

Wednesday <u>Convolution</u>

Phys 89 24 Apr 19

Properties of the Fourier Transform

Convolution

Solving ODEs

Linearity $a_1 f_1(t) + a_2 f_2(t) \overset{FT}{\rightleftharpoons} a_1 F_1(\omega) + a_2 F_2(\omega)$

where $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} f_2(t)dt = F_2(\omega)$

$\rightarrow FT \; f(t)$

Scaling $f(at) \overset{FT}{\rightleftharpoons} \frac{1}{|a|} F(\omega/a)$

Duality $F(t) \overset{FT}{\rightleftharpoons} \frac{1}{2\pi} f(\omega)$

Translation $\cancel{f(t)}\; f(t-t_0) \overset{FT}{\rightleftharpoons} e^{-i\omega t_0} F(\omega)$

Similarly $e^{i\omega_0 t} f(t) \overset{FT}{\rightleftharpoons} F(\omega-\omega_0)$

Conjugation $f^*(t) \overset{FT}{\rightleftharpoons} c^*(-\omega)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} e^{i\omega_0 t} f(t)dt = FT(e^{i\omega_0 t} f(t))$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)t} f(t)dt = F(\omega-\omega_0)$$

$$\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} f^*(t)\,dt \right)^*$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} f(t)\,dt = F(-\omega)$$

Derivatives $\quad \dfrac{d^n f(t)}{dt^n} \rightleftharpoons (i\omega)^n F(\omega)$

$$FT\left(\frac{df}{dt}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \frac{df}{dt}\,dt$$

$$= \frac{1}{2\pi} e^{-i\omega t} f(t) \Big|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt$$

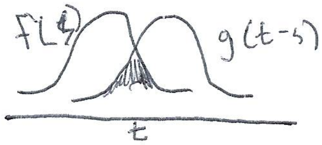
for $f(t) = 0$ as $t \to \pm\infty$

$$= i\omega F(\omega)$$

Convolution of two functions $g(t)$ and $f(t)$

$$h(t) = (f * g)(t) = \int_{-\infty}^{\infty} f(t-s)\,g(s)\,ds = \int_{-\infty}^{\infty} f(s)\,g(t-s)\,ds$$

$f(t)$ $\qquad$ $g(t)$



$s = 0$ overlap

$s > 0$ offset

$$f(t)\,g(t) \overset{FT}{\rightleftharpoons} (F*G)(\omega) \qquad f(t) \rightleftharpoons F(\omega)$$

$$(f*g)(t) \overset{FT}{\rightleftharpoons} 2\pi\, F(\omega)\,G(\omega) \qquad g(t) \rightleftharpoons G(\omega)$$

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt \qquad \cancel{G(\omega)}\; G(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivt} g(t)\,dt$$

$$(F*G)(\omega) = \int_{-\infty}^{\infty} F(\omega-v)\,G(v)\,dv$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\omega-v)t} f(t)\,dt \int_{-\infty}^{\infty} e^{-ivt'} g(t')\,dt\,dv$$

$$= \cancel{\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt\,dt' \int\int}$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dt\,dt'\, e^{-i\omega t} f(t)\,g(t') \underbrace{\int_{-\infty}^{\infty} e^{-iv(t-t')}\,dv}$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt\,dt'\, f(t)\,g(t')\, 2\pi\,\delta(t-t')\, e^{-i\omega t'}$$

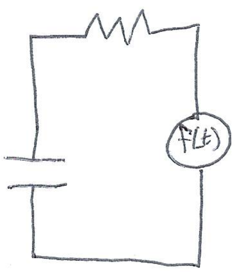
$$\int f(t)\,\delta(t-t') = f(t')$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t')\,g(t')\,e^{-i\omega t'}\,dt'$$

$$= FT(f(t)\,g(t))$$

$$f(t)\,g(t) \rightleftharpoons (F*G)(\omega)$$

$e^{i\omega_0 t} f(t) \rightleftharpoons F(\omega - \omega_0)$ $\qquad$ $\frac{d^n f}{dt^n} \rightleftharpoons (i\omega)^n F(\omega)$



$R\dot{q} + \frac{q}{C} = f(t)$

$FT\left(R\dot{q} + \frac{q}{C}\right) = FT(f(t))$

$q(t) \rightleftharpoons Q(\omega) \qquad f(t) \rightleftharpoons F(\omega)$

$Ri\omega Q(\omega) + \frac{1}{C} Q(\omega) = F(\omega)$

$\left(Ri\omega + \frac{1}{C}\right) Q(\omega) = F(\omega)$

$Q(\omega) = \frac{1}{R} \underbrace{\frac{1}{i\omega + \frac{1}{RC}}} \underbrace{F(\omega)}$

$\frac{1}{2\pi} \frac{1}{i\omega + \frac{1}{RC}} = G(\omega)$ Then $Q(\omega) = \frac{1}{R} \underbrace{G(\omega) F(\omega)}$

$Q(\omega) = \frac{1}{R} 2\pi G(\omega) F(\omega)$

Convolution is involved
For time domain

$q(t) = \frac{1}{R} (f * g)(t)$

$g(t) \rightleftharpoons \frac{1}{2\pi} \frac{1}{i\omega + \frac{1}{RC}}$ $\qquad$ $\frac{1}{i\omega\pi} \rightleftharpoons sgn(t)$

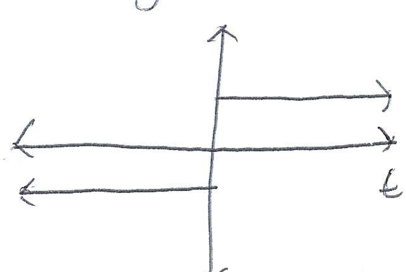
$\pi\, sgn(t) \rightleftharpoons \frac{1}{i\omega}$

FT(sgn(t))

$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} sgn(t)\, dt$.

$sgn(t) = \begin{cases} +1 & t > 0 \\ -1 & t < 0 \end{cases}$

$= \frac{-1}{2\pi} \int_{-\infty}^{0} e^{-i\omega t} dt + \frac{1}{2\pi} \int_{0}^{\infty} e^{-i\omega t} dt$



$= \lim_{a \to 0} \frac{-1}{2\pi} \int_{-\infty}^{0} e^{-i\omega t} e^{at} dt + \frac{1}{2\pi} \int_{0}^{\infty} e^{-i\omega t} e^{-at} dt$

$\ldots \lim_{a \to 0} \frac{1}{\pi} \frac{1}{i\omega - a} = \frac{1}{i\pi\omega}$

$$\frac{1}{i\omega + \frac{1}{RL}} = \frac{1}{i(\omega - \omega_0)} \qquad \omega_0 = \frac{+i}{RL}$$

$$G(\omega) = G'(\omega - \omega_0) = \frac{1}{2\pi} \frac{1}{i\omega + \frac{1}{RL}}$$

$$\frac{\pi}{2\pi} \operatorname{sgn}(t)\, e^{i(i t)/RL} = \frac{\pi}{2\pi} \operatorname{sgn}(t)\, e^{-1/RL}$$

$$q(t) = \left(\frac{1}{R} \frac{1}{2} \operatorname{sgn}(t) e^{-t/RL}\right) * (f(t))$$

$$= \frac{1}{2R} \int_{-\infty}^{\infty} f(t-s) \operatorname{sgn}(s)\, e^{-s/RL}\, ds$$

$$f(t) = \delta(t)$$

$$q(t) = \frac{1}{2R} e^{-t/RL} \qquad \boxed{\text{Try}}$$

Phys 89 26 Apr 19 | Kreyszig Ch 6

C

Laplace Transform of $f(t)$

$$\mathcal{L}(f(t)) = F(p) = \int_0^\infty e^{-pt} f(t)\, dt$$

Note similarity with Fourier Transform

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt \quad \text{and} \quad f(t) = \int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\, d\omega$$

$f(t)$	$F(p)$
1	$1/p$
e^{-at}	$\frac{1}{p+a}$
t^n	$\frac{n!}{p^{n+1}}$
$\cos\omega t$	$\frac{p}{p^2+\omega^2}$
$\sin\omega t$	$\frac{\omega}{p^2+\omega^2}$

$f(t) = 1$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-pt} \cdot dt = \frac{-1}{p} e^{-pt}\Big|_0^\infty = \frac{-1}{p}(0-1) = \frac{1}{p}$$

$$\mathcal{L}^{-1}\left(\frac{1}{p}\right) = 1$$

$f(t) = e^{-at}$ $\quad \mathcal{L}(f(t)) = \int_0^\infty e^{-at} e^{-pt}\, dt = \int_0^\infty e^{-(a+p)t}\, dt = \frac{1}{p+a}$

$f(t) = t^n$

$$\mathcal{L}(f(t)) = \int_0^\infty t^n e^{-pt}\, dt \qquad v = e^{-pt}$$

$$dv = -pe^{-pt}\, dt \quad -\frac{dv}{p} = e^{-pt}\, dt$$

$$= t^n e^{-pt}\Big|_0^\infty + \frac{1}{p}\int_0^\infty n e^{-pt} t^{n-1}\, dt = \frac{n}{p}\int_0^\infty e^{-pt} t^{n-1}\, dt = \frac{n}{p}\mathcal{L}(t^{n-1})$$

$$\downarrow$$
$$0$$

$$\mathcal{L}(t^n) = \frac{n}{p}\mathcal{L}(t^{n-1}) = \frac{n}{p}\cdot\frac{n-1}{p}\mathcal{L}(t^{n-2}) = \frac{n}{p}\,\frac{n-1}{p}\,\frac{n-2}{p}\mathcal{L}(t^{n-3}) = \cdots$$

$$\mathcal{L}(t^n) = \frac{n(n-1)(n-2)\cdots 2\, 1}{p\cdot p\cdot p\cdots p\, p} = \frac{n!}{p^{n+1}}$$

$$f(t) = \cos(wt) = \frac{1}{2}(e^{iwt} + e^{-iwt}) = \mathrm{Re}(e^{iwt})$$

$$\mathcal{L}(F(t)) = \int_0^\infty \cos wt\, e^{-pt}\, dt = \int_0^\infty e^{-pt}\,\mathrm{Re}(e^{iwt})\, dt = \mathrm{Re}\int_0^\infty e^{-pt} e^{iwt}\, dt$$

$$* \int_0^\infty e^{-pt} e^{iwt}\, dt = \int_0^\infty e^{-(p-iw)t}\, dt = \frac{-1}{p-iw} e^{-(p-iw)t}\Big|_0^\infty = \frac{-1}{p-iw}(-1) = \frac{1}{p-iw}$$

$$\frac{p+iw}{(p-iw)(p+iw)} = \frac{p+iw}{p^2+w^2} \qquad \mathrm{Re}\left(\frac{p+iw}{p^2+w^2}\right) = \frac{p}{p^2+w^2} = \mathcal{L}(\cos wt)$$

Shifting Theorem

$$\mathcal{L}(e^{-at}f(t)) \quad \text{know } \mathcal{L}(f(t)) = F(p)$$

$$= \int_0^\infty e^{-at} e^{-pt} f(t)\,dt = \int_0^\infty e^{-(a+p)t} f(t)\,dt = F(p+a)$$

$$\mathcal{F}(e^{i\omega_0 t}f(t)) = F(\omega-\omega_0) \Big\} \text{ Recall}$$

We use this to help solve ODEs

$$y'' + ay' + by = r(t)$$

Take Laplace Transform of both sides

$$\mathcal{L}(y') = \int_0^\infty \underbrace{e^{-pt}}_{u} \underbrace{\frac{dy}{dt}\,dt}_{dv} = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= e^{-pt}y(t)\Big|_0^\infty + p\int_0^\infty e^{-pt}y(t)\,dt \qquad \mathcal{L}(y(t)) = Y(p)$$

$$= -y(0) + p\,\mathcal{L}(y(t)) = pY(p) - y(0)$$

Give particular and homogeneous solutions while Fourier only gives particular solution due to initial conditions in Laplace

$$\mathcal{L}(y'') = \int_0^\infty e^{-pt}\frac{d^2y}{dt^2}\,dt = \int_0^\infty e^{-pt}\frac{d(\dot{y})}{dt}\,dt = p\,\mathcal{L}(\dot{y}) - \dot{y}(0)$$

$$= p(pY(p) - y(0)) - \dot{y}(0) = p^2Y(p) - p(y(0)) - \dot{y}(0)$$

32
4
12 8

$$p^2 Y - pY_0 - \dot{Y}_0 + apY - aY_0 + bY$$

$$\cancel{p^2 Y} \; (p^2 + ap + b)Y - (p+a)Y_0 - \dot{Y}_0 = \mathcal{L}(r(t)) = R(p)$$

If $Y_0 = \dot{Y}_0 = 0$

$$R(p) = (p^2 + ap + b)Y \qquad Y(p) = \frac{R(p)}{p^2 + ap + b}$$

$$y(t) = \mathcal{L}^{-1}(Y(p)) = \mathcal{L}^{-1}\left(\frac{R(p)}{p^2 + ap + b}\right)$$

$$\ddot{y} + 4\dot{y} + 4y = t^2 e^{-2t}$$

$$\mathcal{L}(t^2 e^{-2t}) \; \cancel{\frac{2}{p^3+}} = \frac{2!}{(p+2)^3} \qquad Y_0 = \dot{Y}_0 = 0$$

$$Y(p) = \frac{2!}{(2+p)^3} \cdot \frac{1}{p^2 + 4p + 4} = \frac{2}{(p+2)^5}$$

$$\mathcal{L}(Y) = \mathcal{L}^{-1}\left(\frac{2!\,4!}{(p+2)^5\,4!}\right) = \frac{2!}{4!} t^4 e^{-2t} = \frac{1}{12} t^4 e^{-2t}$$

Phys 89 29Apr19

Ch 6 Kreyszig

$f(t)$	$\mathcal{L}(f(t)) = F(p)$
1	$1/p$
t^n	$\frac{n!}{p^{n+1}}$
e^{-at}	$\frac{1}{p+a}$
$\cos\omega t$	$\frac{p}{p^2+\omega^2}$
$\sin\omega t$	$\frac{\omega}{p^2+\omega^2}$

- Step functions
- convolution in Laplace

$$y'' + y = t \qquad y(0) = y_0 = 1 \quad y'(0) = y_0' = 0$$

$$\mathcal{L}(LHS) = \mathcal{L}(RHS)$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(t)$$

$$p^2 Y - p y_0 - y_0' + Y = \frac{1}{p^2}$$

$$Y(p^2+1) = \frac{1}{p^2} + p \qquad Y = \frac{1+p^3}{p^2(p^2+1)} = \frac{1}{p^2} - \frac{1}{p^2+1} + \frac{p}{p^2+1}$$

$$\frac{A}{p} + \frac{B}{p^2} + \frac{C}{p+i} + \frac{D}{p-i} \qquad Ap(p^2+1) + B(p^2+1) + Cp^2(p-i) + Dp^2(p+i) = 1$$

$$A = 1, \quad B = -1$$

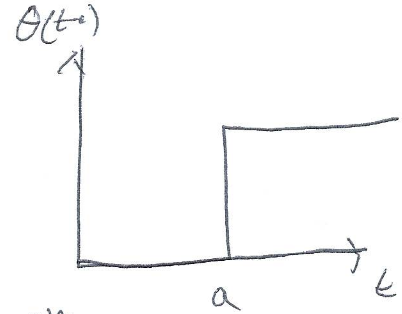
$$y(t) = t - \sin t + \cos t$$

Shifting Theorem $\mathcal{L}\{e^{at} f(t)\} = F(p-a)$

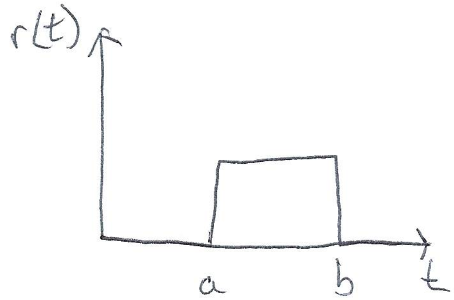
Laplace of an integral $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{p}F(p)$, $F(p) = \mathcal{L}(f(t))$

Step Function (Heaviside function)

$$\theta(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$$

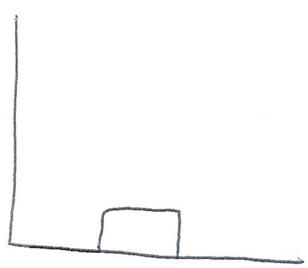


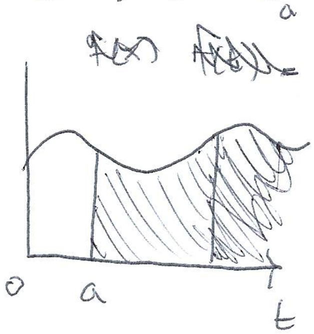
Say $r(t)$ is a pulse of duration $\tau = b-a$



$$r(t) = \theta(t-a) - \theta(t-b)$$

$$\mathcal{L}(\theta(t-a)) = \int_0^\infty e^{-pt}\theta(t-a)\,dt = \int_a^\infty e^{-pt}\,dt = \frac{-1}{p}e^{-pt}\Big|_a^\infty = \frac{1}{p}e^{-ap}$$



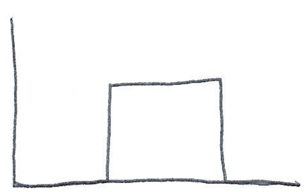


$r(t) = f(t-a)\theta(t-a)$

driving function starts at $a = t$

~~so $\mathcal{L}(r(t)) = e^{-at}\mathcal{L}(f(t)\theta(t))$~~

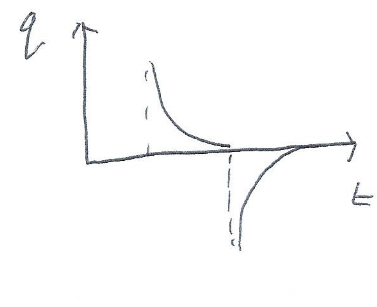
$$R\dot{q} + \frac{q}{C} = V_0(\theta(t-a) - \theta(t-b))$$



So $\mathcal{L}(r(t))$

~~$\mathcal{L}^{-1}(R(p)) = e^{at}\mathcal{L}^{-1}(\mathcal{L}(f(t)\theta(t)))$~~

$$\mathcal{L}^{-1}(R(p)) = e^{-at}\mathcal{L}^{-1}(\mathcal{L}(f(t-a)\theta(t-a)))$$



Convolution

$$\cancel{f*g = \int_0^\infty f(t)\, g(t)}$$

$$f*g = \int_0^\infty f(\tau)\, g(t-\tau)\, d\tau$$

Just as in FT we'll find products of transforms we recognize like $F(p)\,G(p)$

$$\mathcal{L}(f*g) = F(p)\,G(p)$$

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt \qquad G(p) = \int_0^\infty e^{-ps} g(s)\, ds$$

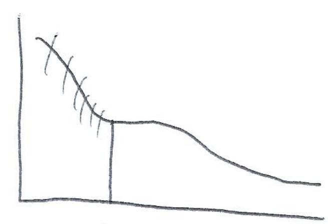
$\tau = s+t$ t is treated as constant for $G(p)$

$$G(p) = \int_t^\infty e^{-p(\tau-t)}\, g(\tau-t)\, d\tau$$

$$F(p)\,G(p) = \int_0^\infty e^{-pt} f(t)\, dt \int_t^\infty e^{-p(\tau-t)}\, g(\tau-t)\, d\tau$$

$$= \int_0^\infty f(t)\, e^{-pt} dt\, e^{pt} \int_t^\infty e^{-p\tau} g(\tau-t)\, d\tau = \int_0^\infty f(t)\, e^{-pt} e^{pt} \int_t^\infty e^{-p\tau} g(t+\tau)\, d\tau\, dt$$

$$= \int_0^\infty f(t) \int_t^\infty e^{-p\tau} g(t+\tau)\, d\tau\, dt = \int_0^\infty f(t) \int_0^\infty e^{-p\tau} g(\tau-t)\,\theta(\tau-t)\, d\tau\, dt$$



$$= \int_0^\infty \int_0^\infty f(t)\, e^{-p\tau} g(\tau-t)\,\theta(\tau-t)\, dt\, d\tau$$

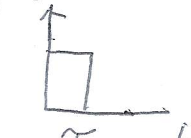
$$= \int_0^\infty e^{-p\tau} \int_0^\infty f(t)\, g(\tau-t)\,\theta(\tau-t)\, dt\, d\tau$$

$=$

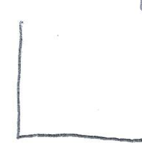
$$\int_0^\infty f(t)\, g(\tau-t)\,\theta(\tau-t)\, dt = \int_0^\tau f(t)\, g(\tau-t)\, dt$$

$$= f*g$$

in t space



in τ space



Phys 89 01May19

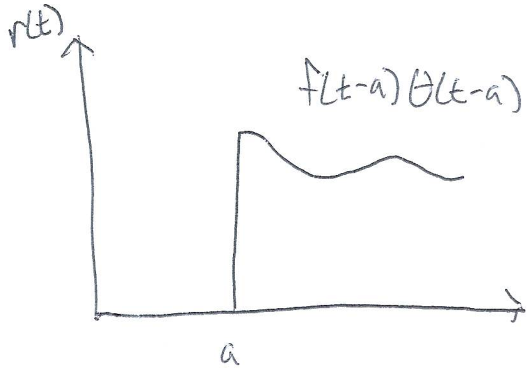
Convolution

$\mathcal{L}(f*g) = F(p)\,G(p)$

$f*g = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$

$\mathcal{L}(\theta(t-a)) = \frac{e^{-ap}}{p}$

$f(t)$	$F(p)$
1	$1/p$
t	$1/p^2$
t^n	$n!/p^{n+1}$
e^{at}	$\frac{1}{p-a}$
$\sin wt$	$\frac{w}{p^2+w^2}$
$\cos wt$	$\frac{p}{p^2+w^2}$
$\theta(t-a)$	$\frac{e^{-ap}}{p}$

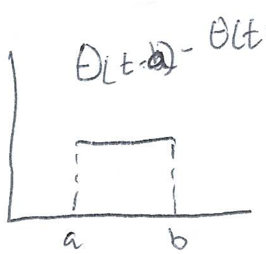


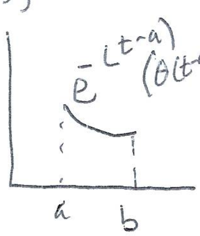
$$\mathcal{L}(f(t-a)\theta(t-a)) = \int_0^\infty e^{-pt} f(t-a)\,\theta(t-a)\,dt$$

$$= \int^\infty e^{-pt} f(t-a)\,dt \qquad \tau = t-a$$

$$= \int_0^\infty e^{-p(\tau+a)} f(\tau)\,d\tau = e^{-pa}\int_0^\infty e^{-p\tau} f(\tau)\,d\tau$$

$$= e^{-pa} F(p)$$





$y'' + 3y' + 2y =$

$$y'' + 3y' + 2y = e^{-(t-1)}\left(\theta(t-1) - \theta(t-2)\right) \qquad y(0)=0 \quad y'(0)=0$$

$$p^2 Y - py(0) - y'(0) + 3(pY - y(0)) + 2Y = \mathcal{L}(r(t)) = R$$

$$Y(p^2+3p+2) = R \Rightarrow Y = \frac{R}{p^2+3p+2} = \frac{R}{(p+2)(p+1)} = R\left(\frac{1}{(p+2)(p+1)}\right)$$

$$= QR \qquad y = q * r = \int_0^t q(\tau)\, r(t-\tau)\, d\tau$$

So reduced our problem to simply finding $q(t) = \mathcal{L}^{-1}(Q(p))$ and $r(t)$

$$A(p+1) + B(p+2) = 1 \qquad Y = R\left(\frac{1}{p+1} - \frac{1}{p+2}\right)$$

$$B + A = 0 \qquad A + 2B = 1$$

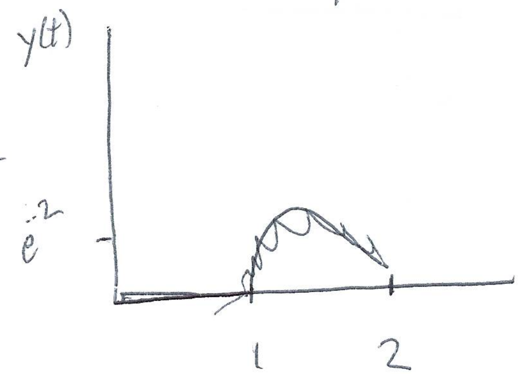
$$B = 1 \qquad \mathcal{L}^{-1}(Q) = e^{-t} - e^{-2t} = q(t)$$

$$A = -1$$

Convolve $r(t)$ and $q(t)$

$$r*g = \int_0^t r(\tau)\, q(t-\tau)\, d\tau = \int_0^t e^{-(\tau-1)}\left(\theta(\tau-1)-\theta(\tau-2)\right)\left(e^{-(t-\tau)} + e^{-2(t-\tau)}\right) d\tau$$

Three time regimes But each has a different functional form since t enters the limits of the integral and the function you are integrating

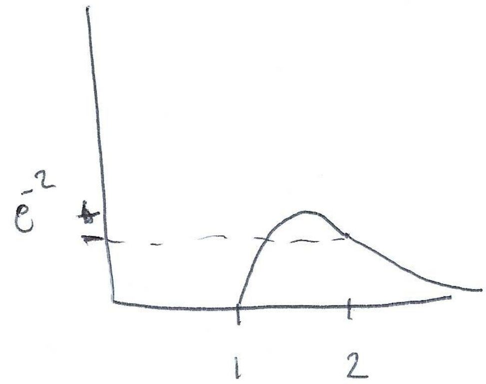


$t<1$ $\quad y(t) = \int_0^1 0\, d\tau = 0$

$2>t>1$ $\quad \int_1^t e^{-(\tau-1)}\left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau$ ← stays at t since we are looking at pulse while it is happening

$$= \int_1^t e^{-t+1} - e^{-2t+\tau+1}\, d\tau = \left(\tau e^{-t+1} - e^{-2t+1} e^{\tau}\right)\Big|_1^t = te^{-t+1} - e^{-t+1} - \left(e^{-2t+1}e^{t} - e^{-2t+1}e\right)$$

$$= e^{-t+1}\left(t-1-1+e^{-t+1}\right) = e^{-t+1}\left(t-2+e^{-t+1}\right)$$



$t > 2$

$$\int_1^2 e^{-(\tau-1)} e^{-(t-\tau)} e^{-2(t-\tau)} \, d\tau$$

$$= e^{-t+1} \tau \Big|_1^2 - e^{-2t+1} e^{\tau} \Big|_1^2$$

$$= e^{-t+1}(2-1) - e^{-2t+1}(e^2 - e^1)$$

$$= e^{-t+1}(1) - (e^{-2t+3} + e^{-2t+2})$$

$$= e^{-t+1}(1 - e^{-t+2} + e^{-t+1})$$

Definition

$$f(y) = \int_{-\infty}^{\infty} f(x)\,\delta(y-x)\,dx$$

for FT

$$\cancel{f(y) = \int_0^t}$$

$$f(\tau) = \int_0^{\infty} f(t)\,\delta(\tau - t)\,dt$$

$$\mathcal{L}(\delta(\tau - t)) = \int_0^{\infty} e^{-pt}\,\delta(t-a)\,dt = e^{-ap}$$

$$y'' + 3y + 2y = \delta(t-1)$$

$$\mathcal{L}(L) = e^{-p}$$

Phys 89 03May19

1[Eigenvalues / Eigenvectors

2[ODE rapid fire techniques

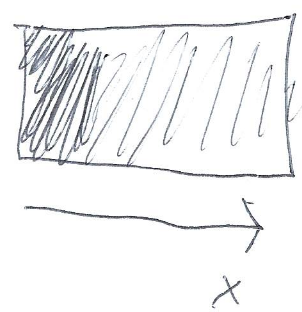
- separation of variables
- method of undetermined coefficients
- reduction of order

3[Power series solutions

4[Fourier transforms

5[Laplace transforms

PDEs



Heat Equation $\quad \dfrac{\partial u}{\partial t} = \alpha \nabla^2 u$

Wave equation $\quad \dfrac{\partial^2 u}{\partial t^2} = V^2 \dfrac{\partial^2 u}{\partial x^2}$ or $= V^2 \nabla^2 u$

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \psi + V\psi$$

function of x, y, z

For example

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$ (only one dimension)
only time and space

Fourier Transform of spatial component

In time and space
$u(x,t)$

Fourier and time space
$c(k,t)$
$\uparrow$ wave number

$$FT(y'') = -k^2 FT(y) = -k^2 c(k,t)$$

$$y'' = \frac{\partial^2 y}{\partial x^2} \qquad FT\left(\frac{\partial^2 u}{\partial t^2}\right) = \frac{\partial^2 c(k,t)}{\partial t^2}$$

$$\frac{\partial^2 c(k,t)}{\partial t^2} = -v^2 k^2 c(k,t)$$

$$\ddot{c} = -v^2 k^2 c \qquad \ddot{c} + v^2 k^2 c = 0$$

By taking FT I get an equation that has derivatives in only one variable

This can be solved as an ODE

Heat Equation

$$\dot{c} = -\alpha^2 k^2 c$$

Schrödinger Equation

$$\tilde{V} = FT(V(x))$$

$$i\hbar \dot{c} = \frac{k^2 \hbar^2}{2m} c + \tilde{V} * c \qquad c = FT(\psi(x,t))$$

Solve the ODE using whatever technique

Then IFT back

2) Separation of variables

(Basically if you have

$u(x,t) \to X(x)T(t)$

Get two ODEs

one in x and another in t

$u(x,t) = X(x)T(t)$ "unentangled" functions

"uncoupled" variables

$\cos(k_1x)e^{i\omega_1 t} + \sin(k_2x)e^{i\omega_2 t} = u(x,t)$

This function cannot be separated but its components can.

<u>Step 1</u> Assume $u(x,t) = X(x)T(t)$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial t^2}(X(x)T(t)) = X(x)\frac{d^2T}{\partial t^2} = v^2\frac{\partial^2}{\partial x^2}(X(x)T(t)) = v^2 T\frac{\partial^2 X}{\partial x^2}$$

$$X\ddot{T} = v^2 T X'' \qquad X'' = \frac{\partial^2 X}{\partial x^2}$$

$$X\ddot{T} - v^2 T X'' = 0$$

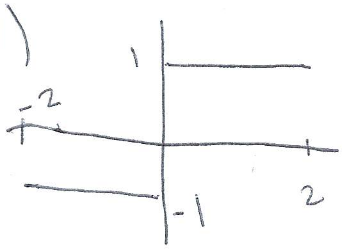
<u>Step 2</u> separate variables divide by XT

$$\frac{\ddot{T}}{T} - v^2 X X'' = 0 \qquad f(t) + g(x) = 0$$

$$\frac{\ddot{T}}{T} = C^2 \qquad v^2 X X'' = C^2 \qquad f(t) = C = -g(x)$$

$$\ddot{T} = \omega^2 T \qquad X'' = \frac{\omega^2}{v^2}X$$

1)



A) $\frac{1}{2\pi n}(1-(-1)^n)$

B) $\frac{1}{\pi n}(1-(-1)^n)$

C) $\frac{1}{2\pi n}$

D) $\frac{1}{\pi n}$

2) $f(x) = \begin{cases} e^{2ix} & \text{if } -1<x<1 \\ 0 & \text{otherwise} \end{cases}$

$$\frac{1}{2\pi}\int e^{2ix}e^{-ikx}\,dx = \frac{1}{2\pi}\int e^{i(2x-kx)}\,dx$$

=

A) $\frac{-1}{4\pi}\sin(w-2)/w$

B) $\frac{1}{4\pi}\sin(w-2)/(w-2)$

C) $\frac{-1}{4\pi}\sin(w-2)/w-2$

D) $\frac{1}{4\pi}\sin w/w$

3) Laplace transform of sinhat

A) $\frac{a}{s^2+a^2}$

B) $\frac{s}{s^2-a^2}$

C) $\frac{a}{s^2-a^2}$

D) $\frac{s}{s^2-a^2}$

4) find $t * e^t$

$$\int_0^t \tau e^{t-\tau}\,d\tau$$

A) $e^t - t - 1$

B) $e^{-t} - t - 1$

C) $e^{-t} - t + 1$

D) $e^t + t + 1$

89 28 Jan 19

Discussion

Classroom empty before class at least 30 mins.

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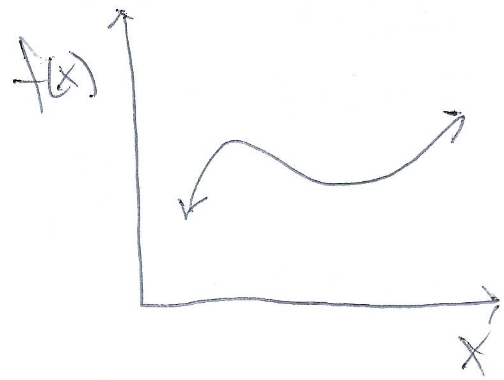
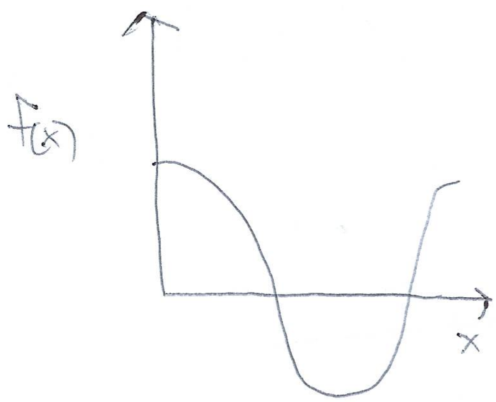
Email: ~~musatyuk~~ musatyuk@berkeley.edu

Taylor Expansion



$f(x) = \cos x$, $\cos(0.001)$ is complicated to solve

Find an approximation polynomial that is easier to solve



$\cos(0) = 1$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{[n]}(a)$$

$\frac{1}{100} \sim (x-a)$ is small

$$f(x) = c_0 + c_1 \frac{1}{100} + c_2 \left(\frac{1}{100}\right)^2 + \dots$$

A few terms are relevant since the larger terms are negligibly small

$a = 0$

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\log(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$(1+x)^n \approx 1 + nx + \frac{1}{2}(n-1)nx^2 + \dots$$

$f(x)g(x)$, $f(x) = e^x$, $g(x) = \cos x$

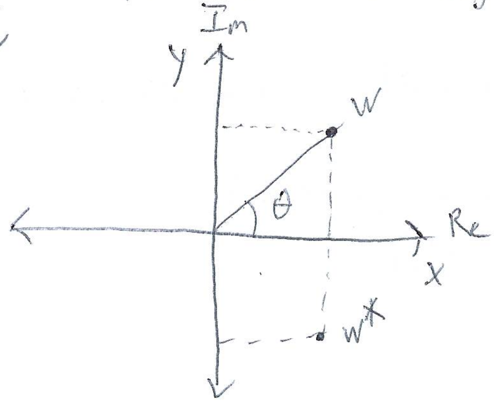
$$f(x)g(x) = e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \dots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)$$

Complex Numbers

$$a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 = 0$$

Always n solutions using complex numbers

$\mathbb{C}$



$Re(z) = x$

$Im(z) = y$

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

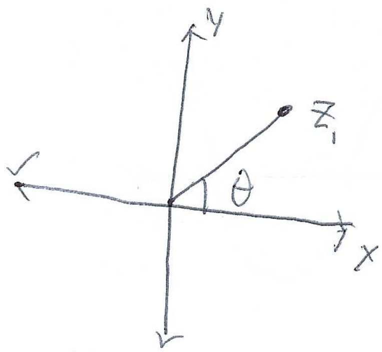
$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$z_1 z_2 = x_1 x_2 + i^2 y_1 y_2 + i(y_1 x_2 + y_2 x_1)$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

switches sign of i

$z^* = x - iy$



$z = r\cos\theta + ri\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$z_1 = x_1 + iy_1$

$\sqrt{x_1^2 + y_1^2} = |z_1| = \sqrt{z_1 z_1^*}$

$\tan\theta = \frac{y}{x}$

$Arg(z) = \theta$

$\theta = \arctan\left(\frac{y}{x}\right)$

- Taylor Expand 2 functions
- $E = \gamma mc^2$, $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

taylor expand

- $z = x+iy$ calculate $\sqrt{zz^*}$
- Taylor expand $e^{i\theta}$, $\cos\theta + i\sin\theta$ Do they match

$\cos x = f(x)$ $\quad f(x) = 1 - \cancel{\frac{x^2}{3!}} \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{[4]}(x) = \cos x$

$\sin x = f(x)$ $\quad f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{[4]}(x) = \sin x$

$f^{[5]}(x) = \cos x$

$$E(v) = \gamma mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{mc^2}{\left(1-\left(\frac{v}{c}\right)^2\right)^{1/2}} = mc^2\left(1 + -\left(\frac{v}{c}\right)^2\right)^{-1/2}$$

$$= \cancel{x} + mc^2\left(1 + \frac{-1}{2}\left(\frac{-v^2}{c^2}\right) + \frac{1}{2}\left(\frac{-3}{2}\right)\left(\frac{+1}{2}\right)\left(\frac{-v^2}{c^2}\right)^2 + \ldots\right.$$

$$= mc^2\left(1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \ldots\right)$$

$$= mc^2 + \frac{mv^2}{2} + \frac{3mv^4}{8c^2} + \ldots$$

$z = x + iy$, find $\sqrt{zz^*}$

$z^* = x - iy$

$|z| = \sqrt{zz^*} = \sqrt{(x+iy)(x-iy)} = \sqrt{x^2+y^2}$

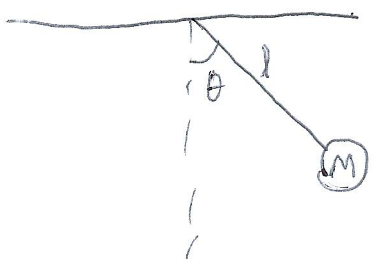
$e^{i\theta}$, $\cos\theta + i\sin\theta$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \dots$$

$$\cos\theta = 1 - \frac{\cancel{\theta}^2}{2!} + \frac{\cancel{\theta}^4}{4!} + \dots = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots$$

$$i\sin\theta = i\theta - \frac{i(\theta)^3}{3!} + \frac{i(\theta)^5}{5!} + \dots = i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} + \dots$$

$e^{i\theta} = \cos\theta + i\sin\theta$



$$\frac{d^2\theta}{dt^2} = \frac{-g}{l}\sin\theta \quad , \theta(t) = ?$$

$$\ddot{\theta} = \frac{-g}{l}\left(\theta - \frac{\theta^3}{3!} + \dots\right)$$

$$\ddot{\theta} = \frac{-g\theta}{l}$$

expand $\frac{1}{1+x^2}$

$$\frac{1}{1-(-x^2)} = 1 + y + y^2 + y^3 + \dots = 1 - x^2 + x^4 - x^6 + x^8 + \dots$$

$$\sum_{n=0}^{\infty} (-1)^n x^{2n}$$

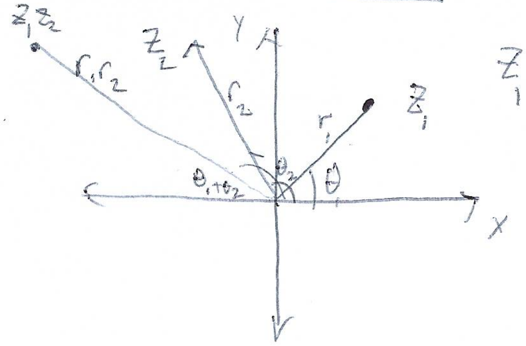
$\frac{1}{2} = 1 + -1 + 1 - 1 + 1 - 1 + \dots$ $\quad x = \pm i$

04 Feb 19 89

Discussion

Classroom is empty hours before class.
Policy says it is not allowed to be in classroom outside of registered hours. Not sure if enforced

Complex numbers



$z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$e^{i\pi} + 1 = 0$

Complex Functions

$f(x) = x^2$ $\quad$ $f(z) = z^2$

$f: \mathbb{R} \to \mathbb{R}$ $\quad$ $f: \mathbb{C} \to \mathbb{C}$

Derivatives

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$f(z) = z^2$

$\Delta z = \varepsilon$

$$\frac{df}{dz} = \lim_{\varepsilon \to 0} \frac{(z+\varepsilon)^2 - z^2}{\varepsilon} = 2(z+\varepsilon) \to 2z$$

$\Delta z = i\varepsilon$

$$\lim_{\varepsilon \to 0} \frac{(z+i\varepsilon)^2 - z^2}{i\varepsilon} = 2z - \varepsilon \to 2z$$

Derivative condition

$f(z)$ $\frac{df}{dz}$ is well defined if $f(z)$ is a function of only z.

$f(z) = z z^*$

not analytic

Analytic Functions

Analytic

$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \to f' = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$

$f(z) = \sin z, \cos z, e^z \to f' = \cos z, -\sin z, e^z$

$f(z) = e^{z^2}$

$Z = x + iy$

$f(z) = u(x,y) + iv(x,y)$

$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

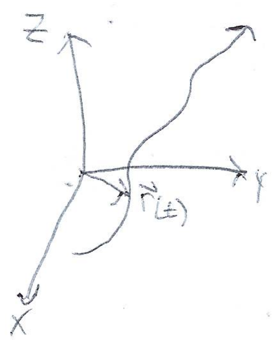
$u(x,y) = x^2 - y^2 \,,\; v(x,y) = 2xy$

Cauchy - Riemann

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \,,\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Vector spaces



$$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\frac{d\vec{r}(t)}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

$$\vec{F} = m\vec{a} \rightarrow \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = m\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}$$

$\vec{V}_1, \vec{V}_2$

$\vec{V}_1 + \vec{V}_2 = \vec{V}_2 + \vec{V}_1$

$\cancel{\alpha\vec{v}}\; \lambda\vec{V}_1 + \vec{V}_2$

$\lambda\vec{V}_1 + \alpha\vec{V}_2$

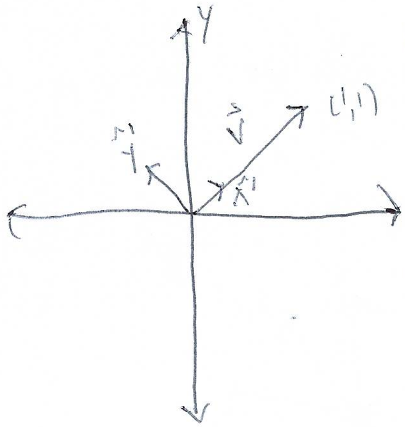
Vector space

- A collection of vectors
- add them together
- multiply them by a constant
- $\vec{V}_1, \vec{V}_2 \in \mathbb{V}$

$\vec{V}_1 + \vec{V}_2 \in \mathbb{V}$

$\lambda\vec{V}_1 \in \mathbb{V}$

Basis



$$\vec{v} = v_1\hat{x} + v_2\hat{y} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{v} = \hat{x} + \hat{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\hat{x}' = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v}' = \hat{x}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{y}' = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

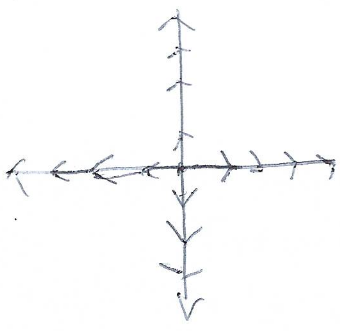
Span

$$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n\}$$

span(↙) = collection of vectors $\vec{w} = \lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 + \dots + \lambda_n\vec{v}_n$

$\lambda_1, \dots, \lambda_n \in \mathbb{R}$

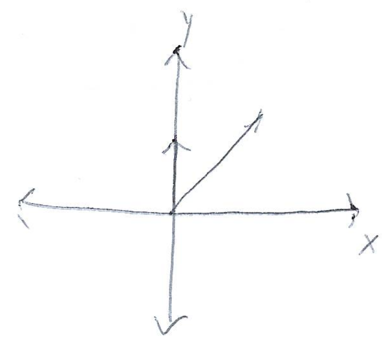
$\mathbb{R}^2$



$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ spans real axis

$$\lambda\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}$$

$$\lambda\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$$



$$\text{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}\right)$$

$$\vec{w} = \lambda_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$\text{span}\left(\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}\right)$$

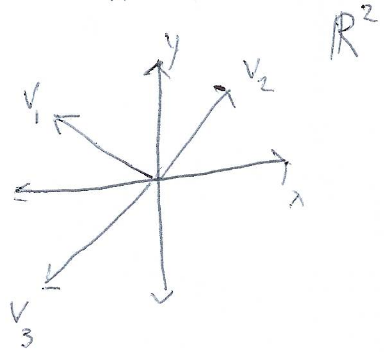
Linear dependence

$\{\vec{v}_1, \dots, \vec{v}_n\}$ linearly independent if the only solution to

$\vec{0} = \lambda_1\vec{v}_1 + \cdots + \lambda_n\vec{v}_n$ consists of $\lambda_1 = \dots = \lambda_n = 0$.

$$\lambda_1\begin{bmatrix}1\\1\end{bmatrix} + \lambda_2\begin{bmatrix}-1\\-1\end{bmatrix} = 0$$

$$\lambda_1 = \lambda_2$$



$\mathbb{R}^n$ n linearly independent vectors

Problems

1) $f(z) = z^2 + 3z = (x+iy)^2 + 3(x+iy) = x^2 + 2ixy - y^2 + 3x + 3iy$

Decompose into $f = u + iv$

$= x^2 - y^2 + 3x + 3iy + 2ixy$

$u(x,y) = x^2 + 3x - y^2$

$v(x,y) = 3y + 2xy$

2) $f(z) = zz^*$ Find $\frac{df}{dz}$ along

$\Delta z = \varepsilon$, $\Delta z = i\varepsilon$ does the limit agree? Does Cauchy riemann hold

$\lim_{\varepsilon \to 0} \frac{f(z+\varepsilon) - f(z)}{\varepsilon}$ $\quad zz^* = x^2 + y^2$

$z + z^*$

$(x + iy + \varepsilon)(x - iy + \varepsilon)$

$x^2 + y^2 + \varepsilon x - i\varepsilon y + \varepsilon^2 + \varepsilon x - i\varepsilon y$

$(x + iy + i\varepsilon)(x - iy - i\varepsilon)$

$\lim_{i\varepsilon \to 0}$ $\quad -z + z^*$

$x^2 + y^2 - i\varepsilon x + i\varepsilon x + \varepsilon^2$

They disagree

$\frac{\partial f_u}{\partial x} = 2x \qquad \frac{\partial f_v}{\partial y} = 0$

3) $f(z) = e^{z^2} = e^{z z^*} = \cancel{\cos z^2 + i \sin z^2}$

$e^{(x+iy)^2} = e^{\cancel{(x^2-y^2)}} = e^{(x^2+2iyx-y^2)} = e^{x^2-y^2} e^{2iy}$

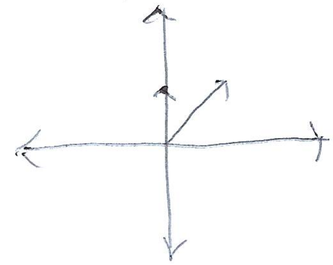
$f(z) = u + iv$

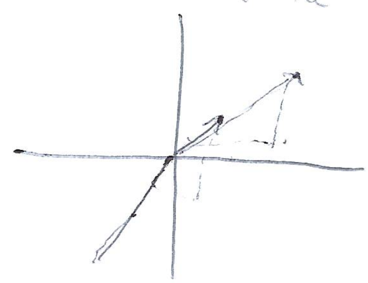
$\cancel{\cos(x^2-y^2)} + (i \sin(2yx) + \cos 2xy) e^{x^2-y^2}$

$\ln f = x^2 - y^2 + 2iy$

4) 3d span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ what does the span of the two sets look like

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix} \right\}$





5) linear independence $v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$

what is the condition for linear independence

$0 = a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3 = \vec{0}$

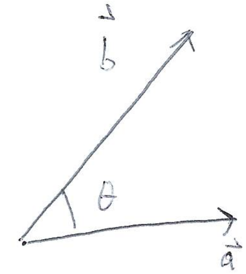
$a_1, a_2, a_3 = 0$

Physics 89 11 Feb 19
Discussion

Dot Product

$\vec{a} \cdot \vec{b} = \#$

$\vec{a} \cdot \vec{a} \geq 0$



$$\left.\begin{array}{l}\vec{a} = (a_1, a_2, a_3) \\ \vec{b} = (b_1, b_2, b_3)\end{array}\right] \text{Cartesian}$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

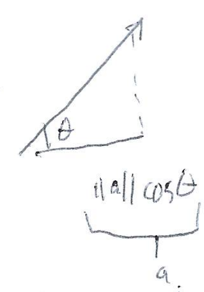
$$\vec{b} = b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3$$

$$\vec{a} \cdot \vec{b} = \vec{a} \cdot \left(\sum_{i=1}^{3} b_i \hat{e}_i \right)$$

$$= \sum_{i=1}^{3} b_i \, \vec{a} \cdot \hat{e}_i$$

$$= \sum_{i=1}^{3} b_i \|a\| \cos\theta_i$$

$$= \sum a_i b_i$$



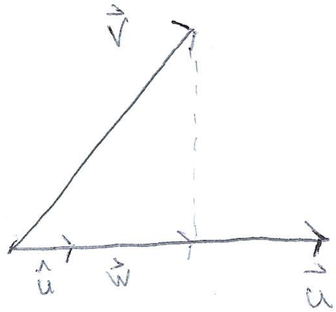
Gram-Schmidt Process

Given $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

Find n orthogonal unit vectors

Projections

$$\vec{w} = \text{proj}_{\vec{u}}(\vec{v}) = (\hat{u} \cdot \vec{v})\hat{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \hat{u}$$



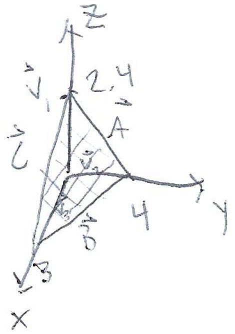
$$\vec{u}_1 = \vec{v}_1 \Rightarrow \hat{e}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{\vec{v}}{\sqrt{\vec{v} \cdot \vec{v}}}$$

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\hat{e}_1}(\vec{v}_2)$$

$$\cancel{\hat{e}_2} = \vec{v}_2 - (\hat{e}_1 \cdot \vec{v}_2)\hat{e}_1$$

$$\hat{e}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|}$$

$$\vec{u}_3 = \vec{v}_3 - \text{proj}_{e_1}(\vec{v}_3) - \text{proj}_{e_2}(\vec{v}_3)$$



$$\vec{V}_1 = \begin{bmatrix} 0 \\ 0 \\ 2{,}4 \end{bmatrix}, \quad \vec{V}_2 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \quad \vec{V}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{A} = \vec{V}_2 - \vec{V}_1 = \begin{bmatrix} 0 \\ 4 \\ -2{,}4 \end{bmatrix}$$

$$\vec{B} = \vec{V}_3 - \vec{V}_2 = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$$

$$\vec{C} = \vec{V}_1 - \vec{V}_3 = \begin{bmatrix} -3 \\ 0 \\ 2{,}4 \end{bmatrix}$$

use Gram-Schmidt on $\{\vec{B}, \vec{A}\}$ to find basis for plane

$$\hat{e}_1 = \frac{\vec{B}}{\|\vec{B}\|} = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \hat{e}_1$$

$$\vec{u}_2 = \vec{A} - (\hat{e}_1 \cdot \vec{A})\hat{e}_1 = \begin{bmatrix} 0 \\ 4 \\ -2{,}4 \end{bmatrix} - \frac{1}{5}(0 - 16 + 0)\hat{e}_1 =$$

$$= \frac{12}{25}\begin{bmatrix} 4 \\ 3 \\ -5 \end{bmatrix} \qquad \hat{e}_2 = \frac{1}{5\sqrt{2}}\begin{bmatrix} 4 \\ 3 \\ -5 \end{bmatrix}$$

Find $\vec{F}_g$ in $\hat{e}_1, \hat{e}_2, \hat{e}_3$ basis

$$\vec{F} = F_1\hat{e}_1 + F_2\hat{e}_2 + F_3\hat{e}_3 = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}$$

$$F_1 = \vec{F}_g \cdot \hat{e}_1 = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}\frac{1}{5} = 0$$

$$F_2 = \vec{F}_g \cdot \hat{e}_2 = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \\ -5 \end{bmatrix}\frac{1}{5\sqrt{2}} = \frac{1}{\sqrt{2}} mg$$

$$\vec{F}_{net} = \frac{1}{\sqrt{2}} mg\, \hat{e}_2 = \frac{mg}{10}\begin{bmatrix} 4 \\ 3 \\ -5 \end{bmatrix}$$

Angle with $-\hat{z}$

$$(-\hat{z}) \cdot \vec{F}_{net} = \|\vec{F}_{net}\| \cos\theta =$$

$$\frac{mg}{2} \qquad \frac{mg}{10}\sqrt{4^2+3^2+5^2} = \frac{\sqrt{2}}{2} mg$$

$$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} = 45^\circ$$

Find $\vec{r}(t)$

$$\vec{r}(t) = \alpha(t)\,\hat{e}_1 + \beta(t)\,\hat{e}_2 + \vec{v}_1$$

$$\vec{a}_{net} = \frac{1}{\sqrt{2}} g\,\hat{e}_2 \qquad \vec{v}(t) = \frac{1}{\sqrt{2}} g t\,\hat{e}_2$$

$$\vec{r}(t) = \frac{1}{2\sqrt{2}} g t^2\,\hat{e}_2 + \vec{r}_0$$

$\searrow \vec{v}_1$

$$= \begin{bmatrix} \frac{1}{5} g t^2 \\ \frac{3}{20} g t^2 \\ 2.4 - \frac{1}{4} g t^2 \end{bmatrix}$$

Inner Product

$$\langle \vec{v}_1, \vec{v}_2 \rangle : V \times V \to F_{\mathbb{R}, \mathbb{C}}$$

3 properties

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
2. Linear in 1st argument

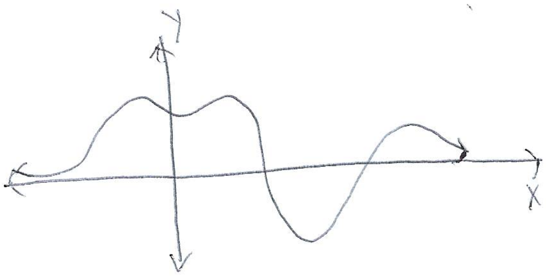
$$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

3. Positive definite

$$\langle x, x \rangle \geq 0 \quad \text{and} \quad \langle x, x \rangle = 0 \iff \vec{x} = \vec{0}$$

- Continuous, bounded functions of 1 variable



$$\int_a^b f(x)g(x)\,dx = \langle f(x), g(x)\rangle$$

$$f(x) = \sum_{i=0}^{\infty} c_i x^i$$

- Functions from $x=-1$ to $x=1$

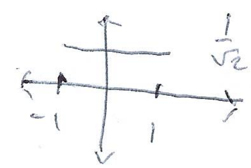
$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

1. $f \leftrightarrow g$ ✓
2. $\int_{-1}^{1} (f(x)+h(x))\,g(x)\,dx = \int_{-1}^{1} f(x)g(x)\,dx + \int_{-1}^{1} h(x)g(x)\,dx$
3. $\int_{-1}^{1} f^2(x)\,dx \geq 0$

- Do G-S to find basis functions

$$\{1, x, x^2, x^3, \ldots\}$$

$\vec{u}_1 = \vec{v}_1 = 1$ $\quad \vec{u}_1 \cdot \vec{u}_1 = \int_{-1}^{1} 1\,dx = 2 \Rightarrow \hat{e}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{1}{\sqrt{2}}$



$\vec{u}_2 = \vec{v}_2 - ((\vec{v}_2)\cdot\hat{e}_1)\hat{e}_1$ $\quad \vec{v}_2 \cdot \hat{e} = \int_{-1}^{1} x\frac{1}{\sqrt{2}}\,dx = \frac{1}{2\sqrt{2}}x^2\Big|_{-1}^{1} = 0$

$\cancel{\vec{v}_2 = x}$

$\hat{e}_2 = \frac{\vec{u}_2}{\sqrt{\vec{u}_2\cdot\vec{u}_2}} = \sqrt{\frac{3}{2}}\,x$ $\quad \vec{u}_2\cdot\vec{u}_2 = \int_{-1}^{1} x^2\,dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{2}{3}$

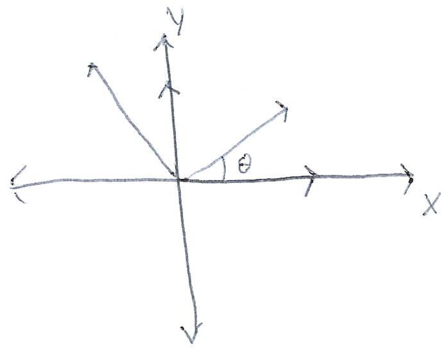
$\vec{u}_3 = \vec{v}_3 - (\vec{v}_3\cdot\hat{e}_1)\hat{e}_1 - (\vec{v}_3\cdot\hat{e}_2)\hat{e}_2$ $\quad v_3\cdot e_1 = \int_{-1}^{1} \frac{x^2}{\sqrt{2}}\,dx = \frac{2}{3\sqrt{2}}$

$= x^2 - \frac{\sqrt{2}x}{3\sqrt{2}} = x^2 - \frac{1}{3}$ $\quad \vec{u}_3\cdot\vec{u}_3 = \frac{8}{45}$

$\hat{e}_3 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$

Matrices

$M\vec{v} = \vec{w}$ M as some operator on a vector



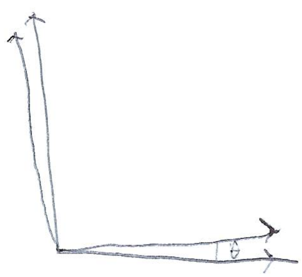
$$x = \begin{bmatrix}1\\0\end{bmatrix} \qquad y = \begin{bmatrix}0\\1\end{bmatrix}$$

$$M\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = \begin{bmatrix}a\\c\end{bmatrix}$$

$$M\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix} = \begin{bmatrix}b\\d\end{bmatrix}$$

$$M = \begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} = R(\theta)$$

- Small angle rotation



$$M\begin{bmatrix}1\\0\end{bmatrix} \approx \begin{bmatrix}1\\ \theta\end{bmatrix} \qquad M \approx \begin{bmatrix}1 & -\theta\\ \theta & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}0 & -\theta\\ \theta & 0\end{bmatrix}$$

$$M\begin{bmatrix}0\\1\end{bmatrix} \approx \begin{bmatrix}-\theta\\ 1\end{bmatrix} \qquad = \mathbb{I} + A$$

$$\lim_{n\to\infty}\left(\mathbb{I} + \frac{\theta}{n}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)^n = e^{\theta\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}$$

$$e^x = 1 + x + \frac{x^2}{2} + \dots = \sum_{i=0}^{\infty}\frac{1}{i!}x^i$$

$$e^M = \mathbb{I} + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots \qquad \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$$e^{\theta\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}} = \mathbb{I} + \theta\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} + \frac{1}{2}\theta^2\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} + \frac{1}{3!}\theta^3\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} + \frac{1}{4!}\theta^4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{\theta^5}{5!}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \dots & -\theta + \frac{1}{3!}\theta^3 - \frac{1}{5!}\theta^5 + \dots\\ \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots & 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \dots\end{bmatrix}$$

$$= \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = L_z =$ generator of rotations

Phys 89 25 Feb 19

Discussion

Review

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

$$\Rightarrow Ax = b = \sum_{j=1}^{n} A_{ij} = b_i$$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$A_{ij} = a_{ij} \qquad Ax = A_{ij}x_j = A^i_j x^j = \sum_{j=1}^{n} A^i_j x^j$$

A is $n \times m$, B is $m \times l$, $(AB)_{kl} = \sum_{m=1} A_{km}B_{ml} = A_{km}B_{ml}$

- No solution
- 1 solution
- many solutions

3 options/types of solutions

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$x_1 = 4$

$x_1 = 5$

$Ax = b$ $\quad A^{-1}$, $A^{-1}A = I$

$A^{-1}Ax = A^{-1}b$

$x = A^{-1}b$

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

turn A to identity

<u>Rank</u> Number of linearly independent row vectors in a matrix A

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \text{rank}(A) = 2$

$B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ $\quad \text{rank}(B) = 1$

A, A^T have the same rank

<u>Image</u>

$A \quad \text{Image}(A) = \{ w \mid Ax = w \}$

<u>Kernel/Null space</u>

$\ker(A) = \{ x \mid Ax = 0 \}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \ker(A) = \left\{ \begin{bmatrix} 0 \\ c \end{bmatrix} \middle| \, c \in \mathbb{R} \right.$

$Ax_1 = b$

$x_0 \in \ker(A) \quad A(x_1 + x_0) = b$

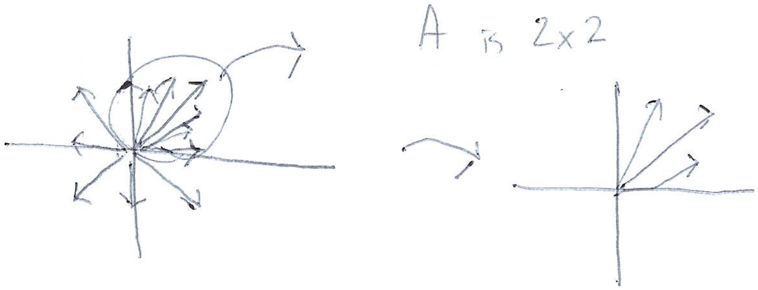
Nullity

dim(kernel)

Number of columns of A = rank(A) + nullity of A

Determinants (square matrices)

A is 2x2



$\det(A) = |A|$

$= a_{11}a_{22} - a_{21}a_{12}$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinants II

$A = \mathrm{diag}(a_{11}, \ldots, a_{nn})$ $\quad |I| = 1$

$|A| = a_{11} \cdots a_{nn}$ $\quad |A^T| = |A|$

nxn matrix has rank n iff $|A| \neq 0$

If $|A| \neq 0$, there exists an inverse matrix

$\det(AB) = \det A \det B$

$\det(A^{-1}) = \dfrac{1}{\det A}$

$\det(cA) = c^n \det A$ A is nxn

Trace

$$\mathrm{Trace}(A) = \mathrm{Tr}\,A = \sum_{i=1}^{n} A_{ii}$$

$$A = \begin{bmatrix} 3 & 5 \\ 7 & 4 \end{bmatrix} \qquad \mathrm{Tr}\,A = 7$$

$$\mathrm{Tr}(cA) = c\,\mathrm{Tr}\,A$$

$$\mathrm{Tr}(A+B) = \mathrm{Tr}\,A + \mathrm{Tr}\,B$$

$$\mathrm{Tr}(A^T) = \mathrm{Tr}\,A$$

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$$

Functions of matrices

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\det(\exp(A)) = \exp(\mathrm{Tr}(A))$$

$$\exp(A) = I + A + \frac{A^2}{2!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(A)^n}{n!}$$

Inverses

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \qquad \det A = 0 \therefore \text{ no inverse}$$

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix} \qquad \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -6 & -2 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 1/3 & -1/6 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 1/3 & -1/6 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & -6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4/6 \\ 0 & -1/6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4/6 \\ 0 & -1/6 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad = \begin{bmatrix} -1/3 & 2/3 \\ 1/3 & -1/6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} C^T \qquad C = \begin{bmatrix} C_{11} & \cdots & \\ \vdots & \ddots & \\ \vdots & & \ddots \end{bmatrix} \qquad C_{2,3} = (-1)^{2+3} (M_{2,3})$$

$$M_{2,3} = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Eigenvalue Problem

$$A = \begin{bmatrix} \cdot & \cdots & \cdot \\ \vdots & \ddots & \vdots \\ \cdot & \cdots & \cdot \end{bmatrix} \qquad \vec{x} \qquad A\vec{x}_\lambda = \lambda \vec{x}_\lambda$$

λ is called an eigenvalue

$\vec{x}_\lambda$ is called an eigenvector

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \vec{x}_{\lambda=3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x}_{\lambda=\pi} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{x}_{\lambda=-1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underbrace{(A - \lambda I)}_{B}\vec{x} = 0 \qquad B\vec{x} = 0 \qquad B^{-1}B\vec{x} = B^{-1}\vec{0}$$

$$\vec{x} = \vec{0}$$

for nontrivial solutions

$\det B = 0$

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (-5-\lambda)(-2-\lambda) - 4 = 0$$

$$10 + 7\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + 7\lambda + 6 = 0$$

$$(\lambda+1)(\lambda+6) = 0$$

$$\lambda = -1, -6$$

$$A\vec{x}_{\lambda=-1} = (-1)\vec{x}$$

$$\vec{x}_{\lambda=-1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\vec{x}_{\lambda=-6} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\mathrm{Tr}A = \sum \lambda$ sum of eigenvalues

$|A| = \prod \lambda = \prod \lambda$ product of eigenvalues

Example Problems

Find eigenvalues, eigenvectors of A, A^{-1} Find A^{-1} if it exists

check that $\mathrm{Tr}A = \sum \lambda$, $\prod \lambda = \det A$

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2-w & -1 \\ -1 & 2-w \end{bmatrix} \qquad (2-w)^2 - 1 = 0$$

$$4 - 4w + w^2 - 1 = 0$$

$$w^2 - 4w + 3 = 0$$

$$(w-3)(w-1) = 0$$

$$w = 1, 3$$

$$w=1 \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$w_2 = 3 \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$w=1 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} -1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -2 & 0 & -1 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \vec{x}_{\lambda=1} = \vec{x}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a-b \\ -a+2b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$a = b$$

$w = 3$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \end{bmatrix} \qquad \begin{bmatrix} 2a-b \\ -a+2b \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \end{bmatrix}$$

$$-b = a$$

$$\vec{x}_{ei} = \begin{bmatrix} a \\ a \end{bmatrix}, \begin{bmatrix} a \\ -a \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1-w & -1 & 0 \\ -1 & 2-w & -1 \\ 0 & -1 & 1-w \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$ no inverse

$$(1-w)((2-w)(1-w)-1) + (1-w)(-1)$$

$$= (1-w)((2-3w+w^2-1)-1) = (1-w)(w^2-3w) = (1-w)w(w-3)$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a-b \\ -a+2b-c \\ -b+c \end{bmatrix} \overset{w=0}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a \\ a \\ a \end{bmatrix}$$

$w = 1, 0, 3$

$w=1$

$$\begin{bmatrix} a-b \\ -a+2b-c \\ -b+c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$a = b$

$b = c$

$$\begin{bmatrix} a \\ 0 \\ -a \end{bmatrix}$$

$b = 0$

$a = -c$

$w = 3$

$$\begin{bmatrix} a-b \\ -a+2b-c \\ -b+c \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \\ 3c \end{bmatrix}$$

$2a = -b$

$2c = -b$

$-a-c = b$

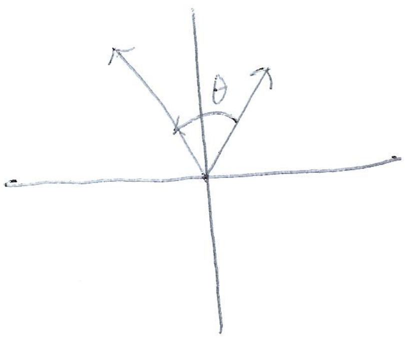
$-2a = b$

$a = c$

b

$$\begin{bmatrix} a \\ -2a \\ a \end{bmatrix}$$

$$\begin{bmatrix} a \\ a \\ a \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix}, \begin{bmatrix} a \\ -2a \\ a \end{bmatrix}$$



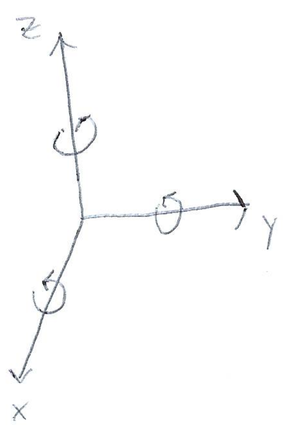
$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{counter clockwise}$$

$$R(-\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{clockwise}$$

$$R(\theta)R(-\theta) = I$$

$$R(2\theta) = R(\theta)R(\theta)$$

$$R(\theta_1)R(\theta_2) = R(\theta_2)R(\theta_1) = R(\theta_1 + \theta_2)$$



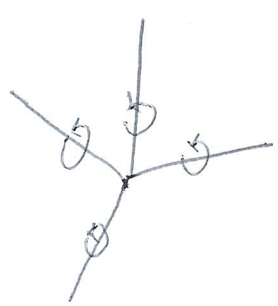
3x3

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta_1)R_x(\theta_2) \neq R_x(\theta_2)R_z(\theta_1)$$

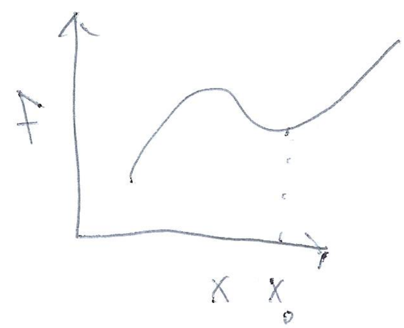


Phys 89 11Mar19

Discussion

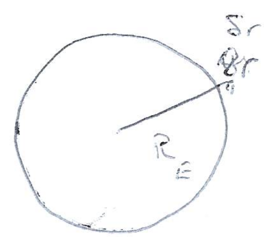
Taylor Series

Approximate a function around some point



$$f(x) = f(a) + \left.\frac{df}{dx}\right|_{x=a}(x-a) + \dots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{[n]}(a)}{n!}(x-a)^n$$



<u>Ex</u> $F(r) = \frac{Gm_1 m_2}{r^2}$

$$F(R_E + \delta_r) = F(R_E) - \frac{2Gm_1 M_E}{R_E^3}(\delta_r) + \mathcal{O}(\delta_r^2)$$

$$= \frac{Gm_1 M_2}{R_E^2}\left(1 - \frac{2\delta r}{R_E}\right) \qquad \frac{GM_E}{R_E^2} = 9.81\,m/s^2$$

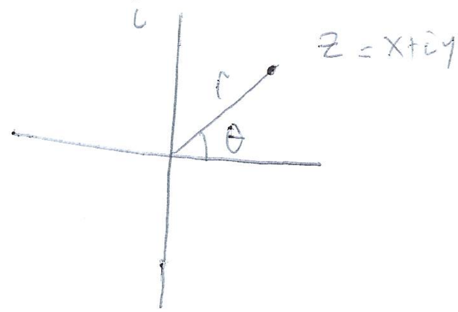
$$F(R_E + \delta_r) = m_1 g\left(1 - \frac{2\delta r}{R_E}\right)$$

$\delta_r = 1000$ meters

$$\left(1 - \frac{2\delta r}{R_E}\right) = 0.9997$$

Complex #'s

$z = x + iy \quad , x, y \in \mathbb{R}$



$z = re^{i\theta}$

$z^* \equiv \bar{z} = x - iy = re^{-i\theta}$

$w = z^n \qquad w = r_w e^{i\theta_w}$

$z = (r_w)^{\frac{1}{n}} e^{i\theta_w/n} e^{\frac{2\pi k i}{n}} \qquad k = \{0, 1, \dots, n-1\}$

Levi-Civita, Kronecker delta, index manipulation

$\varepsilon_{i_1 i_2 i_3 \dots i_n}$ n indices

$i_j \in \{1, \dots, n\}$

$$\varepsilon_{i_1 \dots i_n} = \begin{cases} +1 & \text{if even permutation of } 12\dots n \\ -1 & \text{odd permutation of } 12\dots n \\ 0 & \text{all other cases} \end{cases}$$

$\overset{n=3}{\varepsilon_{113}} = 0 \; , \; \varepsilon_{123} = 1 \; , \; \varepsilon_{213} = -1$

Determinant

$\underset{3\times 3}{A} \qquad |A| = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$

Cross Product

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \vec{v} \times \vec{w} = \varepsilon_{ijk} v_j w_k \hat{e}_i \qquad (\vec{v} \times \vec{w})_1 = v_2 w_3 - v_3 w_2$

$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \qquad \hat{e}_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ ith spot

Determinant

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix} \qquad |A| = \varepsilon_{ij} a_{1i} a_{2j} = a_{11}a_{22} - a_{12}a_{21}$$

Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \qquad \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{j,i=1}^{3} \delta_{ij} a_i b_j = \delta_{ij} a_i b_j$$

$$M_{ij} \qquad \delta_{i4} \delta_{j9} M_{ij} = M_{49} = \delta_{4i} \delta_{9j} M_{ij}$$

$$\cancel{\varepsilon_{ijk} \delta_{ijk} = (\ldots) - (\ldots)}$$

Einstein Summation Notation

$$|A| = \sum \sum \sum \varepsilon_{\cdot\cdot\cdot} a_{\cdot\cdot} a_{\cdot\cdot} a_{\cdot\cdot} = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

$$(AB)_{ik} = A_{ij} B_{jk} = A^i_j B^j_k$$

$$(ABC)_{ik} = A_{ij} B_{jl} C_{lk}$$

Vector Spaces, Orthogonality, Basis

Addition: $v_1, v_2 \in V \quad (v_1 + v_2) \in V$

Inverse: $v_1 \in V,\ \exists v_1^{-1} \quad (v_1 + v_1^{-1}) = \vec{0} \in V$

Scalar multiplication $\lambda \in \mathbb{R} \quad \lambda v_1 \in V$

$\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$

$(\lambda_1 + \lambda_2) v = \lambda_1 v + \lambda_2 v$

Ex

Is the set of all diagonal 3x3 matrices a vector space

$$M = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Orthogonality

$\vec{v}, \vec{w}$ $\quad \vec{v} \cdot \vec{w} = \delta_{ij} v_i w_j = 0 \equiv v^T w$

$\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} -6 \\ 4 \\ 0 \end{bmatrix}$ $\quad \vec{v} \cdot \vec{w} \neq 0$

Linear Independence

n dimension real vector space
n linearly independent vectors

$\{v_i\}$ $\quad \sum_i \lambda_i \vec{v}_i = 0$ $\quad$ If no solutions for $\lambda_i \neq 0$ the collection $\{v_i\}$ is linearly independent

Wronskian

$$W(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

f, g defined on some interval

$\neq 0$ for som x in interval
then f, g are linearly independent

Matrix Algebra

$$M \;,\; M^T \;,\; M^\dagger = (M^*)^T$$

$$\mathcal{O}\mathcal{O}^T = \mathcal{O}^T\mathcal{O} = \mathbb{I}$$

$\mathrm{Tr}\,\mathcal{O}$

$|\mathcal{O}|$

Gaussian Elimination, Matrix Inversion

$$A\vec{x} = \vec{b} \qquad \left[A \middle| \mathbb{I}\right] \longrightarrow \left[\mathbb{I} \middle| A^{-1}\right]$$

$|A| = 0 \Rightarrow$ no inverse

Cofactor

$$C_{ij} = (-1)^{i+j} m_{ij} \qquad m_{ij} = \det\left(i\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right)$$

$$A^{-1} = \frac{1}{|A|} C^T$$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -3 \end{bmatrix} \qquad \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & -5 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2}-\frac{1}{5} & \frac{1}{10} \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{10} & \frac{1}{10} \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

Problems

$\begin{bmatrix} 0 & -4 \\ -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix}$

Transpose

$\begin{bmatrix} +0 & -4 \\ -2 & +1 \end{bmatrix}$

$\begin{bmatrix} 0 & -4 \\ -2 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix}$

1) $w = 17e^{\frac{i\pi}{2}}$

Find z such that $z^3 = w$

$z = (17)^{1/3} e^{\frac{i\pi}{6} + \frac{2\pi n i}{6}} = (17)^{1/3} e^{\frac{i\pi}{6} + \frac{\pi n i}{3}}$

2) $A = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$ det(A) from Levi-Civita

Find A^{-1} with row reduction and cofactors

$|A| = -8$

$|A| = \varepsilon_{ij} a_{1i} a_{2j} = 1(1 \cdot 0) - 1(2 \cdot 4) = -8$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & \frac{1}{4} \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 0 & 2 & 1 & -\frac{1}{4} \\ 1 & 0 & 0 & \frac{1}{4} \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{8} \end{array}\right]$

3) Write out in index notation $Tr(AB)$ $Tr((AB)^T)$ are they equal

$Tr\,AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

$Tr\,AB = A_{ij} B_{jK}$

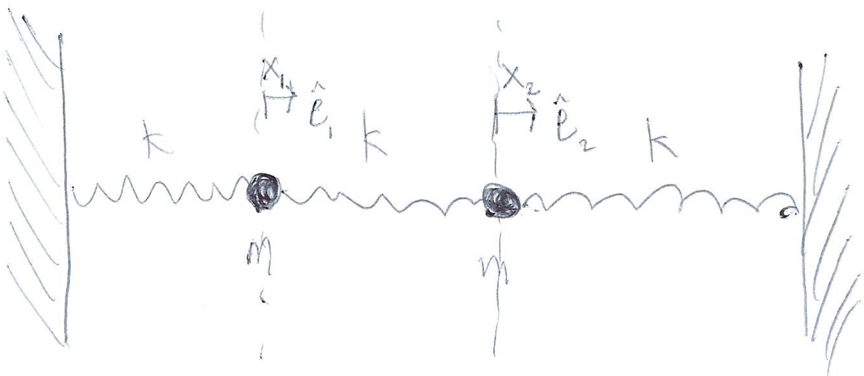
4) Let O be an orthogonal matrix Write $OO^T = \mathbb{I}$ in index notation, what constraints do we have

5) Write $(AB)^T$ in index notation; Show that it equals $B^T A^T$ in index notation

$O_{ij} O_{ji} = \mathbb{I}$

Phys 89 18Mar19

Discussion



Define $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\hat{e}_1 + x_2\hat{e}_2$

a. Find K, symmetric matrix, such that potential energy U is given by $U = \frac{1}{2}\vec{x}^T K \vec{x} = \frac{1}{2}\vec{x}\cdot K\vec{x}$

$$U = U_1 + U_2 + U_3 = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}kx_2^2$$

$$U = kx_1^2 + kx_2^2 - kx_1x_2 \qquad K = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$K\vec{x} = \begin{bmatrix} ax_1 + bx_2 \\ bx_1 + dx_2 \end{bmatrix}$$

$$\vec{x}^T K\vec{x} = ax_1^2 + bx_1x_2 + bx_1x_2 + dx_2^2$$

$$\frac{1}{2}\vec{x}^T K\vec{x} = \frac{1}{2}ax_1^2 + bx_1x_2 + \frac{1}{2}dx_2^2$$

$$a = 2k, \; b = -k, \; d = 2k \qquad K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

b. Find eigenvalues

$$\det(K - \lambda I) = 0$$

$$\begin{vmatrix} 2k-\lambda & -k \\ -k & 2k-\lambda \end{vmatrix} = (2k-\lambda)^2 - k^2 = 0$$

$$\lambda_1 = k$$
$$\lambda_2 = 3k$$

c. Find eigenvectors and normalize them, $\hat{s}_1$, $\hat{s}_2$

$$\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = k \begin{bmatrix} a \\ b \end{bmatrix}$$

$2ka - bk = ka$ $\quad ka = kb \quad a = b \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \hat{s}_1$

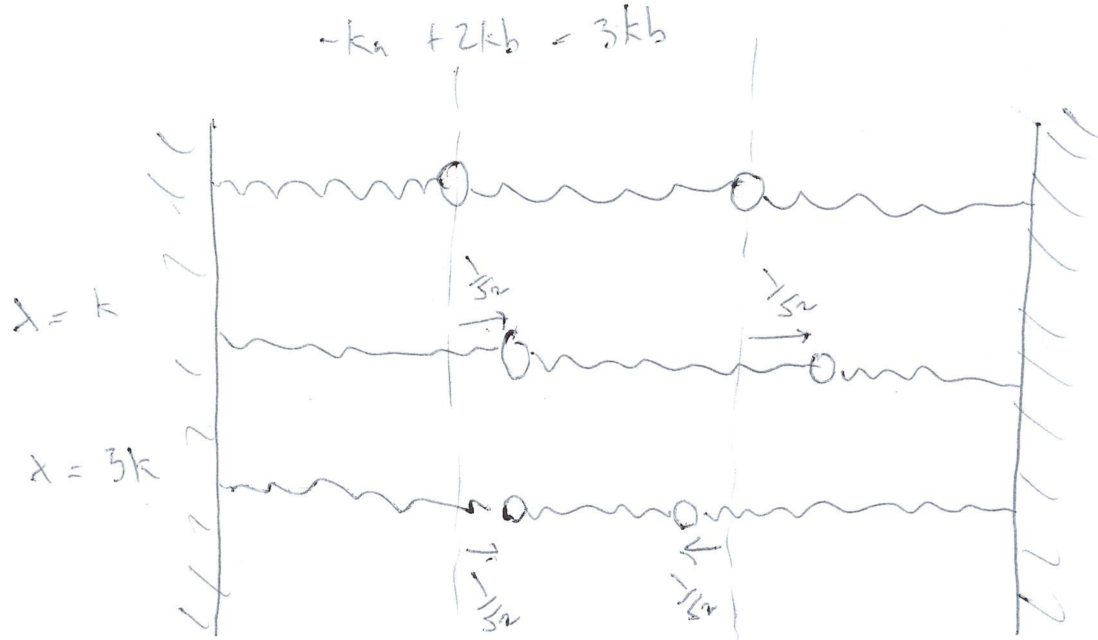
$-ka + 2kb = kb \quad -ka + kb = 0$

$$\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 3k \begin{bmatrix} a \\ b \end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \hat{s}_2$$

$ka = -kb$

$2ka - kb = 3ka$

$-ka + 2kb = 3kb$



e) $\vec{F} = M\ddot{\vec{x}}$ $\quad M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$

Show $\vec{F} = -K\vec{x}$

$\cancel{f_1 = -kx_1 \quad f_2}$

$$f_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$$

$$f_2 = -kx_2 + k(x_1 - x_2) = -2kx_2 + kx_1$$

$$\vec{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -K\vec{x}$$

As an aside:

In general $\vec{F}(x) = -\dfrac{\partial U}{\partial x}$

$$U = \tfrac{1}{2}\vec{x}^T K\vec{x} = \tfrac{1}{2}\vec{x}\cdot K\vec{x}$$

compare: $U = \tfrac{1}{2}kx^2$

$$-\frac{\partial U}{\partial x} = -kx$$

$$f_i = -\frac{\partial U}{\partial x_i} = -\frac{\partial}{\partial x_i}\left(\tfrac{1}{2}\vec{x}^T K\vec{x}\right)$$

$$= -\frac{1}{2}\frac{\partial}{\partial x_i}(x^T)Kx - \frac{1}{2}x^T K\frac{\partial x}{\partial x_i}$$

$$= -\frac{1}{2}\vec{\mathbb{I}}_i^T Kx - \frac{1}{2}\vec{x}^T K\vec{\mathbb{I}}_i$$

some number

$\vec{\mathbb{I}}^T = \vec{\mathbb{I}}$

$$f_i = -\vec{\mathbb{I}}_i^T Kx$$

picks out i component of x

$$\vec{F} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = -\begin{bmatrix} (Kx)_1 \\ (Kx)_2 \\ \vdots \\ (Kx)_n \end{bmatrix} = -K\vec{x}$$

$$\vec{F} = M\ddot{\vec{x}} = -K\vec{x}$$

$$\ddot{\vec{x}} = -M^{-1}K\vec{x} = -A\vec{x}$$

d) $A = M^{-1}K = \frac{1}{m}\mathbb{I}K = \frac{1}{m}K$

$$K\vec{v} = \lambda\vec{v}$$

$A\vec{v} = \frac{1}{m}K\vec{v} = \frac{1}{m}\lambda\vec{v}$ $\quad\vec{v}$ is an eigenvector for A with eigenvalue $= \frac{\lambda}{m}$

So $\left(\hat{s}_1, \frac{k}{m}\right)$

$\left(\hat{s}_2, \frac{3k}{m}\right)$

if $\vec{x} = a(t)\hat{s}_1 = a(t)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$\ddot{\vec{x}} = \ddot{a}(t)\hat{s}_1 \qquad \ddot{\vec{x}} = -A\vec{x} = -A(a(t)\hat{s}_1) = -a(t)A\hat{s}_1$

$= -\frac{k}{m}a(t)\hat{s}_1$

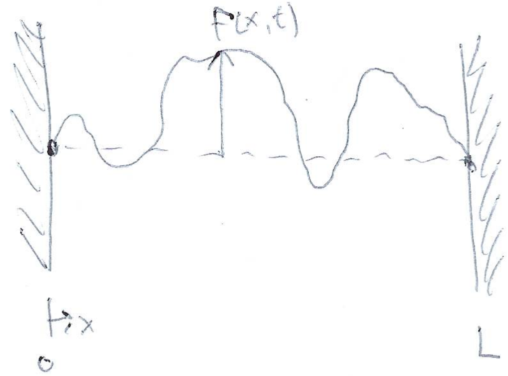
$\ddot{a} = -\frac{k}{m}a \qquad a(t) = c_1\cos\left(\sqrt{\frac{k}{m}}t\right) + c_2\sin\left(\sqrt{\frac{k}{m}}t\right)$

if $\vec{x} = b(t)\hat{s}_2$

$\ddot{\vec{x}} = \ddot{b}\hat{s}_2 = -\frac{3k}{m}b\hat{s}_2 \qquad \ddot{b} = -\frac{3k}{m}b$

$$b(t) = c_3\cos\left(\sqrt{\frac{3k}{m}}t\right) + c_4\sin\left(\sqrt{\frac{3k}{m}}t\right)$$

Waves on a string of length L



$$\frac{\partial^2 f(x,t)}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 f(x,t)}{\partial x^2} = \alpha^2 \frac{\partial^2 f}{\partial x^2}$$

$f(x=0,t)=0$

$f(x=L,t)=0$

Can write as

$$\ddot{\vec{f}} = W\vec{f} = \frac{T}{\rho}\frac{\partial^2}{\partial x^2} f$$

a) Show W is linear

$$W(a\vec{f}+b\vec{g}) = \frac{T}{\rho}\frac{\partial^2}{\partial x^2}(a f(x,t) + b g(x,t))$$

$$= a\frac{T}{\rho}\frac{\partial^2 f}{\partial x^2} + b\frac{T}{\rho}\frac{\partial^2 g}{\partial x^2}$$

$$= aW\vec{f} + bW\vec{g}$$

b) Recall $\vec{f}\cdot\vec{g} = \int_0^L f(x,t)\,g(x,t)\,dx$

show if $f(0) = f(L) = 0$

$g(0) = g(L) = 0$

then $(W\vec{f})\cdot\vec{g} = \vec{f}\cdot(W\vec{g})$

$$\int_0^L \alpha^2 \frac{\partial^2 f}{\partial x^2} g(x)\,dx \Rightarrow \int_0^L f(x)\,\alpha^2\frac{\partial^2 g}{\partial x^2}\,dx$$

$v' \qquad u$

$$\underbrace{\alpha^2 g(x)\frac{\partial f}{\partial x}\Big|_0^L}_{\text{goes to zero}} - \int_0^L \alpha^2 \frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\,dx = -\int_0^L \alpha^2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\,dx = \underbrace{-\alpha^2 f g'\Big|_0^L}_{\text{goes to zero}} + \int_0^L \alpha^2 f\frac{\partial^2 g}{\partial x^2}\,dx$$

$$= \int_0^L \alpha^2 f \frac{\partial^2 g}{\partial x^2}\,dx = \vec{f}\cdot W\vec{g}$$

This means W is symmetric

Symmetric matrices have real ~~eigenvectors~~ eigenvalues and its eigenvectors are orthogonal

c) Find normalized eigenfunctions (eigenvectors) and eigenvalues

Want to solve: $W\vec{f} = \lambda \vec{f}$

$$\alpha^2 \frac{\partial^2 f}{\partial x^2} = \lambda f$$

Try $f(x) = A\cos(kx) + B\sin(kx)$

$f''(x) = -k^2 A\cos(kx) - k^2 B\sin(kx) = -k^2 f$

works if $\lambda = -\alpha^2 k^2$

Plug in boundary conditions to restrict A, B, and/or k

$f(0) = 0 \qquad A\cos(0) + B\sin(0) = 0$

$\qquad A = 0$

$f(L) = 0 \qquad \cancel{A\cos L + B\sin}$ $\qquad n \in \{0, 1, 2, \dots\}$

$\qquad A\sin kL = 0 \quad \Rightarrow \quad kL = n\pi$

$V_n \cdot V_m = \delta_{nm}$

Current list

$V_0 = B_0 \sin(0) = 0 \qquad \lambda_0 = 0$

$V_1 = B_1 \sin\left(\frac{\pi x}{L}\right) \qquad \lambda_1 = -\alpha^2 \frac{\pi^2}{L^2}$

$V_2 = B_n \sin\left(\frac{2\pi x}{L}\right) \qquad \lambda_2 = -\alpha^2 \frac{4\pi^2}{L^2}$

$\vdots$

$V_n = B_n \sin\left(\frac{n\pi x}{L}\right) \qquad \lambda_n = -\alpha^2 \frac{n^2\pi^2}{L^2}$

Normalize $\int_0^L B_n^2 \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$

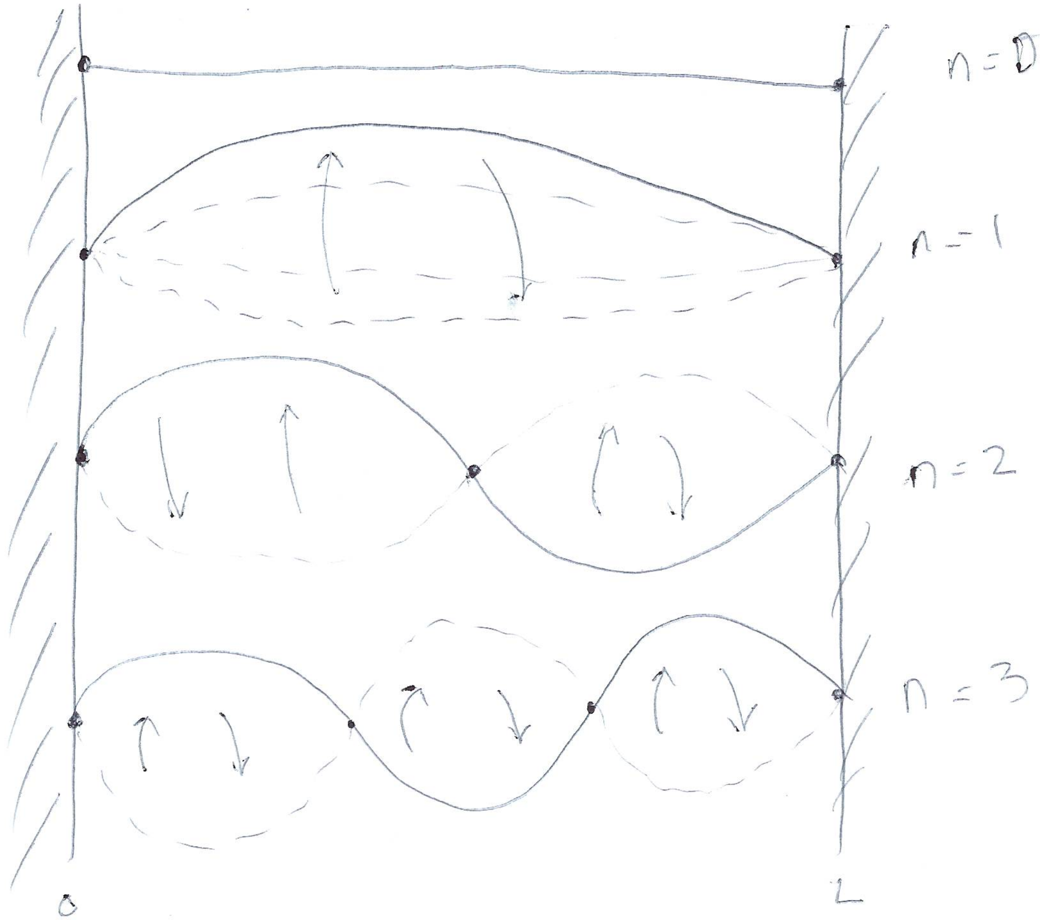
$B_n^2 \frac{L}{2} = 1 \rightarrow B_n = \sqrt{\frac{2}{L}}$

$\int_0^L B_n B_m \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \delta_{nm}$

Eigen system

$$\left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), -\frac{I}{\rho} \frac{n^2\pi^2}{L^2} \right\}$$

$$\{ \vec{V}_n, \lambda_n \} \quad n \in \mathbb{Z} \geq 0$$



do Standing waves

$$\ddot{\vec{f}} = W\vec{f} = \frac{T}{\rho}\frac{\partial^2 f}{\partial x^2} \quad \text{if } f(x,t) = a(t)\, v_n(x)$$

$$\ddot{f} = \ddot{a}\, v_n = W(a\, v_n) = a\, W v_n = a(t)\, \lambda_n v_n$$

$$\ddot{a}\, v_n = a\, \lambda_n v_n \qquad \ddot{a}(t) = \lambda_n a(t) \qquad \lambda_n = -\omega_n^2$$

$$a(t) = c_n \cos\omega_n t + d_n \sin\omega_n t$$

$$f(x,t) = (c_n \cos\omega_n t + d_n \sin\omega_n t)\, v_n(x)$$

<u>Ex</u> wave starts from rest

General $g(x,t) = \sum_{n=0}^{\infty} f_n(x,t)$

Ex wave starts from rest

General $y(x,t) = \sum_{n=\infty}^{\infty} f_n(x,t)$

From rest $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$

$$\left[\frac{\partial}{\partial t}\sum_{n=\infty}^{\infty} f_n\right]\Bigg|_{t=0} = \sum_{n=0}^{\infty}\left(-c_n \sin(\omega_n t)\omega_n + d_n \cos(\omega_n t)\omega_n\right)V_n(x)\Big|_{t=0} = 0$$

$$\sum_{n=\infty}^{\infty} d_n \omega_n = 0$$ all d_n's positive

then d_n's are zero

$$g(x,t) = \sum_{n=-\infty}^{\infty} c_n \cos(\omega_n t) V_n(x)$$

Phys 89 01 Apr 19
Discussion

$(A^T)^{-1} = (A^{-1})^T$

wrong

$\mathbb{I} = \mathbb{I}$

$AA^{-1} = \mathbb{I}$ True

$(AA^{-1})^T = \mathbb{I}^T$

$\underbrace{(A^{-1})^T}_{\text{Inverse of } A^T} A^T = \mathbb{I}$

$(A^T)^{-1} = (A^{-1})^T$

$Ax = b$

Goal: Find A^{-1}

$y^{[4]} + xy'' = 0 \quad y = f(x)$

$y' = y \qquad y(x) = ce^x$

<u>Classical Mechanics</u>

$F = m\ddot{x} \qquad F = mg \ , \ g = \ddot{x} + \gamma\dot{x} + \beta\dot{x}^2$

$x(0) = x_0 \ , \ \dot{x}(0) = 0$

<u>EM</u>

$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$

Central Force

$$m\ddot{r} = mr^2\dot{\phi}^2 - \frac{\partial u}{\partial r}$$

$\not{y}''$ Linear First Order ODEs

$y(x)$ $\quad a_0(x)\,y(x) + a_1(x)\,y'(x) + a_2(x)\,y''(x) + \cdots + a_n(x)\,y^{[n]}(x) = b(x)$

Differential operator $D = a_0(x) + a_1(x)\frac{d}{dx} + a_2(x)\frac{d^2}{dx^2} + \cdots$

$$D\,y(x) = b(x)$$

$$D = a_0(x) + a_1(x)\frac{d}{dx}$$

$$y' + p(x)\,y = r(x)$$

Trick 1: Separable ODEs

$$y' = \frac{dy}{dx} \qquad g(y)\frac{dy}{dx} = f(x)$$

$$y' = 1 + y^2 \Rightarrow \int \frac{dy}{1+y^2} = \int dx \Rightarrow \arctan y = x + C$$

$$y = \tan(x + C)$$

Boundary conditions fix C

Trick 2: Exact ODEs

$$M(x,y)\,y' + N(x,y) = 0$$

Goal: Find $u(x,y)$ $\quad \frac{du}{dx} = N(x,y) \quad \frac{du}{dy} = M(x,y)$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad u(x,y) = C$$

$$du = N(x,y)\,dx + M(x,y)\,dy = 0$$

Bernoulli Equation

$y' + p(x)y = g(x)y^n \qquad n = \text{any real number}$

$u = (y(x))^{1-n}$

$\frac{du}{dx} = (1-n)(g(x) - p(x)\, y^{1-n})$

$\frac{du}{dx} + (1-n)p(x)u = (1-n)g(x)$

Practice Problems

1) $y' = -2xy \quad , \quad y(0) = 1$

$\frac{dy}{y} = -2x\,dx \qquad \ln y = -x^2 + c$

$y = Ce^{-x^2} \qquad y(0) = 1 = C$

$y = e^{-x^2}$

2) $y'(-\sin y \cosh x) + (\cos y \sinh x + 1) = 0$

$-\sin y \cosh x\, dy + (\cos y \sinh x + 1)dx = 0$

$\cos y \cosh x + x \overset{?}{=} u$

$\frac{du}{dy} = -\sin y \cosh x$

$u = x + \cos y \cosh x = C$

Example $\cos(x+y)\,dx + (3y^2+2y+\cos(x+y))\,dy = 0$

$$\underbrace{\cos(x+y)}_{M(x,y)}\,dx + \underbrace{(3y^2+2y+\cos(x+y))}_{N(x,y)}\,dy = 0$$

$\frac{\partial u}{\partial x}$ $\qquad$ $\frac{\partial u}{\partial y}$

$$y^3+y^2+\sin(x+y) \stackrel{?}{=} u(x,y)$$

$$\frac{\partial u}{\partial x} = \cos(x+y) \quad \therefore \text{ yes}$$

$$u = y^3+y^2+\sin(x+y) = C$$

$$\frac{dy}{dx} + p(x)\,y(x) = q(x)$$

$$I(x) = \int_0^x p(x')\,dx' \qquad \frac{dI}{dx} = p(x)$$

Solution

$$y(x) = e^{-I(x)}\int_0^x q(x')\,e^{I(x')}\,dx' + c\,e^{-I(x)}$$

$$\frac{dy}{dx} = -p(x)e^{-I(x)}\int_0^x q(x')\,e^{I(x')}\,dx' + e^{-I(x)}q(x)e^{I(x)} + -c\,p(x)e^{-I(x)}$$

$$= -p(x)e^{-I(x)}\int_0^x q(x')\,e^{I(x')}\,dx' + q(x) - c\,p(x)\,e^{-I(x)}$$

$$\frac{dy}{dx} + p(x)\,y(x) = q(x)$$

Phys 89 08 Apr. 19
Discussion

$$\boxed{\begin{array}{c} y' + p(x)\,y = r(x) \\ \text{easy mode} \end{array}}$$

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

$r(x) = 0$ homogeneous

$r(x) \neq 0$ inhomogeneous

Given $y(x_0) = y_0$

$y'(x_0) = y_0'$

$y(0) = a \quad , \quad y'(0) = b$

Examples $y'' + y = 0$ $\quad Y_1 = e^{x} \;,\; Y_2 = e^{-x}$

$Y(x) = C_1 e^{x} + C_2 e^{-x} \qquad y(0) = 1 \;,\; y'(0) = 1$

$Y(0) = 1 = C_1 + C_2$

$Y'(0) = 1 = C_1 - C_2$

$C_1 = 1 \;,\; C_2 = 0$

Constant Coefficients

$$y'' + a y' + b y = 0$$

Try $y(x) = Ce^{\lambda x}$

$$y'(x) = C\lambda e^{\lambda x}$$
$$y''(x) = C\lambda^2 e^{\lambda x}$$

$$C\lambda^2 e^{\lambda x} + aC\lambda e^{\lambda x} + Cbe^{\lambda x} = 0$$
$$c(C\lambda^2 + aC\lambda + Cb) = 0$$
$$(\lambda^2 + a\lambda + b) = 0$$

for λ $\lambda_{1,2} = \dfrac{-a \pm \sqrt{a^2-4b}}{2}$

λ_1, λ_2 real and different

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

$\lambda_1 = \lambda_2$, real

$a^2 = 4b$

$$y(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}$$

λ_1, λ_2 imaginary

$a^2 - 4b < 0$ $\quad \lambda_{1,2} = \dfrac{-a}{2} \pm i\omega \quad \omega = \dfrac{\sqrt{|a^2-4b|}}{2}$

$$y(x) = C_1 e^{(-\frac{a}{2} + i\omega)x} + C_2 e^{(-\frac{a}{2} - i\omega)x}$$
$$= e^{-\frac{a}{2}x}\left(\tilde{C}_1 \cos(\omega x) + \tilde{C}_2 \sin \omega x\right)$$

$m\ddot{x} + kx = 0$

Solve for $x(t)$

$x = e^{\lambda t}$

$m\lambda^2 + k = 0$

$\lambda^2 = \frac{-k}{m} \Rightarrow \lambda = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$

$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$

$= A\cos\omega_0 t + B\sin\omega_0 t$

Undetermined Coefficients

$y'' + ay' + by = 0 \qquad Y_h = C_1 y_1 + C_2 y_2$

$y'' + ay' + by = r(x) \qquad y = Y_p + Y_h$

Can solve this when $r(x) = e^{\gamma x}, x^n, \cos x, \sin x, \ldots$

$r(x)$	Try
$Ce^{\gamma x}$	$Ke^{\gamma k}$
Cx^n	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_0$
$c_1 \cos(x\omega)$ $c_1 \sin(x\omega)$	$K_1 \cos\omega x + K_2 \sin\omega x$
$Ce^{\gamma x}\cos\omega x$	$e^{\gamma x}(K_1\cos\omega x + K_2 \sin\omega x)$

$$y'' + y' + y = x^2$$

$$y_p = k_2 x^2 + k_1 x + k_0$$

$$y_p'' = 2k_2 \qquad y_p' = 2k_2 x + k_1$$

$$2k_2 + 2k_2 x + k_1 + k_2 x^2 + k_1 x + k_0 = x^2$$

$$x^2(k_2) + x(2k_2 + k_1) + (2k_2 + k_1 + k_0) = x^2$$

$$k_2 = 1 \qquad 2k_2 = -k_1 \qquad 2 - 2 + k_0 = 0$$

$$k_1 = -2 \qquad k_0 = 0$$

$$Y_p = x^2 - 2x$$

$$y'' + p(x)y' + q(x)y = r(x) \qquad Y_h = c_1 y_1 + c_2 y_2$$

$$y_p(x) = -y_1(x)\int \frac{y_2(x)\,r(x)}{W(x)}\,dx + y_2 \int \frac{y_1(x)\,r(x)}{W(x)}\,dx$$

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

Ex $\quad y'' + y = \dfrac{1}{\cos x} \qquad y_1 = \cos x, \; y_2 = \sin x$

$$W = 1$$

$$y_p = -\cos x \int \frac{\sin x}{\cos x}\,dx + \sin x \int \frac{\cos x}{\cos x}\,dx$$

$$= -\cos x\,(\ln|\cos x| + c_1) + \sin x\,(x + c_2)$$

part of homogeneous solution and can be neglected

1) Damped SHO

$$m\ddot{x} + 2m\gamma\dot{x} + m\omega_0^2 x = 0$$

Solve for $x(t)$ find conditions on γ, ω_0 for different oscillator behavior

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

try $x = e^{\lambda x}$ $\quad \lambda^2 e^{\lambda x} + 2\gamma\lambda e^{\lambda x} + \omega_0^2 \cancel{x} e^{\lambda x} = 0$

$\dot{x} = \lambda e^{\lambda x}$ $\quad e^{\lambda x}(\lambda^2 + 2\gamma\lambda + \omega_0^2) = 0$

$\ddot{x} = \lambda^2 e^{\lambda x}$

$$\lambda = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$x = Ae^{-\gamma + \sqrt{\gamma^2 - \omega_0^2}} + Be^{-\gamma - \sqrt{\gamma^2 - \omega_0^2}}$$

oscillates is $\cancel{\gamma^2 \omega_0^2}$ $\gamma^2 < \omega_0^2$

critically dampened if $\gamma^2 = \omega_0^2$

Over dampened if $\gamma^2 > \omega_0^2$

$$m\ddot{x} + 2m\gamma\dot{x} + m\omega_0^2 x = F_0 \cos\omega t$$

$x = C\cos\omega t + D\sin\omega t$

$\dot{x} = -C\omega\sin\omega t + D\omega\cos\omega t$

$\ddot{x} = -C\omega^2\cos\omega t - D\omega^2\sin\omega t$

$F_0\cos\omega t = -m\omega^2(C\cos\omega t + D\sin\omega t) + \omega 2m\gamma(-C\sin\omega t + D\cos\omega t) + m\omega_0^2(C\cos\omega t + D\sin\omega t)$

$= \cos\omega t(-m\omega_0^2 C + 2m\gamma\omega_0 D + m\omega_0^2 C)$

$= \cos\omega t(2m\gamma\omega_0 D) = F_0\cos\omega t$

$$D = \frac{F_0}{2m\gamma\omega_0}$$

$$x_p = \frac{F_0}{2m\gamma\omega_0}\sin\omega_0 t$$

2) $x'' + 25x = 24\sin t \quad , x(0) = 1 \quad x'(0) = 1$

$\cancel{y_1 = \cos 5x}$ $\quad x_1 = \cos 5t$

$\cancel{y_2 = \sin 5x}$ $\quad x_2 = \sin 5t$

$x_h = A\cos 5t + B\sin 5t$

$\cancel{x_h(0) = 1 = A \quad x_h = \cos 5t + B\sin 5t}$

$x = C\cos t + D\sin t$

$\dot{x} = -C\sin t + D\cos t$

$\ddot{x} = -C\cos t - D\sin t$

$-C\cos t - D\sin t + 25(C\cos t + D\sin t) = 24\sin t$

$24C\cos t + 24D\sin t = 24\sin t$

$C = 0 \qquad D = 1$

$x = A\cos 5t + B\sin 5t + \sin t$

$A = 1$

$x = \cos 5t + \sin t + B\sin 5t$

$\dot{x} = -5\sin 5t + \cos t + 5B\cos 5t$

$\dot{x}(0) = 1 = 1 + 5B \qquad B = 0$

$x = \cos 5t + \sin t$

3) $x'' + x = \cos wt$ $\qquad x(0) = 0$

$\qquad\qquad\qquad\qquad\qquad x'(0) = 0$

$X = A\cos wt + B\sin wt$

$X_p = Ct\cos wt + Dt\sin wt$

$\dot{X}_p = C\cos wt - Cwt\sin wt + D\sin wt + Dtw\cos wt$

$\ddot{X}_p = -Cw\sin wt - Cw\sin wt - Cw^2t\cos wt + Dw\cos wt + Dw\cos wt - Dtw^2\sin wt$

$= -2Cw\sin wt + 2Dw\cos wt - Cw^2t\cos wt - Dtw^2\sin wt$

$-2Cw\sin wt + 2Dw\cos wt - Cw^2t\cos wt - Dtw^2\sin wt + Ct\cos wt + Dt\sin wt = \cos wt$

$\cos wt(2Dw - Cw^2t + Ct) + \sin wt(-2Cw - Dtw^2 + Dt) = \cos wt$

$2Dw - Cw^2t + Ct = 1 \qquad -2Cw - Dtw^2 + Dt = 0$

$2Dw - w^2t\left(\frac{Dt(1-w^2)}{2w}\right) + \left(\frac{Dt(1-w^2)}{2w}\right)t = 1 \qquad Dt(1-w^2) = 2Cw$

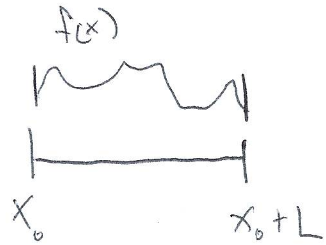
$2Dw - \frac{wDt^2(1-w^2)}{2} + \frac{Dt^2(1-w^2)}{2w} = 1 \qquad C = \frac{Dt(1-w^2)}{2w}$

$D\left(\frac{4w^2 - w^2t^2(1-w^2) + t^2(1-w^2)}{2w}\right) = 1 = D\left(\frac{4w^2 - 2w^2t^2 + w^4t^2 + t^2}{2w}\right)$

$D = \frac{2w}{4w^2 - 2w^2t^2 + w^4t^2 + t^2}$

Phys 89 15 Apr 19
Discussion

Fourier Series
- Finite Interval L



$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_0 n x} \qquad k_0 = \frac{2\pi}{L}$$

$$\vec{f} = \sum_{n=-\infty}^{\infty} c_n \hat{e}_n \qquad \hat{e}_n = e^{ink_0 x}$$

$$\vec{f}\cdot\vec{g} = \frac{1}{L}\int_{x_0}^{x_0+L} f^*(x)\,g(x)\,dx \qquad \frac{1}{L} = \frac{k_0}{2\pi}$$

$$\hat{e}_n\cdot\hat{e}_m = \frac{1}{L}\int_{x_0}^{x_0+L} e^{-ink_0x} e^{imk_0x}\,dx = \frac{1}{L}\int_{x_0}^{x_0+L} e^{i(m-n)k_0x}\,dx = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$$

as $L\to\infty$, $k_0\to 0$

all values become acceptable

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ink_0x} \xrightarrow[nk_0\to k]{L\to\infty} f(x) = \int_{-\infty}^{\infty} c(k)\,e^{ikx}\,dk \qquad k \approx k_0 n$$

$c(k)$ is like c_n

e^{ikx} are the new basis vectors

$\vec{f} = \sum c_n \hat{e}_n$ but now n is continuous

$$\vec{f}\cdot\vec{g} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(x)\,g(x)\,dx$$

Look at $\boxed{\hat{e}_n \cdot \hat{e}_m} \longrightarrow \boxed{\hat{e}_{k_1} \cdot \hat{e}_{k_2}}$

$$\hat{e}_{k1} \cdot \hat{e}_{k2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik_1x} e^{ik_2x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k_2-k_1)x}\,dX = \begin{cases} \text{something } k_1 = k_2 \\ \text{nothing } k_1 \neq k_2 \end{cases}$$

$$= \delta(k_1 - k_2)$$

$$c_n = \hat{e}_n \cdot \tilde{f} = \frac{1}{L}\int_{x_0}^{x_0+L} e^{-ink_0x} f(x)\,dx$$

$$c(k) = \hat{e}_k \cdot \tilde{f} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk_2\, e^{-ikx} c(k_2)\, e^{ik_2x} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dk_2\, e^{i(k_2-k)x} c(k_2)$$

$$= \int_{-\infty}^{\infty} dk_2\, \delta(k_2 - k) c(k_2) = c(k)$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$$f(x) = \int_{-\infty}^{\infty} c(k) e^{ikx}\,dk \qquad c(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx}\,dx$$

$$\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dx \qquad \int_{-\infty}^{\infty} \delta(k)\, g(k-k_0)\,dk = g(k_0)$$

Fourier Transform is a change of basis from x to k

$$f(x_0) = \int_{-\infty}^{\infty} \delta(x - x_0) f(x)\,dx$$

or

$$f(x_0) = \int_{-\infty}^{\infty} e^{ikx_0} c(k)\,dk \text{ where } c(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx}\,dx$$

Fourier Transform is good because basis vectors are functions
eigenvectors of differential operators in X

$$\frac{d}{dx} f(x) = g(x)$$

$$\frac{d}{dx}(x^2+3x+2) = 2x+3$$

$$\{1, x, x^2, \dots\}$$

$$\frac{d}{dx}(x^3) = 3x^2 \qquad \frac{d}{dx}\vec{e}_3 = 3\vec{e}_2$$

e^{ikx} basis

$$\frac{d}{dx} e^{ikx} = ike^{ikx}$$

$$\frac{d}{dx}\hat{e}_k = ik\hat{e}_k$$

$$\frac{d^2}{dx^2} e^{ikx} = \frac{d}{dx} ike^{ikx} = (ik)^2 e^{ikx}$$

$$\frac{d^n}{dx^n}(e^{ikx}) = (ik)^n e^{ikx}$$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\int_{-\infty}^{\infty} c(k)\, e^{ikx}\, dk = \int_{-\infty}^{\infty} c(k)\left(\frac{d}{dx} e^{ikx}\right) dk = \int_{-\infty}^{\infty} ik\, c(k)\, e^{ikx}\, dk$$

$$\left(2\frac{d^2}{dx^2} + \frac{d}{dx}\right) f(x) = g(x)$$

Know $g(x)$

$$\left(2\frac{d^2}{dx^2} + \frac{d}{dx}\right) \int c(k) e^{ikx} dk = \int b(k) e^{ikx} dk$$

$$\int_{-\infty}^{\infty} \left(2(ik)^2 + ik\right) c(k) e^{ikx} dk = \int b(k) e^{ikx} dk$$

$$b(k) = c(k)\left(2(ik)^2 + ik\right)$$

$$c(k) = \frac{b(k)}{-2k^2 + ik}$$ Plug back into transform

$$f(x) = \int_{-\infty}^{\infty} \frac{b(k)}{-2k^2 + ik} e^{ikx} dk$$

Use: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$ $\quad c(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \frac{1}{2\pi} e^{-\frac{\sigma^2 k^2}{2}}$

$\frac{\partial f}{\partial t} = a^2 \frac{\partial^2 f}{\partial x^2}$ $\quad f(x,0) = \frac{1}{\quad}$

$$\frac{\partial f}{\partial t} = a^2 \frac{\partial^2 f}{\partial x^2} \qquad f(x,0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$$

a) Find ODE for $c(k,t)$

$$f(x,t) = \int_{-\infty}^{\infty} c(k,t)\, e^{ikx}\, dk$$

$$\frac{\partial f}{\partial t} = \int_{-\infty}^{\infty} \frac{dc}{dt} e^{ikx}\, dk \qquad a^2 \frac{\partial^2 f}{\partial x^2} = a^2 \int_{-\infty}^{\infty} (ik)^2 c(k,t)\, e^{ikx}\, dk$$

$$\Rightarrow \int_{-\infty}^{\infty} \dot{c}\, e^{ikx}\, dk = a^2 \int_{-\infty}^{\infty} -k^3 c(k,t) e^{ikx}\, dk$$

$$\boxed{\dot{c} = -a^2 k^2 c(k,t)}$$

b) Find $c(k,t)$ with $c(k,0) = c_0(k)$

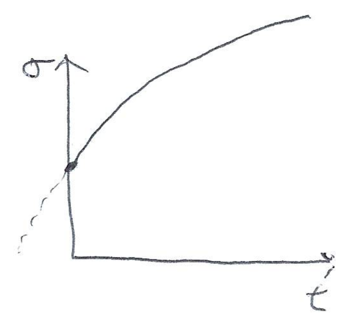
$$c(k,t) = A e^{-a^2 k^2 t}$$

$$c(k,t) = c_0(k)\, e^{-a^2 k^2 / t}$$

c) Plug in initial conditions to find $c_0(k)$ and then $c(k,t)$

$$c_0(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} e^{-ikx}\, dx = \frac{1}{2\pi} e^{-\frac{\sigma^2 k^2}{2}}$$

$$\boxed{c(k,t) = \frac{1}{2\pi} e^{-(a^2 t + \sigma^2) k^2/2}}$$

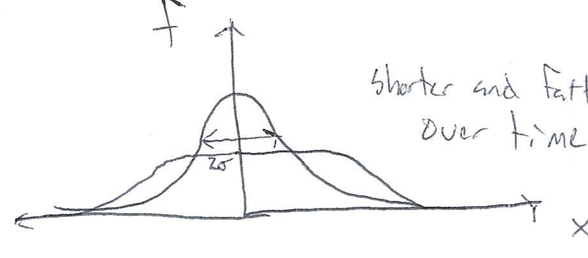


d) Find $f(x,t)$

$$f(x,t) = \int_{-\infty}^{\infty} c(k,t)\, e^{ikx}\, dk = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-(a^2 t + \sigma^2) k^2/2 + ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma^2 + 2a^2 t}} e^{\frac{-x^2}{2(\sigma^2 + 2a^2 t)}}$$

$$\sigma^2(t) = \sigma^2 + 2a^2 t$$



Shorter and fatter over time

Parseval's Theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 2\pi \int_{-\infty}^{\infty} |c(k)|^2 dk$$

$$\int_{-\infty}^{\infty} f^*(x) f(x) dx = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, c^*(k_1) e^{-ik_1x} c(k_2) e^{ik_2x}$$

$$\int_{-\infty}^{\infty} c^*(k) e^{-ikx} dk \int_{-\infty}^{\infty} c(k) e^{ikx} dk = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dk_1 dk_2\, c^*(k_1) c(k_2) e^{i(k_2-k_1)x}$$

$$= 2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_1 dk_2\, \delta(k_2-k_1)\, c^*(k_1) c(k_2)$$

$$= 2\pi \int_{-\infty}^{\infty} dk_1\, c^*(k_1) c(k_1)$$

$$= 2\pi \int_{-\infty}^{\infty} |c(k)|^2 dk$$

Convolution

What if $c(k) = a(k)\, b(k)$?

$$g(x) = \int_{-\infty}^{\infty} a(k) e^{ikx} dk \qquad h(x) = \int_{-\infty}^{\infty} b(k) e^{ikx} dk$$

$$f(x) = \int_{-\infty}^{\infty} a(k) b(k) e^{ikx} dk$$

$$= \int_{-\infty}^{\infty} a(k_0)\, \delta(k_0-k)\, b(k) e^{ikx} dk\, dk_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} a(k_0) \int dx_1\, e^{i(k_0-k)x_1}\, b(k) e^{ikx} dk\, dk_0$$

$$\cancel{= \frac{1}{2\pi} \int\int_{-\infty}^{\infty} a(k_0)}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int\int dk\, dk_0\, dx_1\, a(k_0) b(k) e^{ik(x-x_1)} e^{ik_0x_1}$$

$$\cancel{= \int\int dk_0\, dx_1\, a(k_0) b(k)\, \delta(x-x_1) e^{ik_0x_1}}$$

$$\frac{1}{2\pi} \int dk\, dx_1\, g(x_1) b(k) e^{ik(x-x_1)}$$

$$= \boxed{\frac{1}{2\pi} \int dx_1\, g(x_1) h(x-x_1)}$$

Phys 89 22Apr19

Discussion

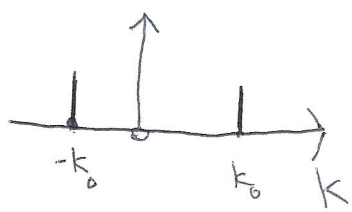
$$f(x) = \int_{-\infty}^{\infty} c(k) e^{ikx} dk \qquad c(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

FT of $\cos(k_0 x)$ $\qquad \int e^{iau} du = 2\pi\, \delta(a)$

$$c(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(k_0 x) e^{-ikx} dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik_0x} + e^{-ik_0x}}{2} e^{-ikx} dx = \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{i(k_0-k)x} + e^{i(-k_0-k)x} dx$$

$$= \frac{1}{2}\left(\delta(k_0-k) + \delta(-k_0-k)\right)$$



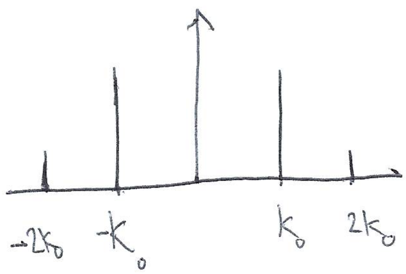
FT of $\frac{1}{2}[\delta(x_0-x) + \delta(-x-x_0))$

$$c(k) = \frac{1}{2\pi}\int \frac{1}{2}\delta(x_0-x)e^{-ikx} + \frac{1}{2}\delta(-x_0-x)e^{-ikx} dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{2}e^{-ikx_0} + \frac{1}{2}e^{ikx_0}\right) = \frac{1}{2\pi}\cos(kx_0)$$

FT of $\cos(k_0 x) + \frac{1}{4}\cos(2k_0 x)$

$$c(k) = \frac{1}{2}\left(\delta(k_0 - k) + \delta(-k_0 - k)\right) + \frac{1}{4}\left(\delta(2k_0 - k) + \delta(-2k_0 - k)\right)$$



- shifts in coordinates
  - $f(x) \longleftrightarrow c(k)$

Check $c_\Delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x+\Delta) e^{-ikx}\, dx$ $\quad y = x + \Delta \quad x = y - \Delta$

$dx = dy$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) e^{-ik(y-\Delta)}\, dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) e^{-iky} e^{ik\Delta}\, dy$$

$$= e^{ik\Delta}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) e^{-iky}\, dy\right) = e^{ik\Delta} c(k)$$

Small shift

$\Delta = \varepsilon \ll 1$

$$c_\varepsilon(k) = e^{ik\varepsilon} c(k) \approx (1 + ik\varepsilon) c(k)$$

IFT

$$g(x) = \int_{-\infty}^{\infty} (1 + ik\varepsilon) c(k) e^{ikx}\, dk$$

$$= f(x) + \int_{-\infty}^{\infty} i\varepsilon k\, c(k) e^{ikx}\, dk$$

$$= f(x) + i\varepsilon \int_{-\infty}^{\infty} c(k) \frac{d}{dx} e^{ikx} \frac{1}{i}\, dk = f(x) + \varepsilon \frac{d}{dx}\int_{-\infty}^{\infty} c(k) e^{ikx}\, dk$$

$$= f(x) + \varepsilon \frac{d}{dx} f(x) \approx f(x+\varepsilon)$$

- Recall

$$\frac{d^n}{dx^n} \leftrightarrow (ik)^n$$

$$g(x) = \int_{-\infty}^{\infty} e^{ik\Delta} c(k) e^{ikx} \, dk$$

$$= \int_{-\infty}^{\infty} \left(1 + ik\Delta + \frac{1}{2}(ik\Delta)^2 + \frac{1}{3!}(ik\Delta)^3 + \cdots\right) c(k) e^{ikx} \, dk$$

$$= \int_{-\infty}^{\infty} c(k)\left(1 + \Delta \frac{d}{dx} + \frac{1}{2}\Delta^2 \frac{d^2}{dx^2} + \frac{1}{3!}\Delta^3 \frac{d^3}{dx^3} + \cdots\right) e^{ikx} \, dk$$

$$= \left(1 + \Delta \frac{d}{dx} + \frac{\Delta^2}{2} \frac{d^2}{dx^2} + \cdots\right) \underbrace{\int_{-\infty}^{\infty} c(k) e^{ikx} \, dk}_{f(x)}$$

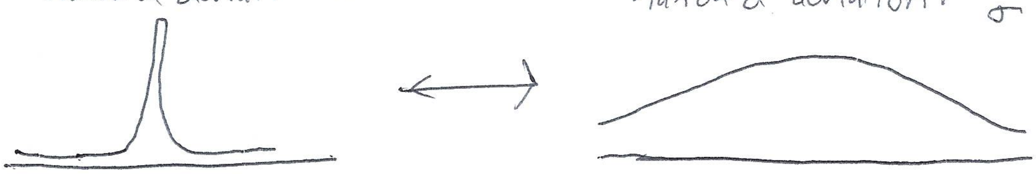
$$= e^{\Delta \frac{d}{dx}} f(x) = f(x+\Delta)$$

- Intuition

Localized $\longleftrightarrow$ delocalized

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \longleftrightarrow c(k) = \frac{1}{2\pi} e^{-\sigma^2 k^2/2}$$

standard deviation: σ standard deviation: $\frac{1}{\sigma}$



Best possible localization for both is $\sigma = 1$ for both

- Related to diagonalizing 2 matrices

x basis $\quad \delta(x-x_0) \quad e^{+ik_0x}$

K basis $\quad e^{-ikx_0} \quad \delta(k-k_0)$

$[x,k] \neq 0$

- Which operators are we diagonalizing?
  - $\hat{K} = \frac{1}{i}\frac{d}{dx}$

$$\underbrace{\frac{1}{i}\frac{d}{dx}}_{M} \underbrace{e^{ik_0x}}_{V} = \underbrace{k_0}_{\#} \underbrace{e^{ik_0x}}_{V} \qquad \{\overset{\lambda}{k_0}, \overset{V}{e^{ik_0x}}\}$$

still in x basis

$\delta(k-k_0)$ in k basis

- $\hat{X} = x$

x basis $\delta(x-x_0)$

v^TMw

$$\int g^*(x)\, x\, \delta(x-x_0)\, dx$$
$$= g^*(x_0)x_0 = x_0 \int g^*(x)\delta(x-x_0)$$
$$= \int g^*(x)\, x_0\, \delta(x-x_0)\, dx \qquad \text{True for any } g$$

$$v^TMw = \alpha v^Tw$$
$$Mw = \alpha w$$
$$x\delta(x-x_0) = x_0\delta(x-x_0)$$

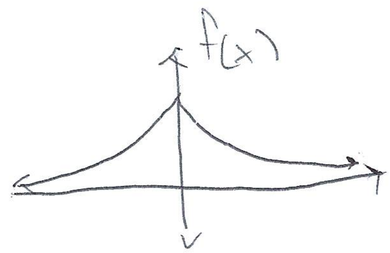
Commute?

$$[\hat{K}, \hat{X}] f(x)$$

$$= \left(\frac{1}{i}\frac{d}{dx}x - x\frac{1}{i}\frac{d}{dx}\right) f(x)$$

$$= \left(\frac{1}{i} f(x) + \frac{1}{i} x f'(x) - \frac{1}{i} x f'(x)\right) = \frac{1}{i} f(x)$$

$$[\hat{K}, \hat{X}] = \frac{1}{i} \mathbb{I}$$



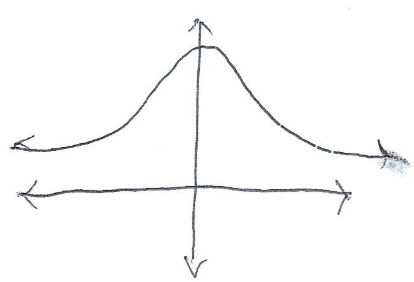
Example: $f(x) = \frac{2}{\alpha} e^{-\alpha|x|}$

$$C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{\alpha} e^{-\alpha|x|} e^{-ikx} dx$$

$$= \frac{1}{\pi\alpha}\left(\int_{-\infty}^{0} e^{\alpha x - ikx} dx + \int_{0}^{\infty} e^{-\alpha x - ikx} dx\right)$$

$$= \frac{1}{\pi\alpha}\left(\left.\frac{e^{(\alpha - ik)x}}{\alpha - ik}\right|_{-\infty}^{0} + \left.\frac{e^{-(\alpha+ik)x}}{-\alpha - ik}\right|_{0}^{\infty}\right)$$

$$= \frac{1}{\pi\alpha(\alpha - ik)} + \frac{1}{\pi\alpha(\alpha + ik)} = \frac{2}{\pi}\frac{1}{\alpha^2 + k^2} = C(k) \quad \text{Lorentzian}$$



$$\hat{H}\psi(x) = E\psi(x)$$

$$\left(a^2 \frac{d^2}{dx^2} - V(x)\right)\psi(x) = E\psi(x)$$

Laplace Transforms

$\int_{-\infty}^{\infty} x^2 e^{ikx} dx$ integral is difficult to deal with

$K = iK$ $\quad \int_{-\infty}^{\infty} x^2 e^{-Kx} dx$

$$F(s) = \int_0^{\infty} y(t) e^{-st} dt$$

Example $\quad 1 \to \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^{\infty} = \frac{1}{s}$

$e^{at} \to \int_0^{\infty} e^{at} e^{-st} dt = \int_0^{\infty} e^{(a-s)t} dt = \frac{e^{(a-s)t}}{a-s}\Big|_0^{\infty} = \frac{1}{s-a}$

$f'(t) \to \int_0^{\infty} \underset{v'}{f'(t)} \underset{u}{e^{-st}} dt$

$= f(t)e^{-st}\Big|_0^{\infty} + s\int_0^{\infty} f(t) e^{-st} dt = -f(0) + \int_0^{\infty} s f(t) e^{-st} dt$

$= -f(0) + s\int_0^{\infty} f(t) e^{-st} dt = -f(0) + sF(s)$

$$f''(t) \to \int_0^\infty f'' e^{-st}\,dt = -f'(0) + s\int_0^\infty f'(t) e^{-st}\,dt$$

$$= -f'(0) + s\left(f e^{-st}\Big|_0^\infty + s\int_0^\infty f e^{-st}\,dt \right)$$

$$= -f'(0) - s f(0) + s^2 F(s)$$

$$t^p \to \frac{\Gamma(p+1)}{s^{p+1}} \quad , \quad \Gamma(p+1) = p!$$

$p \geq -1$

Example

$$y'' - 10y' + 9y = 5t \qquad y(0) = -1 \;, y'(0) = 2$$

$$(s^2F - sy(0) - y'(0)) - 10(sF - y(0)) + 9F = 5$$

$$(s^2 - 10s + 9)F(s) + s - 12 = \frac{5}{s^2}$$

$$F(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

$$A = \frac{50}{81} \;, B = \frac{5}{9} \;, C = \frac{31}{81} \;, D = -2$$

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} + 2e^{t}$$

$$\delta(t-t_0) \to e^{-st_0}$$

$$\sin(at) = \frac{1}{2}\left(e^{iat} - e^{-iat}\right) = \frac{a}{s^2+a^2}$$

$$\cos(at) = \frac{s}{s^2+a^2}$$

$$\Theta(t-c) = \frac{e^{-cs}}{s} = \int_c^{\infty} e^{-st}\,dt$$

$$\int_0^t f(t-\tau)\,g(\tau)\,d\tau \to F(s)G(s)$$

$$y'' + y = \delta(t-4) \qquad y(0) = 0,\ y'(0) = 1$$

$$\cancel{\left(s^2 - sy(0) - y'(0)\right)F(s) + F(s) = e^{-4s}}$$

$$\cancel{s^2F(s) = e^{-4s} \to F(s) = \frac{e^{-4s}}{s^2}}$$

$$\left(s^2F(s) - sy(0) - y'(0)\right) + F(s) = e^{-4s}$$

$$(s^2+1)F(s) = 1 + e^{-4s} \to F(s) = \frac{1}{1+s^2} + \frac{e^{-4s}}{1+s^2}$$

$$\frac{1}{1+s^2} \to \sin t \qquad \frac{e^{-4s}}{1+s^2} \to \frac{e^{-4s}}{s}\left(\frac{s}{1+s^2}\right)$$

$$y(t) = \sin t + \int_0^t \Theta(t-4-\tau)\cos(\tau)\,d\tau$$

$$= \cancel{\sin t + \int_0^t \Theta(t-4)}$$

$$= \sin t + \int_0^t \Theta(t-4-\tau)\cos(\tau)\,d\tau \qquad = \Theta(t-4)\cos t$$

Phys 89 29 Apr 19
Discussion

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \qquad \int_{\mathbb{R}} \delta(x)\,dx = 1$$

$$\int_{\mathbb{R}} dx\, \delta(x) f(x) = f(0) \Rightarrow \int_{\mathbb{R}} \delta(x-a) f(x)\,dx = f(a)$$

$$\int_{-10}^{0} dx\, \delta(x-1) f(x) = 0$$

$$\begin{tikzcd} \text{Problem} \arrow[r] \arrow[d, "\text{hard}"'] & \text{Fourier Transform} \arrow[d, "\text{easy}"] \\ \text{Solution} & \text{Fourier Transform Solution} \arrow[l, "\text{inverse FT}"'] \end{tikzcd}$$

<u>Fourier Transform</u>

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k)\, e^{ikx}\, dk$$

$$f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx = \mathcal{F}\{f(x)\}$$

$$\mathcal{F}^{-1}\{f(k)\} = f(x)$$

$$(f * g)(x) = \int_{-\infty}^{\infty} dx' f(x') g(x-x')$$

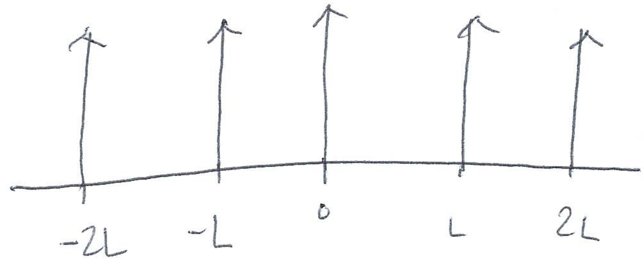
$$f * g = \mathcal{F}^{-1}\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\}$$

$$\mathcal{F}\left\{\frac{df}{dx}\right\} = ik\,\mathcal{F}\{f\} = +ikf(k)$$

$$\mathcal{F}\{f^{(n)}\} = (ik)^n f(k)$$

Dirac Comb

$$Ш_L(x) = \sum_{n=-\infty}^{\infty} \delta(x - nL)$$



a) f $\quad f(x+L) = f(x)$

Define $\{\hat{e}_n = e^{2\pi i n x/L}\}$

$(f+g)(x) = (f+g)(x+L)$

$cf(x) = cf(x+L)$

Define inner product $\langle \vec{f} | \vec{g} \rangle$

$$\vec{f} \cdot \vec{g} \equiv \frac{1}{L} \int_0^L f^*(x)\, g(x)\, dx$$

$\vec{f} \cdot \vec{f} \geq 0$ only 0 if $f = 0$

$$\vec{f} \cdot \vec{f} = \int_0^L dx\, |f(x)|^2$$

$$(\alpha f) \cdot g = \alpha^* (f \cdot g)$$

$$\frac{1}{L} \int_0^L (\alpha f)^* g\, dx = \frac{\alpha^*}{L} \int_0^L f^* g\, dx = \alpha^* (f \cdot g)$$

$$f \cdot g = (g \cdot f)^*$$

Show that $\{\hat{e}_n = e^{2\pi i n x/L}\}$ is orthogonal with our inner product

$$\hat{e}_n \cdot \hat{e}_m = \delta_{nm}$$

$$\frac{1}{L} \int_0^L dx\, e^{\frac{2\pi i x (n-m)}{L}} = (\cdots)\left(e^{2\pi i (m-n)} - 1\right) = 0 \text{ when } n \neq m$$

b) $\vec{f} = \sum_n c_n \hat{e}_n = \text{III}_L(x)$

Show that $c_n = \hat{e}_n \cdot f$

$$\hat{e}_n \cdot \vec{f} = \sum_m c_m \hat{e}_n \cdot \hat{e}_m = \sum_m c_m \delta_{mn} = c_n$$

$$\hat{e}_n \cdot \vec{f} = \frac{1}{L} \sum_{m=-\infty}^{\infty} \int_0^L dx\, \delta(x - mL) e^{2\pi i n x/L} = \frac{1}{L}$$

$$= \cancel{\frac{1}{L}\sum_{n=-\infty}^{\infty}} \quad \vec{f} = \frac{1}{L} \sum_{m=-\infty}^{\infty} e^{2\pi i n x/L} = \text{III}_L(x)$$

c) $$f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\sum_n \int \delta(x - nL) e^{-ikx} dx \right)$$

$$\boxed{f(k) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{-iknL}} \qquad f(k) e^{-ikL} = f(k)$$

d) $f(x)$ find convolution $(\text{III}_L * f)(x)$

$$= \int_{\mathbb{R}} dx'\, \text{III}_L(x') f(x - x') = \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} dx'\, \delta(x' - nL)\, f(x - x')$$

$$= \sum_{n=-\infty}^{\infty} f(x - nL)$$

e)

e) $\dot{y} + \gamma y = v(t)$, $y = y(t)$

$\mathcal{F}\{y\} = c(w)$ $\mathcal{F}\{\dot{y}\}$

$\mathcal{F}\{v(t)\} = f(w)$

$\mathcal{F}\{\dot{y} + \gamma y = v(t)\} \Rightarrow iw\, c(w) + \gamma\, c(w) = f(w)$

$c(w)(\gamma + iw) = f(w)$ $\qquad \mathcal{F}\{e^{-\gamma t}\theta(t)\} = \dfrac{1}{\gamma + iw}$

$$c(w) = \frac{f(w)}{\gamma + iw}$$

$$\mathcal{F}^{-1}\{c(w)\} = \mathcal{F}^{-1}\left\{\frac{f(w)}{\gamma + iw}\right\} = \cancel{\mathcal{F}^{-1}\left\{\mathcal{F}\{v(t)\}\,\mathcal{F}\left\{\frac{1}{\gamma + iw}\right\}\right\}}$$

$$= \mathcal{F}^{-1}\left\{\mathcal{F}\{v(t)\}\,\mathcal{F}\{e^{-\gamma t}\theta(t)\}\right\}$$

$$= e^{-\gamma t}\theta(t) * v(t)$$

$$\boxed{y_p(t) = \int d\tau\, e^{-\gamma\tau}\theta(\tau)\, v(t-\tau)} \qquad v(t) = V_0\,\delta(t)$$

$$y_p(t) = \int d\tau\, e^{-\gamma\tau}\theta(\tau)\, V_0\,\delta(t-\tau) = V_0\, e^{-\gamma t}\theta(t)$$

$$v(t) = V_0\, Ш_T(t)$$

$$y_p(t) = \sum_n V_0\, e^{-\gamma(t-nT)}\,\theta(t-nT)$$

Laplace Problem

$\hat{H}\psi = E\psi \qquad H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - eV(x)$

Goal: solve for $\psi(x)$ with laplace

$\mathcal{L}(f(x)) = f(s) = \int_0^\infty e^{-sx} f(x)\,dx \qquad V(x) = V(x+a)$

a) $V(x+na) = V(x)$

$V(\overset{x'}{x+a}) = V(x'+a) = V(x'+2a) = \cdots = V(x'+na)$

Show that $\mathcal{L}(V(x)) = \frac{1}{1-e^{-sna}}\int_0^a dx\, V(x)\, e^{-sx}$

$\int_{na}^{(n+1)a} dx\, V(x) e^{-sx} = \int_0^a dx'\, V(x') e^{-s(x'+na)} \qquad x = x'+na$

$= e^{-sna}\int_0^a dx\, V(x) e^{-sx} \qquad \mathcal{L}(V) = \int_0^a (\ldots) + \int_a^{2a}(\ldots) + \cdots = \sum_{n=0}^{\infty}\int_{na}^{(n+1)a} dx\, e^{-sx} V(x)$

$= \sum_{n=0}^{\infty} e^{-sna}\int_0^a dx\, V(x) e^{-sx} = \frac{1}{1-e^{-sa}}\int_0^a e^{-sx} V(x)\,dx$

c) $\psi(x) = e^{ikx} u(x) \qquad u(x+na) = u(x)$

$\psi(x+na) = e^{ikna}\psi(x) \qquad \psi'(x+na) = e^{ikna}\psi'(x)$

d) $u(x) = \frac{ma^2 e}{\hbar^2} V(x) \qquad k^2 = \frac{2m}{\hbar^2}E$

$\hat{H}\psi - E\psi = \psi''(x) + \left(\frac{2}{a^2}u(x) + k^2\right)\psi(x) = 0$

e) $\mathcal{L}(\hat{H}\psi - E\psi) = 0$

$$\mathcal{L}(\psi'') = s^2\psi(s) - s\psi'(0) - \psi'(0)$$

$$s^2\psi(s) - s\psi(0) - \psi'(0) + \frac{2}{a^2}\mathcal{L}(u(x)\psi(x)) + k^2\psi(s) = 0$$

$$\cancel{\psi(s)(s^2+k^2)}$$

$$\psi(s) = \frac{1}{s^2+k^2}\left(s\psi(0) + \psi'(0) - \frac{2}{a^2}\mathcal{L}(u(x)\psi(x))\right)$$

f) $u(x) = a u_0 \sum_n \delta(x-na)$ $\quad \mathcal{L}(u(x)\psi(x)) = a u_0 \sum_n \int_0^\infty dx\, \delta(x-na)\psi(x) e^{-sx}$

$$= a u_0 \sum_{n=0}^{\infty} \psi(na) e^{-sna}$$

$$\psi(s) = \frac{1}{s^2+k^2}\left(s\psi(0) + \psi'(0) - \frac{2}{a^2} a u_0 \sum_{n=0}^{\infty} \psi(na) e^{-sna}\right)$$

$$\cancel{\psi(x) = \psi_0 \cos kx + \psi'_0 \sin}$$

$$\psi(x) = \psi(0)\cos(kx)\Theta(x)$$

$$\psi(x) = \Theta(x)\left(\psi(0)\cos(kx) + \frac{\psi'(0)}{k}\sin(kx) - \frac{2u_0}{ak}\sum_{n=0}^{\infty} \psi(0) e^{ikna} \sin(k(x-na))\right)$$

Problems FT

Show 1) $\mathcal{F}\left[f^{(n)}(x)\right] = (i\omega)^n f(\omega)$

2) $\mathcal{F}(\delta(x)) = \frac{1}{\sqrt{2\pi}}$

3) $\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/4a}$

1) $\cancel{f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ik\omega x} dx}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int f(\omega) e^{i\omega x} d\omega$$

$$\frac{df}{dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega)(i\omega) e^{i\omega x} d\omega$$

$$\frac{d^2f}{dx^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega)(i\omega)^2 e^{i\omega x} d\omega \Rightarrow \frac{d^nf}{dx^n} = \frac{(i\omega)^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) e^{i\omega x} dx$$

$$= (i\omega)^n f(\omega) = \frac{d^nf}{dx^n}$$

2) $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} e^0 = \frac{1}{\sqrt{2\pi}}$

3) $\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/4a}$