Analytic Mechanics Lecture 6

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Changing Mass Problem

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \frac{d(m\boldsymbol{v})}{dt} = \left(\frac{dm}{dt}\right)\boldsymbol{v} + m\left(\frac{dv}{dt}\right) = \left(\frac{dm}{dt}\right)\boldsymbol{v} + m\dot{\boldsymbol{v}}$$

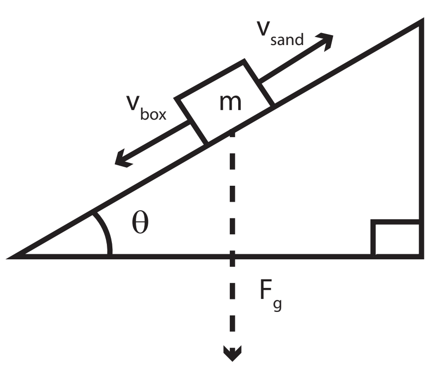

Figure 1

There is a box filled with sand moving down a frictionless slope with velocity $\boldsymbol{v}$. Sand leaks out of the box moving at a velocity $-\boldsymbol{v}$. The sand leaks out at a constant rate so our mass is

$$m(t) = m_0 - kt$$

At $t = 0$ the box system is of mass m_0 and the velocity is $\boldsymbol{v}(0) = 0$. At time t, the momentum of the system is

$$\boldsymbol{p_i} = \boldsymbol{p}(t) = m\boldsymbol{v}$$

At time $t + dt$ the momentum is

$$\boldsymbol{p_f}(t + dt) = (m + dm)(\boldsymbol{v} + d\boldsymbol{v}) + dm(\boldsymbol{v} - \boldsymbol{v})$$

Rate of change over time of this momentum will then be

$$\frac{(m + dm)(v + dv) + dm(0) - mv}{dt} = F = mg\sin(\theta)$$

We can ignore the $dmdv$ term because it is very small.

$$m\frac{dv}{dt} = -\Big(\frac{dm}{dt}\Big)v + mg\sin(\theta)$$

$$(m_0 - kt)\frac{dv}{dt} = kv + (m_0 - kt)g\sin(\theta)$$

$$\frac{d}{dt}((m_0 - ky)) = (m_0 - kt)g\sin(\theta)$$

We can integrate this to find the velocity.

$$\int_0^v d((m_0 - kt)v') = \int_0^t ((m_0 - kt)g\sin(\theta))dt'$$

$$(m_0 - kt)v = (m_0 - \frac{kt}{2})tg\sin(\theta)$$

$$v(t) = \frac{(m_0 - \frac{kt}{2})tg\sin(\theta)}{m_0 - kt}$$

Oscillations

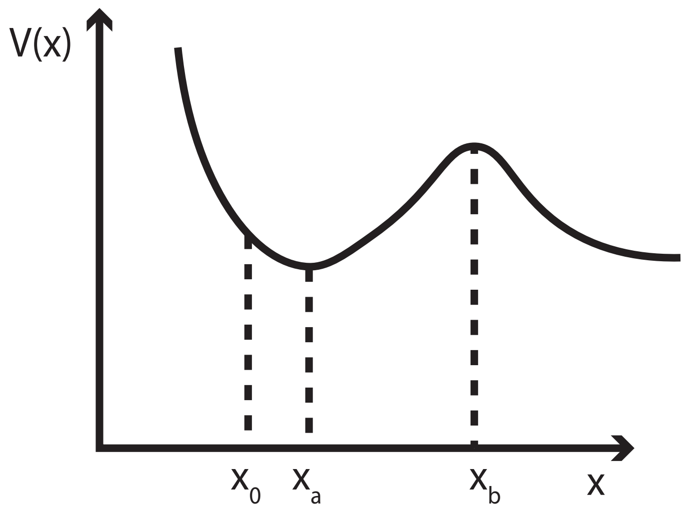

Figure 2

We have some potential energy $V(x)$ and since it behaves nicely, we can taylor expand it at $x_a = x_0$.

$$V(x_a) = V(x_0) + (x_a - x_0)\Big|_{x_0} \frac{\partial V}{\partial x_a} + \frac{1}{2}(x_a - x_0)^2\Big|_{x_0 \frac{\partial^2 V}{\partial x_a^2}} + \dots$$

We define $V(x_0) = 0$, $x = x_a - x_0$

$$V(x) = x\frac{\partial V}{\partial x}\Big|_{x_0} + \frac{1}{2}x^2\frac{\partial^2 V}{\partial x^2}\Big|_{x_0} + \dots$$

If x_0 is a local extremum then

$$\frac{\partial V}{\partial x}\Big|_{x_0} = 0$$

Which leaves us, after ignoring all terms third order or higher

$$V(x) = \frac{1}{2}x^2\frac{\partial^2 V}{\partial x^2}\Big|_{x_0}$$

We have two conditions for the stability of the equilibrium.

1. If x_0 is a stable equilibrium, the second partial derivative will be positive.

$$\frac{\partial^2 V}{\partial x^2}\bigg|_{x_0} > 0$$

2. If x_0 is an unstable equilibrium, the second partial derivative will be negative.

$$\frac{\partial^2 V}{\partial x^2}\bigg|_{x_0} < 0$$

The total energy of a mass m moving around x_0 would then be

$$E = \frac{1}{2}m\dot{x}^2 + V(x) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

With

$$k = \frac{\partial^2 V}{\partial x^2}\bigg|_{x_0}$$

If energy is conserved, we get

$$\frac{dE}{dt} = 0 = \frac{1}{2}m(2\dot{x}\ddot{x}) + \frac{1}{2}k(2x\dot{x})$$

$$\frac{dE}{dt} = \dot{x}(m\ddot{x} + kx) = 0$$

If $\dot{x} = 0$, the object is at rest. otherwise

$$\ddot{x} = -\frac{kx}{m}$$

If we say

$$\omega_0^2 = \frac{k}{m}$$

Then

$$x = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

Which is the solution for a simple harmonic oscillator.

General Oscillatory Motion

We want to be able to describe more complicated oscillatory motion than just the simple harmonic oscillator so we need to add a couple more types of forces.

1. Damping force:

$$F_b(x) = -b\dot{x}$$

In what we deal with for now, $b > 0$ to remove impurity.

2. Driving force:

$$F_d(x) = F_0\cos(\omega t)$$

General Equation of Motion

Here we are going to try to find the general equation of motion for some type of general oscillator. A general equation will come of the form

$$x_g(t) = x_p(t) + x_h(t)$$

x_p is the particular solution for our given driving force and x_h is the solution to the homogeneous form of our equation.

$$m\ddot{x} = -kx + F_b + F_d = -kx - b\dot{x} + F_0\cos(\omega t)$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos(\omega t)$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

with

$$\gamma = \frac{b}{m}$$

$$\omega_0^2 = \frac{k}{m}$$

Particular Solution

Let's let

$$F_0\cos(\omega t) = Re[F_0 e^{i\omega t}]$$

We can then say that our particular solution is

$$x_p = Re[\widetilde{A}e^{i\omega t}], \tilde{A} \in \mathbb{C}$$

We now want to solve for $\widetilde{A}$

$$\widetilde{A}(i\omega)^2 e^{i\omega t} + \gamma\widetilde{A}(i\omega)e^{i\omega t} + \omega_0^2\widetilde{A}e^{i\omega t} = \left(\frac{F}{m}\right)e^{i\omega t}$$

$$\widetilde{A} = \frac{\left(\frac{F_0}{m}\right)}{(\omega_0^2 - \omega^2) + i\gamma\omega} = |\widetilde{A}|e^{i\delta}$$

$$\delta = \arctan\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right)$$

$$|\widetilde{A}| = \frac{\left(\frac{F_0}{m}\right)}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$$