Analytic Mechanics Lecture 5

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Conservative Forces

Previously we found that

$$W = \int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{l} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$$

A conservative force is a force, when integrated over, is path independent. This means that no matter what path we take, our integral will always end up the same. This can be very powerful as it will allow us to abuse coordinate systems to evaluate integrals. Another way of saying this is that the line integral of the force over a closed path is zero.

$$W = \oint \boldsymbol{F} \cdot d\boldsymbol{l} = 0$$

We can use stoke's theorem to relate this line integral to a surface integral

$$\oint \boldsymbol{F} \cdot d\boldsymbol{l} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{a}$$

For this surface integral to be zero, it means that the force has no curl. Using a vector identity, we can rewrite the curl of $\boldsymbol{F}$ in a more useful way.

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} U) = 0$$

$$\boldsymbol{F} = -\boldsymbol{\nabla} U$$

This means that we can attribute the gradient of a scalar field to a conservative force. I used U because this gradient field is potential energy.

Conservation of Energy

If we have some conservative force $\boldsymbol{F}$

$$\Delta U = -\int_a^b \boldsymbol{F} \cdot d\boldsymbol{l} = -\Big(\frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2\Big)$$

$$U_a + \frac{1}{2}mv_a^2 = U_b + \frac{1}{2}mv_b^2$$

If we defined our mechanical energy to be

$$E = U + \frac{1}{2}mv^2$$

The previous relation tells us that energy is conserved with a conservative force.

Example 1

Consider a force $\boldsymbol{F} = k\hat{\boldsymbol{l}}$ with k being constant.

$$W = \int_a^b \boldsymbol{F} \cdot d\boldsymbol{l} = k\int_a^b \hat{\boldsymbol{l}} \cdot d\boldsymbol{l} = k\int_a^b dl$$

This is path dependent so this force is not conservative.

Example 2

Now let's consider the gravitational force

$$\boldsymbol{F} = -G\frac{mM}{r^2}\hat{\boldsymbol{r}}$$

$$W = \int_a^b \boldsymbol{F} \cdot d\boldsymbol{l} = -GmM\int_a^b \frac{1}{r^2}\hat{\boldsymbol{r}} \cdot d\boldsymbol{r}$$

$$= GmM\Big(-\frac{1}{r}\Big)\Big|_a^b = GmM\Big(\frac{1}{r_a} - \frac{1}{r_b}\Big)$$

This implies that our potential energy is

$$U(r) = -\frac{GmM}{r}$$

Power

A force can be decomposed into a conservative part and a non-conservative part

$$\boldsymbol{F} = \boldsymbol{F_c} + \boldsymbol{F_{nonc}}$$

We can rewrite our equation for work to get our line integral to be a time integral.

$$W = \frac{1}{2}m(v_2^2 - v_1^2) = \int_a^b \boldsymbol{F} \cdot d\boldsymbol{l} = \int_{t_1}^{t_2} \boldsymbol{F} \cdot \boldsymbol{v} dt$$

$$= \int_{t_1}^{t_2} \boldsymbol{F_c} \cdot \boldsymbol{v} dt + \int_{t_1}^{t_2} \boldsymbol{F_{nonc} \cdot v dt}$$

$$= \Delta U + \int_{t_1}^{t_2} \boldsymbol{F_{nonc}} \cdot \boldsymbol{v} dt$$

Rearranging this and using E we find

$$\Delta E = \int_{t_1}^{t_2} \boldsymbol{F_{nonc}} \cdot \boldsymbol{v} dt = \int_{E_1}^{E_2} dE$$

We can differentiate this result in respect to time and get

$$\frac{d}{dt} \int_{E_1}^{E_2} dE = \frac{d}{dt} \int_{t_1}^{t_2} \boldsymbol{F_{nonc}} \cdot \boldsymbol{v} dt$$

$$\frac{dE}{dt} = \boldsymbol{F_{nonc}} \cdot \boldsymbol{v} = \text{Power}$$