PHYS 112 Homework Assignment 1

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Problem 1

(Kittel and Kroemer 2.4) It has been said that "six monkeys, set to strum unintelligently on typewriters for millions of years, would be bound in time to write all the books in the British Museum." This statement is nonsense, for it gives a misleading conclusion about very, very large numbers. Could all the monkeys in the world have typed out a single specified book in the age of the universe?

Suppose that 10^{10} mokeys have been seated at typewriters throughout the age of the universe, $10^{18}s$. This number of monkeys is about three times greater than the present human population of the earth. We suppose that a monkey can hit 10 typewriter keys per second. A typewriter may have 44 keys; we accept lowercase letters in place of capital letters. Assuming that Shakespeare's *Hamlet* has 10^5 characters, will the monkeys hit upon *Hamlet*?

a) Show that the probability that any given sequence of 10^5 characters typed at random will come out in the correct sequence (the sequence of *Hamlet*) is of the order of

$$\left(\frac{1}{44}\right)^{100000} = 10^{-164345}$$

where we have used $\log_{10}(44) = 1.64345$.

b) Show that the probability that a *monkey-Hamlet* will be typed in the age of the universe is approximately $10^{-164316}$. The probability of *Hamlet* is therefore zero in any operational sense of an event, so that the original statement at the beginning of this problem is nonsense: one book, much less a library, will never occur in the total literary production of the monkeys.

Solution a) The probability of hitting any specific key on the typewriter will be $1/44$ since there are 44 keys and each key is unique. If we are going to look at a single permutation of

100000 characters at random, our probability of a specific sequence to be

$$\left(\frac{1}{44}\right)^{100000}$$

We are going to take the logarithm of this number to make it easier to deal with.

$$\log_{10}\left(\left(\frac{1}{44}\right)^{100000}\right) = 100000\log_{10}\left(\frac{1}{44}\right) = -100000\log_{10}(44) = -164345$$

By undoing the logarithm, this gives us the equality we were looking for

$$\left(\frac{1}{44}\right)^{100000} = 10^{-164345}$$

b) Since the universe is about 10^{18} seconds old, and at a rate of 10 keys per second, each monkey will type about 10^{19} keys. We already found the probability of *Hamlet* showing up in a sequence of 10^5 characters so our probability for a single monkey typed it will be

$$P(\text{one monkey}) = 10^{19}10^{-164345} = 10^{-164326}$$

Since we have 10^{10} monkeys, our total probability will be

$$P(10^{10}\text{ monkeys}) = 10^{10}10^{-164326} = 10^{-164316}$$

which is what we were after. This is practically zero so our team of celestial monkeys have failed.

Problem 2

Consider a biased coin that outputs heads with probability θ and tails with probability $(1-\theta)$ on each "toss," where $0 \le \theta \le 1$ is an input parameter. You're using this coin to generate a random walk in 1 dimension. The starting position of the weak is $x = 0$. With every heads flip you move +1 in the x direction, and with tails you move -1.

a) Find the average x position after N flips of the coin.

b) Find the standard deviation in the position after N flips.

c) Using a) and b), find the Gaussian approximation of the position distribution after a large number of flips $N \gg 1$.

Solution **a)** Our probability can be described using the binomial distribution with $x = H - T$ so

$$P(H) = \frac{N!}{H!T!}\theta^H(1-\theta)^T = \frac{N!}{H!(N-H)!}\theta^H(1-\theta)^{N-H}$$

To make our equation easier to write, I am going to define h to be the probability of heads and t to be the probability of tails. The average amount of Heads can be calculated with

$$\langle H\rangle = \sum_{H=0}^{N}\frac{N!}{H!(N-H)!}h^H t^{N-H} H$$

We can see that

$$Hh^H = h\frac{\partial}{\partial h}(h^H)$$

which allows us to manipulate our expectation value.

$$\sum_{H=0}^{N}\frac{N!}{H!(N-H)!}h^H t^{N-H} H = \sum_{H=0}^{N}\frac{N!}{H!(N-H)!}\Big(h\frac{\partial}{\partial h}(h^H)\Big)t^{N-H}$$

$$= h\frac{\partial}{\partial h}\Big(\sum_{H=0}^{N}\frac{N!}{H!(N-H)!}h^H t^{N-H} = h\frac{\partial}{\partial h}(h+1-h)^N = hN(1)^{N-1} = hN$$

This will lead us to the expectation value of tails to be

$$\langle T\rangle = N(1-h)$$

so our average x will be

$$\langle x\rangle = \langle H\rangle - \langle T\rangle = N\theta - N + N\theta = N(2h-1)$$

b) Our standard deviation can be described by

$$\sigma = \sqrt{\langle (x-\langle x\rangle)^2\rangle} = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

Lets start off by seeing how the deviation is for a particular direction.

$$\sigma_H = \left\langle H^2\right\rangle - \langle H\rangle^2$$

$$\left\langle H^2\right\rangle = \sum_{H=0}^{N}\frac{N!}{H!(N-H)!}h^H t^{N-H} H^2$$

We can use differentiation to simplify this down into something workable like we did in the previous part.

$$H^2 h = \Big(h^2\frac{\partial}{\partial h}\Big)(h^H)$$

So our expectation value simplifies down to

$$\big\langle H^2 \big\rangle = \Big(h^2\frac{\partial}{\partial h}\Big)^2 \sum_{H=0}^{N} \frac{N!}{H!(N-H)!} h^H t^{N-H} = \Big(h^2\frac{\partial}{\partial h}\Big)^2 (h+t)^N$$

$$= \Big(h\frac{\partial}{\partial h}\Big) hN(h+t)^{N-1} = hN(h+t)^{N-1} + h^2N(N-1)(h+t)^{N-2}$$

$$= hN + h^2N(N-1) = hN + h^2N^2 - h^2N = hN(1+hN-h) = h^2N^2 + htN$$

$$= \langle H \rangle^2 + htN$$

Heading back towards the problem we are trying to solve.

$$x - \langle x \rangle = 2H - N - 2\langle H \rangle + N = 2(H - \langle H \rangle)$$

We will then square this and then take an average after using the piece we found previously in this part.

$$(x - \langle x \rangle)^2 = 4(H - \langle H \rangle)^2$$

$$\big\langle (x - \langle x \rangle)^2 \big\rangle = 4\big\langle (H + \langle H \rangle)^2 \big\rangle = 4htN = \sigma^2$$

This means that our standard deviation is

$$\sigma_x = 2\sqrt{htN}$$

c) The gaussian distribution is of the form

$$G = \Big(\frac{1}{\sqrt{2\pi\sigma^2}}\Big) e^{-\frac{(x-\langle x \rangle)^2}{2\sigma^2}}$$

Since $x = 2H - 1$ our Gaussian will be

$$G(x) = \Big(\frac{1}{\sqrt{8\pi htN}}\Big) e^{-\frac{(x-N(2h-1))^2}{8htN}}$$

Problem 3

Simulate the random walk considered in the previous problem using the parameter $\theta = 0.75$. Plot the distribution of position in the cases $N = \{50, 100, 150\}$ as a histogram. In each case sample over more than 1000 random walks to generate the histogram. In each case compare the theoretically predicted (approximate) Gaussian distribution to the simulated result by plotting the Gaussian on top of the histogram. Make sure they are both normalized in the same way so they can be compared on the same axes.

Solution I sampled about 2000 random walks to be certain my distribution would look pretty.

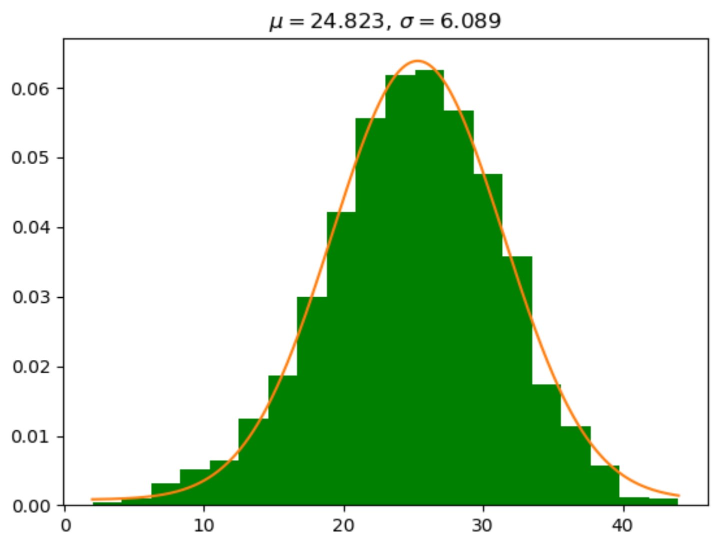

Figure 1: $N = 50$

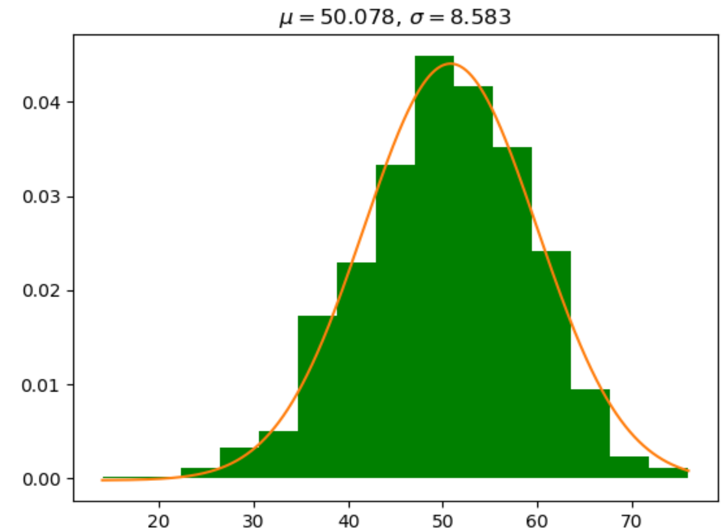

Figure 2: $N = 100$

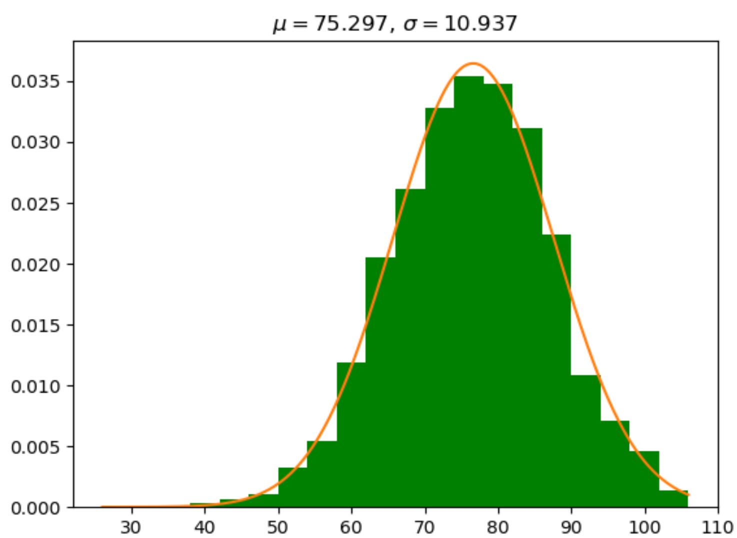

Figure 3: $N = 150$

Problem 4

The one dimensional Ising model consists of $N+1$ spins that interact with their nearest neighbors on a chain so that the total energy is

$$E = \frac{\epsilon}{2}\sum_{i=1}^{N}(1-\sigma_i\sigma_{i+1})$$

Here $\sigma_i = \pm 1$ is the spin on the site i, which can take two values. Our goal is to compute the entropy $S(E)$. At first this looks complicated, because the energy is not a sum of independent variables, until we notice that we can view this problem in another way. Instead of thinking in terms of the $N+1$ spins on the chain we can think in terms of the N links between nearby spins. A link on which the spins are anti-aligned is called a domain wall because it separates a domain with spin up on one side of it from a domain of spin down on the other side. In this language, the total energy is just proportional to the number of domain walls n, i.e. $E = n\epsilon$.

a) Show that the number of states with n domain walls is

$$\Omega(n) = \frac{2(N!)}{n!(N-n)!}$$

What is the reason for the factor 2?

b) Use the Stirling approximation to compute the entropy $S(E)$ as done in class for the case of the independent spins. ***Info: note that the two results are almost identical, so the specific heat as a function of temperature is also expected to behave in the same way as in the simple paramagnet.***

Solution **a)** If we have N links between nearby spins, we have N possible locations for domain walls. We want to choose n domain walls which will give us

$$\Omega = \frac{N!}{n!(N-n)!}$$

But this can't be right because there are two different types of domain walls. We could have an (up, down) for a (down, up) domain wall so we have twice as many states than that first guess.

$$\Omega = \frac{2(N!)}{n!(N-n)!}$$

b) Our entropy function will be

$$S(E) = k\ln(\Omega) = k(\ln(2) + \ln(N!) - \ln(n!) - \ln((N-n)!))$$

Using Stirling's approximation which is

$$\ln(N!) = N\ln(N) - N$$

we get

$$S(E) = k(\ln(2) + N\ln(N) - N - n\ln(n) + n - (N-n)\ln(N-n))$$
$$= k(\ln(2) + N\ln(N) - N - \frac{E}{\epsilon}\ln\left(\frac{E}{\epsilon}\right) + \frac{E}{\epsilon} - (N - \frac{E}{\epsilon})\ln\left(N - \frac{E}{\epsilon}\right))$$
$$= k(\ln(2) + N\ln(N) - N - \frac{E}{\epsilon}(\ln(E) - \ln(\epsilon)) + \frac{E}{\epsilon} - (N - \frac{E}{\epsilon})(\ln(\epsilon N - E) - \ln(\epsilon))$$
$$= k(\ln(2) + N\ln(N) - N + N\ln(\epsilon) - \frac{E}{\epsilon}\ln(E) + \frac{E}{\epsilon})$$
$$S(E) = k(\ln(2) + N(\ln(N\epsilon) - 1) + \frac{E}{\epsilon}(1 - \ln(E)))$$

Problem 5

Einstein's oversimplified model of a harmonic solid consists of N independent quantum harmonic oscillators all with the same frequency ω. The total energy is then

$$E = \bar{h}\omega\sum_{i=1}^{N} n_i$$

In class we derived the number of states with q excitation quanta, i.e. $q = \sum_i n_i$,

$$\Omega(q) = \frac{(N+q-1)!}{(N-1)!q!}$$

a) Using Stirling's formula find the experession for the entropy $S(E)$ for the limit of large N and q.

b) Compute the temperature $T(E)$.

c) Invert the result of b) to get $E(T)$ and from it derive the heat capacity $C_v(T)$ of the Einstein solid. Obtain simplified expressions for the low temperature limit. What should the temperature be for this expression to be valid? Obtain a simplified expression for the high temperature limit.

Solution a)

$$S(E) = k\ln(\Omega)$$
$$= k\ln\left(\frac{(N+q-1)!}{(N-1)!q!}\right)$$

Since q and N are large, the one isn't going to be doing much so we can pretend it doesn't exist.

$$S(E) = k\ln\left(\frac{(N+q)!}{N!Q!}\right)$$
$$= k(\ln((N+q)!) - \ln(N!) - \ln(q!))$$

We can then use Stirling's Approximation to simplify this further.

$$S(E) = k((N+q)\ln(N+q) - (N+q) - N\ln(N) + N - q\ln(q) + q)$$

$$= k((N+q)\ln(N+q) - N\ln(N) - q\ln(q))$$

Replacing q to give an explicit energy dependence gives us

$$S(E) = k((N + \frac{E}{\hbar\omega})\ln\left(N + \frac{E}{\hbar\omega}\right) - N\ln(N) - \frac{E}{\hbar\omega}\ln\left(\frac{E}{\hbar\omega}\right))$$

b) We know our temperature is related to the entropy by

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{E,N}$$

Differentiating our entropy in respect to E gives us

$$\frac{\partial S}{\partial E} = k(\frac{1}{\hbar\omega}\ln\left(N + \frac{E}{\hbar\omega}\right) + (N + \frac{E}{\hbar\omega})\frac{1}{\hbar\omega}\frac{1}{N + \frac{E}{\hbar\omega}} - \frac{1}{\hbar\omega}\ln\left(\frac{E}{\hbar\omega}\right) - \frac{E}{\hbar\omega}\frac{\hbar\omega}{E}\frac{1}{\hbar\omega})$$

$$\frac{1}{T} = \frac{k}{\hbar\omega}\ln\left(\frac{N\hbar\omega + E}{E}\right)$$

This means our temperature as a function of energy is just the reciprocal.

$$T(E) = \left(\frac{k}{\hbar\omega}\ln\left(\frac{N\hbar\omega + E}{E}\right)\right)^{-1}$$

c) We can start where we left off and try to rearrange $T(E)$ to get $E(T)$.

$$\frac{1}{T} = \frac{k}{\hbar\omega}\ln\left(\frac{N\hbar\omega + E}{E}\right)$$

$$\frac{\hbar\omega}{kT} = \ln\left(\frac{N\hbar\omega}{E}\right)$$

$$e^{\frac{\hbar\omega}{kT}} = \frac{N\hbar\omega}{E} + 1$$

$$E(T) = \frac{N\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}$$

Now that we have that, we can solve for our heat capacity $C_v(T)$ which is just

$$C_v(T) = \left(\frac{\partial E}{\partial T}\right)_{N,V}$$

$$C_v(T) = -\frac{N\hbar\omega}{(e^{\frac{\hbar\omega}{kT}} - 1)^2}\frac{\hbar\omega}{k}e^{\frac{\hbar\omega}{kT}}\left(\frac{-1}{T^2}\right) = \frac{N\hbar^2\omega^2}{kT^2}\frac{e^{\frac{\hbar\omega}{kT}}}{(e^{\frac{\hbar\omega}{kT}} - 1)^2}$$

At low temperatures, say $kT \ll \hbar\omega$, this goes like

$$C_v(T_{Low}) = \frac{N\hbar^2\omega^2}{kT^2e^{\frac{\hbar\omega}{kT}}}$$

For high temperatures, say $kT \gg \hbar\omega$, the heat capacity goes like

$$C_v(T_{High}) = \frac{N\hbar^2\omega^2}{kT^2}$$