Analytic Mechanics Homework Assignment 3

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Problem 1

As shown in Figure 1, a pendulum is suspended from the cusp of a cycloid cut in a rigid support. The path described by the bob of the pendulum is cycloidal:

$$x = a(\phi - \sin(\phi)) \qquad , \qquad y = a(\cos(\phi) - 1)$$

where a is a constant such that the length of the pendulum $l = 4a$, and ϕ is the angle of rotation of the circle generating the cycloid. Show that the oscillation is exactly isochronous with a frequency $\omega_0 = \sqrt{g/l}$, independent of the amplitude. Here, isochronous means the period of the pendulum is independent of the amplitude (or total energy).

Solution

We want to find the path the bob takes over time, and to do that we can use the pythagorean theorem

$$ds^2 = dx^2 + dy^2$$

with

$$dx = a(1 - \cos(\phi))d\phi$$

$$dy = -a\sin(\phi)d\phi$$

Plugging this in and solving for ds we get

$$ds^2 = (a^2(1 - \cos(\phi))^2 + a^2\sin^2(\phi))d\phi^2$$

$$= a^2(1 - 2\cos(\phi) + \cos^2(\phi) + \sin^2(\phi))d\phi^2$$

$$= 2a^2(1 - \cos(\phi))d\phi^2$$

$$= 4a^2 \sin^2\left(\frac{\phi}{2}\right) d\phi^2$$

$$ds = 2a \sin\left(\frac{\phi}{2}\right) d\phi$$

We can find our velocity by differentiating this with respect to time.

$$v = \frac{ds}{dt} = 2a \sin\left(\frac{\phi}{2}\right) \frac{d\phi}{dt} = -4a \frac{d}{dt} (\cos\left(\frac{\phi}{2}\right)$$

The acceleration will then be

$$\dot{v} = -4a \frac{d^2}{dt^2} (\cos\left(\frac{\phi}{2}\right))$$

Now gravity isn't pulling at the pendulum bob parallel to the path it is traveling. It is actually pulling the bob at some angle, and we can find its proportional strength by differentiating the vertical component of the path by the entire path.

$$\frac{dy}{ds} = \frac{-a \sin(\phi)}{2a \sin()\frac{\phi}{2})} = -\frac{1}{2} \frac{2 \sin\left(\frac{\phi}{2}\right) \cos\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi}{2}\right)} = -\cos\left(\frac{\phi}{2}\right)$$

We can then pull up our force equation and plug everything we found into it.

$$F = m\dot{v} = mg \cos\left(\frac{\phi}{2}\right) = -4am \frac{d^2}{dt^2} (\cos\left(\frac{\phi}{2}\right))$$

Rearranging, this gives us

$$\frac{d^2}{dt^2} (\cos\left(\frac{\phi}{2}\right)) + \frac{g}{4a} \cos\left(\frac{\phi}{2}\right) = 0$$

Let $\eta = \cos\left(\frac{\phi}{2}\right)$ and using what we were given for $\omega_0 = \sqrt{\frac{g}{4a}}$ we get our standard differential equation for oscillatory motion

$$\ddot{\eta} + \omega_0^2 \eta = 0$$

And from this we can see that the oscillation is isochronous.

Problem 2

One end of a spring with force constant k and negligible mass is hanging from a fixed support. A particle of mass m is attached at rest at the other end of the spring. At $t = 0$, a constant downward force F is applied to the mass and acts for

a time t_0. Show that, after the force is removed, the displacement of the mass from its equilibrium position $(x = x_0)$ is

$$x = x_0 = \frac{F}{k}[\cos(\omega_0(t - t_0)) - \cos(\omega_0 t)]$$

where $\omega_0^2 = k/m$.

Solution

Our force equation is

$$F_{net} = m\ddot{x} = -kx$$

Rearranging, we get

$$\ddot{x} + \frac{k}{m}x = 0$$

This is a differential equation has solution

$$x = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

with $\omega_0^2 = \frac{k}{m}$. The sine solution can be trashed because we know at $t = 0$ there will be some displacement so all we need is the cosine. At $t = 0$ the force we have applied is F, which comes in the form of a spring force.

$$F = -kx$$

Rearranging this we get

$$x = -\frac{F}{k} = A\cos(\omega_0(0)) = A$$

Having solved for A, we now have our position equation

$$x(t) = -\frac{F}{k}\cos(\omega_0 t)$$

At some previous time t_0 when the force was applied, the position will be

$$x(t - t_0) = -\frac{F}{k}\cos(\omega_0(t - t_0))$$

This difference in position after the force is applied is then

$$x(t) - x(t - t_0) = \frac{F}{k}(\cos(\omega_0(t - t_0)) - \cos(\omega_0 t))$$

Which is exactly what we are looking for.

Problem 3

One end of a massless spring with natural length a and spring constant k is attached to a horizontal table. The spring can rotate freely on the table without friction. A point mass m connected to the other end of the spring slides without friction on the table. The net force on the mass is a central force $F(r) = -k(r-a)$.

a) Find and sketch both the potential energy $V(r)$ and the effective potential $V_{eff}(r)$.

b) What is the angular velocity ω_0 required for a circular orbit with radius r_0?

c) Derive the frequency for small oscillation ω about the circular orbit with r_0.

Solution

a) We can find the potential energy V by integrating the force over the distance from the origin since it is spherically symmetric.

$$V = -\int_a^r (-k(r' - a))dr' = \frac{1}{2}k(r-a)^2$$

This is a pretty nice parabola. We can find the effective by adding an extra term

$$V_{eff} = V + \frac{l^2}{2mr^2} = \frac{1}{2}k(r-a)^2 + \frac{l^2}{2mr^2}$$

Graphing these two together gives something like this.

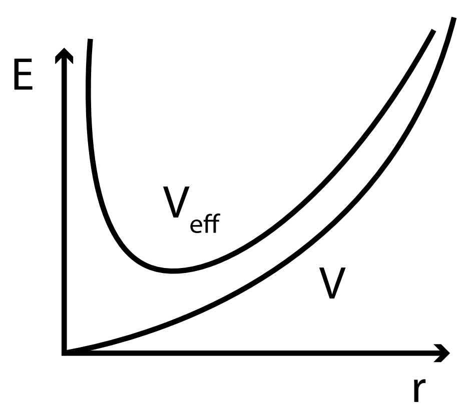

Figure 1

b) To find the angular velocity, we just need to equate the central force we have with the centripetal force.

$$k(r_0 - a) = m\omega_0^2 r_0$$

$$\omega_0 = \pm\sqrt{\frac{k(r_0 - a)}{mr_0}}$$

c) We are going to have to taylor expan our effective potential

$$V_{eff} = V(r_0) + (r - r_0)\frac{\partial V}{\partial r}\Big|_{r_0} + \frac{1}{2}(r - r_0)^2\frac{\partial^2 V}{\partial r^2}\Big|_{r_0} + \dots$$

The first derivative of the potential must be zero and the second derivative of our potential is

$$\frac{\partial^2 V}{\partial r^2} = k + \frac{3l^2}{mr^4}$$

This gives us a potential of

$$V_{eff} = \frac{1}{2}k(r_0 - a)^2 + \frac{l^2}{2mr_0^2} + \frac{1}{2}(r - r_0)^2(k + \frac{3l^2}{mr_0^4})$$

$$= \frac{1}{2}k((r_0 - a)^2 + (r - r_0)^2) + \frac{l^2}{2mr_0^2}(1 + \frac{3(r - r_0)^2}{r_0^2})$$

Since the first derivative is zero

$$k(r_0 - a) = \frac{l^2}{mr^3}$$

$$\frac{kr_0(r_0 - a)}{2} = \frac{l^2}{2mr_0^2}$$

This allows us to rewrite the potential as

$$V_{eff} = \frac{1}{2}k((r_0 - a)^2 + (r - r_0)^2) + \frac{kr_0(r_0 - a)}{2}(1 + \frac{3(r - r_0)^2}{r_0^2})$$

$$= \frac{1}{2}k((r - r_0)^2 + (r_0 - a)^2 + \frac{3(r - r_0)^2}{r_0})$$

$$= \frac{1}{2}k(r - r_0)^2(1 + \frac{3(r_0 - a)}{r_0})$$

We want to solve for k and to do that we just need to set our central force to equal the centripetal force.

$$k(r_0 - a) = m\omega_0^2 r$$

$$k = \frac{m\omega_0^2 r_0}{r_0 - a}$$

Plugging this into what we found previously

$$V_{eff} = \frac{1}{2}\frac{m\omega_0^2 r_0}{r_0 - a}(r - r_0)^2(1 + \frac{3(r_0 - a)}{r_0})$$

Looking at the second term we can simplify

$$\frac{3m\omega_0^2(r - r_0)^2}{2} = \frac{3k(r_0 - a)(r - r_0)^2}{2r_0}$$

From this we get our frequency we were looking for

$$\omega = \sqrt{\frac{3k(r_0 - a)}{mr_0}}$$

Problem 4

A damped linear oscillator, originally at rest in its equilibrium position $x = 0$, is subjected to a force given by

$$\frac{F(t)}{m} = \begin{cases} 0, & t < 0 \\ a \times \frac{t}{\tau}, & 0 < t < \tau \\ a, & t > \tau \end{cases}$$

where a is a constant. Find the response function $x(t)$. By allowing τ approaches zero, show that the solution becomes that for a step function.

Solution

A damped oscillator will have a differential equation of the form

$$\ddot{x} + 2\beta\dot{x}\omega_0^2 x = F(t)$$

The solution of the homogeneous version of this equation is

$$x = e^{-\beta t}(A\cos(\omega_1 t) + B\sin(\omega_1 t))$$

with $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. Looking at $0 < t < \tau$, we can guess the particular solution is of the form

$$x_p = Ct + D$$

Plugging this into our differential equation we find

$$2\beta C + \omega_0^2 Ct + D\omega_0^2 = \frac{at}{\tau}$$

Combining similar terms we find that

$$C = \frac{a}{\omega_0^2}$$

$$D = -\frac{2\beta a}{\omega_0^4 \tau}$$

Adding this to our homogeneous solution gives us

$$x = e^{-\beta t}(A\cos(\omega_1 t) + B\sin(\omega_1 t)) + \frac{at}{\omega_0^2 \tau} - \frac{2\beta a}{\omega_0^4}$$

Differentiating with respect to time gives us

$$\dot{x} = -\beta e^{-\beta t}(A\cos(\omega_1 t) + B\sin(\omega_1 t)) + e^{-\beta t}(-A\omega_1 \sin(\omega_1 t) + B\omega_1 \cos(\omega_1 t)) + \frac{a}{\omega_0^2 \tau}$$

With initial conditions of $x = 0, \dot{x} = 0$ we get

$$x(0) = A - \frac{2\beta a}{\omega_0^4 \tau} = 0$$

$$\dot{x}(0) = -\beta A + B\omega_1 + \frac{a}{\omega_0^4}$$

This gives us

$$A = \frac{2\beta a}{\omega_0^4 \tau}$$

$$B = \frac{2\beta^2 a}{\omega_1 \omega_0^4 \tau} - \frac{a}{\omega_0^2 \omega_1 \tau}$$

Plugging this into our position function we have

$$x(t) = e^{-\beta t}\Big(\frac{2\beta a}{\omega_0^4 \tau}\cos(\omega_1 t) + \Big(\frac{2\beta^2 a}{\omega_1 \omega_0^4 \tau} - \frac{a}{\omega_1 \omega_0^2 \tau}\Big)\sin(\omega_1 t)\Big) + \frac{at}{\omega_0^2 \tau} - \frac{2\beta a}{\omega_0^4}$$

For $t > \tau$ we can rewrite the force as

$$F = \frac{at}{\tau} - \frac{a(t-\tau)}{\tau}$$

Which will have basically the same solution as the previous part except won't have the exponential decay and has a phase shift in the trigonometric functions.

$$x = e^{-\beta t}\frac{2\beta a}{\omega_0^4\tau}(\cos(\omega_1 t) - e^{-\beta t}\cos(\omega_1(t-\tau)) + \Big(\frac{2\beta^2 a}{\omega_1\omega_0^4\tau} - \frac{a}{\omega_1\omega_0^2\tau}\Big)e^{-\beta t}(\sin(\omega_1 t) - e^{\beta t}\sin(\omega_1(t-\tau))) + \frac{a}{\omega_0^2}$$

We can start taylor expanding our transcendental functions if τ gets arbitrarily close to zero which gives us

$$\begin{aligned}
x = \frac{2\beta a}{\omega_0^4\tau}(\cos(\omega_1 t) - (1+\beta\tau)(\cos(\omega_1 t + \omega_1\tau\sin(\omega_1 t)))& \\
+\Big(\frac{2\beta^2 a}{\omega_1\omega_0^4\tau} - \frac{a}{\omega_1\omega_0^2\tau}\Big)e^{\beta t}(\sin(\omega_1 t - (1+\beta\tau)(\sin(\omega_1 t) - \omega_1\tau\cos(\omega_1 t))) + \frac{a}{\omega_0^2}& \\
= \frac{a}{\omega_0^2} - \frac{a}{\omega_0^2}e^{-\beta t}\cos(\omega_1 t) - \Big(\frac{2\beta\omega_1 a}{\omega_0^4} - \frac{\beta a}{\omega_1\omega_0^2} + \frac{2\beta^3 a}{\omega_1\omega_0^4}\Big)e^{-\beta t}\sin(\omega_1 t)& \\
= \frac{a}{\omega_0^2} - \frac{a}{\omega_0^2}e^{-\beta t}\cos(\omega_1 t) - \Big(\frac{2\beta(\omega_0^2-\beta^2)a - \beta a\omega_0^2 + 2\beta^3 a}{\omega_1\omega_0^4}\Big)e^{-\beta t}\sin(\omega_1 t)& \\
= \frac{a}{\omega_0^2} - \frac{a}{\omega_0^2}e^{-\beta t}\cos(\omega_1 t) - \frac{\beta a}{\omega_1\omega_0^2}e^{-\beta t}\sin(\omega_1 t)&
\end{aligned}$$

Which is the response of a step function.

Problem 5

A car with a mass 1000 kg settles 1 cm closer to the road for every additional 100 kg of passengers. This fact allows us to determine the 'spring' constant k of the car. Now the car is driven with a constant horizontal component of speed of 20 km/h over a washboard road with sinusoidal bumps. The amplitude and wavelength of the sine curve are 5.0 cm and 20 cm respectively. The separation between the front and back wheels is 2.4 m. Find the amplitude of oscillation of the car, assuming it moves vertically as an undamped driven harmonic oscillator. Neglect the mass of the wheels and springs. Assume the wheels are always in contact with the road.

Solution

We can start with our force equation for the system

$$F = m\ddot{y} = \alpha\cos(\gamma t) - mg + ky$$

$$\ddot{y} = \alpha\cos(\gamma t) - g + \omega_0^2 y$$

with α being the amplitude of the road, $\omega_0^2 = k/m$ and γ being the frequency of the road.

$$\alpha = 0.05k$$
$$\gamma = 2\pi f = \frac{2\pi v}{\lambda} = 175$$

Our driving force is sinusoidal and vertically shifted so our guess for a particular solution will be something of that form.

$$y_p = A + B\cos(\gamma t) + C\sin(\gamma t)$$
$$\ddot{y_p} = -B\gamma^2\cos(\gamma t) - C\gamma^2\sin(\gamma t)$$

Plugging this into our force equation gives us

$$-B\gamma^2\cos(\gamma t) - C\gamma^2\sin(\gamma t) = \alpha\cos(\gamma t) - g + \omega_0^2(A + B\cos(\gamma t) + C\sin(\gamma t))$$

Looking at similar terms, we get

$$A = \frac{g}{\omega_0^2}$$
$$B = -\frac{\alpha}{\omega_0^2 + \gamma^2}$$
$$C = 0$$

This gives us

$$y_p = \frac{g}{\omega_0^2} - \frac{\alpha}{\omega_0^2 + \gamma^2}\cos(\gamma t)$$

Our answer will then be

$$\text{Amplitude} = L = \frac{\alpha}{\omega_0^2 + \gamma^2}$$

We can find k to be

$$k - \frac{mg}{\Delta x} = 98100$$

Which ultimately gives us the amplitude to be

$$L = 1.2 \times 10^{-4} \text{ meters}$$

Problem 6

Ehrenfest's famous Principle of Adiabatic Invariance states that the ratio of the energy of an oscillatory motion to the frequency stays constant. This principle can be demonstrated with a simple pendulum. Consider a simple pendulum executing small-amplitude oscillations. The string with negligible mass is shortened an amount Δl very slowly, so that the change in amplitude from one oscillation to the next is negligible. Show that under these conditions, the ratio of the energy to the frequency remains constant.

Solution

The tension on the bob of the pendulum is

$$T = mg(1 - \cos(\theta)) = 2mg\sin^2\left(\frac{\theta}{2}\right) = \frac{mg\theta^2}{2}$$

Since we are dealing with small angles, the sine function was approximated to first order. The energy of our system can be described with

$$E = \frac{mgl\phi^2}{4}$$

where we swapped our θ with ϕ, the maximum angle from the pendulum bob to the vertical. From this we can see that

$$dT = \frac{E}{2l}dl$$

The frequency is

$$\omega = \sqrt{\frac{g}{l}}$$

We can differentiate it to find

$$d\omega = -\frac{1}{2}\sqrt{g}\frac{dl}{l^{\frac{3}{2}}}$$

$$\frac{d\omega}{\omega} = -\frac{1}{2}\frac{dl}{l}$$

From this we find

$$\frac{dE}{E} = -\frac{d\omega}{\omega}$$

Which is what we are looking for.