Analytic Mechanics Lecture 4

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(Weak) Equivalence Principle

The ratio of inertial mass and gravitational mass has been calculate to high precision

$$\frac{m_i}{m_g} = 1 \pm 10^{-12}$$

This suggests that the inertial mass and gravitational mass are probably equivalent and that the mass is independent of composition.

$$m_i = m_g = m$$

Dropping a Particle with Air Resistance and Gravity

We can consider a particle falling with only two forces acting on it: air resistance and gravity. We can then say the net force is

$$\boldsymbol{F_{tot}} = \boldsymbol{F_g} + \boldsymbol{F_r} = (mg - mkv^2)\hat{\boldsymbol{y}} = m\boldsymbol{a} = ma\hat{\boldsymbol{y}}$$

We are going to find our equation of motion to get a better feel for what is going on in this system.

$$ma = mg - mkv^2$$

$$a = g - kv^2 = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}$$

This last piece can be rewritten as

$$v\frac{dv}{dy} = \frac{1}{2}\frac{dv^2}{dy}$$

This gives us our general solution (using $\beta = \frac{-g}{k}$)

$$v^2 = \beta e^{-2ky} - \beta$$

Momentum

$$\boldsymbol{F} = m\boldsymbol{a} = m\frac{d\boldsymbol{v}}{dt}$$

If m is constant, we can rewrite this equation as

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$$

Impulse

Impulse will be the change in momentum.

$$\Delta\boldsymbol{p} = \boldsymbol{p_f} - \boldsymbol{p_i} = \int_{t_i}^{t_f} \boldsymbol{F} dt = \text{impulse}$$

If we had some force acting on some object for a short period of time

$$\boldsymbol{F} = \begin{cases} \boldsymbol{F_0} & [t, t+\Delta t] \\ 0 & otherwise \end{cases}$$

The impulse will be the integral of this equation.

$$\int_t^{t+\Delta t} \boldsymbol{F_0} dt = \boldsymbol{F_0}\Delta t$$

To find the distance travelled, we need to find the average force.

$$\langle F \rangle = \frac{\int_t^{t+\Delta t} F_0 dt}{\Delta t} = \frac{F_0 \Delta t}{\Delta t} = F_0$$

The average acceleration would then be

$$\langle a \rangle = \frac{F_0}{m}$$

We can then find the distance travelled as

$$\Delta x = \frac{1}{2} \langle a \rangle (\Delta t)^2 = \frac{F_0}{2m}(\Delta t)^2$$

$$\lim_{t\to\infty} \Delta x = 0$$

This means that the object doesn't move, even if our applied force is not zero.

Work and Energy

Work can be defined as a force $\boldsymbol{F}$ acting on an object along some path Γ.

$$W = \int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{l}$$

$d\boldsymbol{l}$ is our line element is expressed in cartesian coordinates:

$$d\boldsymbol{l} = dx_i \hat{\boldsymbol{x}}$$

Cylindrical:

$$d\boldsymbol{l} = dr\hat{\boldsymbol{r}} + rd\hat{\boldsymbol{\phi}} + dz\hat{\boldsymbol{z}}$$

Spherical:

$$d\boldsymbol{l} = dr\hat{\boldsymbol{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin(\theta)d\phi\hat{\boldsymbol{\phi}}$$

Let's complete the integral and see if we can get work as a function of velocity.

$$W = \int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{l} = \int_\Gamma \frac{d(m\boldsymbol{v})}{dt} \cdot \boldsymbol{v}dt = m\int_\Gamma \frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v}dt = m\int_\Gamma d\boldsymbol{v} \cdot \boldsymbol{v}$$

$$= \frac{d(\boldsymbol{v} \cdot \boldsymbol{v})}{dt} = \boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt} + \frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v}$$

Using a magical trick where we can "cancel out" the dt's, we get

$$2d\boldsymbol{v} \cdot \boldsymbol{v} = d(\boldsymbol{v} \cdot \boldsymbol{v})$$

$$W = \frac{m}{2}\int_\Gamma d(\boldsymbol{v} \cdot \boldsymbol{v}) = \frac{-1}{2}m(v_b^2 - v_a^2)$$

We can see now that work is the change in kinetic energy.