Analytic Mechanics Homework Assignment 4

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21Feb20

Problem 1

An undamped harmonic oscillator with mass m is driven by a force $F(t) = F_0 \sin(\omega t)$, F_0 being a constant, which is switched on at time $t = 0$.

a) Find the response function $x(t)$ for $t > 0$ with the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$.

b) Find $x(t)$ for $\omega = \omega_0$ by taking the limit $\omega \to \omega_0$ in the answer for part a.

c) Sketch your result for $x(t)$.

Solution

a) We start with our differential equation

$$\ddot{x} = \omega_0^2 x = \frac{F_0}{m} \sin(\omega t)$$

The solution to the homogeneous equation will be in the form

$$x_h = A \sin(\omega_0 t) + B \cos(\omega_0 t)$$

We will have problems with this equation if $\omega = \omega_0$ because the simplest ansatz will be linearly dependent on the homogeneous solution. In order to find a solution, we will need to split up our solutions into one where the driving force is resonant with the oscillator and one where the driving force is not resonant with the oscillator. Let's start where $\omega \neq \omega_0$

$$\text{ansatz} = x = A \sin(\omega t) + B \cos(\omega t)$$

$$\dot{x} = A\omega \cos(\omega t) - B\omega \sin(\omega t)$$

$$\ddot{x} = -A\omega^2 \sin(\omega t) - B\omega^2 \cos(\omega t)$$

Plugging this all into our main equation we get

$$-A\omega^2 \sin(\omega t) - B\omega^2 \cos(\omega t) + \omega_0^2(A\sin(\omega t) + B\cos(\omega t)) = \frac{F_0}{m}\sin(\omega t)$$

This gives us

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$
$$B = 0$$

So the particular solution we have for a non-resonant oscillator is

$$x_1 = \frac{F_0}{m(\omega_0^2 - \omega^2)}\sin(\omega t) + A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

Using our initial conditions we get

$$x_1(0) = 0 = B$$

$$\dot{x_1} = \frac{F_0\omega}{m(\omega_0^2 - \omega^2)}\cos(\omega t) + A\omega_0\cos(\omega_0 t)$$
$$\dot{x_1}(0) = 0 = \frac{F_0\omega}{m(\omega_0^2 - \omega^2)} + A\omega_0$$
$$A = -\frac{F_0\omega}{m\omega_0(\omega_0^2 - \omega^2)}$$

So the solution for the system not at resonance is

$$x_1 = \frac{F_0}{m(\omega_0^2 - \omega^2)}\sin(\omega t) - \frac{F_0\omega}{m\omega_0(\omega_0^2 - \omega^2)}\sin(\omega_0 t)$$

We still want to find out what happens if $\omega = \omega_0$, so we will have to use a trick to find a solution that is linearly independent to the homogeneous solution. We can do this by throwing in a t.

$$x_2 = x_h + x_h$$
$$\dot{x_2} = \dot{x_h} + x_h + t\dot{x_h}$$
$$\ddot{x_2} = \ddot{x_h} + \dot{x_h} + \dot{x_h} + t\ddot{x_h} = 2\dot{x_h} + \ddot{x_h} + t\ddot{x_h}$$

Plugging this into our main equation gives us

$$2\dot{x_h} + \ddot{x_h} + t\ddot{x_h} + \omega_0^2(x_h + tx_h) = 2\dot{x_h}$$

We were able to reduce it to this as the homogeneous solution is equal to zero, and we have multiple copies of it in our expression. So now we have

$$2\dot{x_h} = 2A\omega\cos(\omega t) - 2B\omega\sin(\omega t) = \frac{F_0}{m}\sin(\omega t)$$

This gives us

$$A = 0$$
$$B = -\frac{F_0}{2m\omega}$$

So the solution at resonance is

$$x_2 = -\frac{F_0 t}{2m\omega}\cos(\omega t) + A\sin(\omega t) + B\cos(\omega t)$$

Plugging in our initial conditions gives

$$x_2(0) = 0 = B$$

$$\dot{x_2} = -\frac{F_0}{2m\omega}\cos(\omega t) + \frac{F_0\omega t}{2m\omega}\sin(\omega t) + A\omega\cos(\omega t)$$
$$\dot{x_2}(0) = 0 = A\omega - \frac{F_0}{2m\omega}$$
$$A = \frac{F_0}{2m\omega^2}$$

So the solution at resonance is

$$x_2 = -\frac{F_0 t}{2m\omega}\cos(\omega t) + \frac{F_0}{2m\omega^2}\sin(\omega t)$$

b) The solution is what we found at the end of part a.

$$x = -\frac{F_0 t}{2m\omega}\cos(\omega t) + \frac{F_0}{2m\omega^2}\sin(\omega t)$$

c)

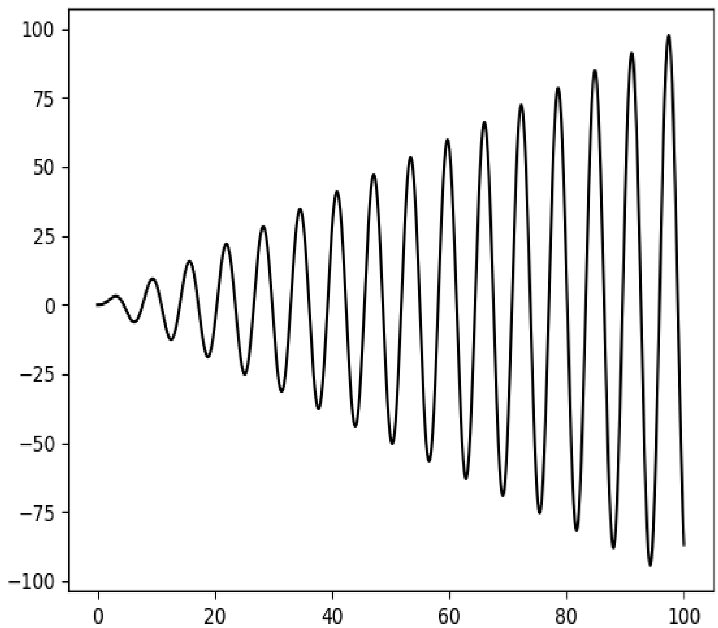

Figure 1: Position vs time for the resonant solution with coefficients set to 1.

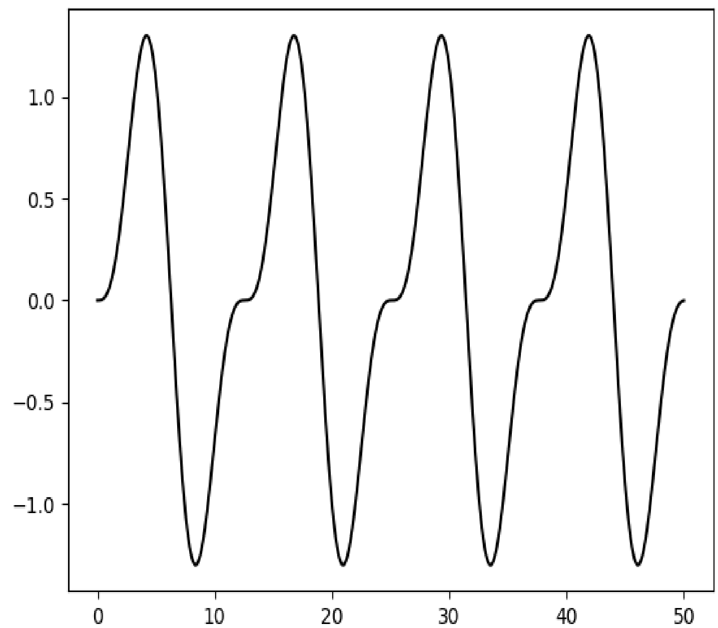

Figure 2: Position vs time for the non-resonant solution with the driving force having half the frequency of the natural frequency.

Problem 2

A heavy particle of mass m is hung at the lower end of a light vertical elastic spring of natural length l_0 and modulus (spring constant) $2mg$, where g is the acceleration due to gravity. At time $t = 0$ the system is in the equilibrium. Then the upper end of the spring is made to execute vertical oscillation so that its downward displacement at time t is $a\sin(\omega t)$, where a is a constant and $\omega^2 = 2g/l_0$. Find the displacement of the particle at this instant.

Solution

This problem runs very similarly to the previous problem. Our differential equation will be

$$\ddot{x} + \frac{2g}{m}x = a\sin(\omega t) = \ddot{x} + \omega_0^2 x$$

with

$$\omega_0^2 = \frac{2g}{m}$$

The homogeneous solution is of the form

$$x_h = A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

Using what we found in problem 1, the particular solution is

$$x_p = \frac{a}{\omega_0^2 - \omega^2}$$

We can then solve for our general solution's coefficients

$$x = \frac{a}{\omega_0^2}\sin(\omega t) + A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

$$x(0) = 0 = B$$

$$\dot{x} = \frac{a\omega}{\omega_0^2 - \omega^2}\cos(\omega t) + A\omega_0\cos(\omega_0 t)$$

$$\dot{x}(0) = 0 = \frac{a\omega}{\omega_0^2} + A\omega_0$$

$$A = -\frac{a\omega}{\omega_0(\omega_0^2 - \omega^2)}$$

So the displacement function of our mass is

$$x = \frac{a}{\omega_0^2 - \omega^2}\sin(\omega t) - \frac{a\omega}{\omega_0(\omega_0^2 - \omega^2)}\sin(\omega_0 t)$$

Problem 3

P and Q are two particles, each with mass m, attached to the two ends of a light inextensible string of length $2l$, which passes through a small smooth hole O in a smooth horizontal table. P is free to slide on the table, Q hangs freely below the hole O. Initially OQ is of length l, and P is projected from rest, at right angle to OP with speed $\sqrt{8gl/3}$. Show that in the ensuing motion Q will just reach the hole.

Solution

At first we want to look at the change in speed of P. Angular momentum should be conserved, so

$$L_i = L_f = mvl = mv'(2l)$$

$$v' = \frac{1}{2}\sqrt{\frac{8gl}{3}}$$

The change in kinetic energy, assuming no loss, will be equal to the change in gravitational potential of the raising Q.

$$\frac{1}{2}m(v^2 - v'^2) = mgh$$

$$h = \frac{1}{2g}\Big(\frac{8gl}{3} - \frac{2gl}{3}\Big) = l$$

So Q will reach the hole.

Problem 4

A uniform rod AB of mass M and length $2a$ lies at rest on a smooth horizontal table. An impulse J is applied at A in the plane of the table and perpendicular to the rod. Determine the velocity of the centroid and the angular velocity of the rod.

Solution

Due to the conservation of linear momentum, we have

$$J = \Delta p = Mv$$

$$v = \frac{J}{M}$$

The rod will also start to rotate around its center, and using angular momentum conservation

$$\Delta L = Ja = I\omega = \frac{1}{3}M\omega a^2$$

$$\omega = \frac{3J}{Ma}$$

Problem 5

A perfectly rough circular hoop of radius a rolls on a horizontal table with speed v towards and normal to the edge of the table. What is the value of the angle θ that the hoop can roll down the step without losing contact. Here θ is the angle between the vertical and the line connecting the contact point to the center of the hoop at the corner of the step.

Solution

As the hoop travels off the step, gravity causes some torque on the hoop, so we have some change in the speed of the hoop, as well as the change in gravitational potential energy.

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mga = \frac{1}{2}mv_1^2 + \frac{1}{2}I\omega_1^2 + mg\cos(\theta)$$

Since the moment of inertia for the hoop is $I = ma^2$, we can rewrite this expression with only regards to linear velocity.

$$\frac{1}{2}mv^2 + \frac{1}{2}\frac{ma^2v^2}{a^2} + mga = \frac{1}{2}mv_1^2 + \frac{1}{2}\frac{ma^2v_1^2}{a^2} + mga\cos(\theta)$$

$$mv^2 + mga = mv_1^2 + mga\cos(\theta)$$

When the hoop loses contact with the step, the normal force is zero, and the only force we have acting on the hoop is gravity.

$$mg\cos(\theta) = \frac{mv_1^2}{a}$$

$$v_1 = \sqrt{ag\cos(\theta)}$$

Plugging this into the energy relation we found earlier gives us

$$mv^2 + mga = mag\cos(\theta) + mg\cos(\theta)$$

$$\cos(\theta) = \frac{v^2}{2ag} + \frac{1}{2}$$

$$\theta = \arccos\left(\frac{v^2}{2ag} + \frac{1}{2}\right)$$

Problem 6

A ball is projected up from a point O of a plane inclined at an angle α to the horizontal. The direction of the projection makes an angle β with the inclined surface, and the path lies in a vertical plane through a line of greatest slope. If the ball returns to O at the n-th impact, shown that

$$(1 - e_r)\cot(\alpha)\cot(\beta) = 1 - e_r^n$$

where e_r is called the coefficient of restitution between the ball and the plane. In general, e_r is defined as the ratio of the final relative velocity to the initial relative velocity between two objects after they collide. Its value is between 0 (inelastic collision) and 1 (elastic collision).

Solution

We can start by rotating our reference frame by α to simplify our equations of motion. Our new coordinates are

$$v_{\parallel} = v\cos(\beta)$$
$$v_{\perp} = v\sin(\beta)$$
$$g_{\parallel} = g\sin(\alpha)$$
$$g_{\perp} = g\cos(\alpha)$$

Let's find the time the ball hits the inclined plane for the first time.

$$\Delta y = vt\sin(\beta) - \frac{1}{2}gt^2\cos(\alpha) = 0$$

$$t = \frac{2v\sin(\beta)}{g\cos(\alpha)}$$

After every bounce, some energy will be lost as long as the collision isn't totally elastic. If we summed up all n bounces, we get

$$t_{tot} = \frac{2v\sin(\beta)}{g\cos(\alpha)}\sum_{i=0}^{n} e_r^i = \frac{2v\sin(\beta)}{g\cos(\alpha)}\frac{(1-e_r^n)}{(1-e_r)}$$

Now, we want to find the time it takes for the ball to return home, so we look at the parallel kinematics. We can do this all in one go because the parallel component of speed is unaffected by the loss in energy due to the collision, so there is no need to reduce each iterative bounce's parallel speed.

$$\Delta x = vt\cos(\beta) - \frac{1}{2}gt^2\sin(\alpha) = 0$$

$$t_{tot} = \frac{2v\cos(\beta)}{g\sin(\alpha)}$$

We can now equate our two total times and get

$$\frac{2v\cos(\beta)}{g\sin(\alpha)} = \frac{2v\sin(\beta)(1-e_r^n)}{g\cos(\alpha)(1-e_r)}$$

or

$$(1-e_r)\cot(\alpha)\cot(\beta) = 1 - e_r^n$$