Analytic Mechanics Homework Assignment 2

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Problem 1

(Taylor 3.11, 3.13) A rocket resting on the ground takes off vertically in a gravitational field g.

a) Show that the equation

$$m\dot{v} = -\dot{m}v_{ex} - mg$$

where m is the mass of the rocket at time t, $\dot{m}$ is the rate at which the rocket is ejecting fuel, and v_{ex} is the speed of the ejected mass relative to the rocket.

b) Assuming $\dot{m} = k$ with k being positive and constant such that $m = m_0 - kt$, determine the speed of the rocket v as a function of time.

c) Determine the height of the rocket as a function of time.

d) Suppose the intial mass of the rocket is 2×10^6 kg, the final mass after two minutes is 1×10^6 kg, the exhaust speed v_{ex} is 3 km/s, what is the height of the rocket after two minutes in flight?

Solution a) At first, let's look at the momentum of our rocket system, ignoring some outside force. Our momentum at some time t will be

$$p(t) = mv$$

At a later time, the mass and the speed of the rocket will change, and we will have to take into account the fuel's momentum as it decouples from the rocket. This gives us

$$p(t + dt) = (m + dm)(v + dv) - dm(v - v_{ex})$$

$$= mv + mdv + vdm + dmdv - vdm + dmv_{ex}$$

$dmdv$ is really small compared to everything else, and can be considered to be zero, which let's use simplify this equation to

$$p(t + dt) = mv + mdv + v_{ex}dm$$

Our change of momentum will then be

$$dp = p(t + dt) - p(t) = mv + mdv + v_{ex}dm - mv = mdv + v_{ex}dm$$

This is where our external force comes in. Without an external force acting our rocket, this change in momentum would be zero, but since our external force is not zero, our change in momentum will be

$$dp = \int F_g dt = mdv + v_{ex}dm$$

Now all we have to do is differentiate everything in respect to t to get our desired equation.

$$\frac{dp}{dt} = m\frac{dv}{dt} + v_{ex}\frac{dm}{dt} = -mg$$

$$m\dot{v} = -\dot{m}v_{ex} - mg$$

b) Plugging in our givens into our previous equation gives

$$(m_0 - kt)\dot{v} = -kv_{ex} - (m_0 - kt)g$$

$$\dot{v} = -\frac{kv_{ex}}{m_0 - kt} - g$$

We want to solve for v so let's integrate both sides in respect to t.

$$v = v_{ex}\ln\left(\frac{m_0}{m_0 - kt}\right) - gt$$

c) To find our height over time, we will need to integrate our speed function from the previous part in respect to time. First, let's manipulate the speed function in a way that is easier to integrate without making mistakes.

$$v = v_{ex}\ln\left(\frac{m_0}{m_0 - kt}\right) - gt = v_{ex}\ln\left(\frac{1}{1 - \frac{kt}{m_0}}\right) - gt$$

$$= -v_{ex}\ln\left(1 - \frac{kt}{m_0}\right) - gt$$

Now we are ready to integrate.

$$h(t) = \int vdt = -\int (v_{ex}\ln\left(1 - \frac{kt'}{m_0}\right) - gt')dt'$$

$$= \frac{v_{ex}m_0}{k}(1 - \frac{kt'}{m_0})(\ln\left(1 - \frac{kt'}{m_0}\right) - 1)\Big|_0^t - \frac{1}{2}gt'^2\Big|_0^t$$

$$= \frac{v_{ex}m_0}{k}(1 - \frac{kt}{m_0})(\ln\left(1 - \frac{kt}{m_0}\right) - 1) - \frac{v_{ex}m_0}{k}(-1) - \frac{1}{2}gt^2$$

$$= \frac{v_{ex}m_0}{k}\left(\frac{m}{m_0}\right)(\ln\left(\frac{m}{m_0}\right) - 1) + \frac{v_{ex}m_0}{k} - \frac{1}{2}gt^2$$

$$= \frac{v_{ex}m}{k}\ln\left(\frac{m}{m_0}\right) - \frac{v_{ex}m}{k} + \frac{v_{ex}m_0}{k} - \frac{1}{2}gt^2$$

and since $m_0 - m = kt$

$$h(t) = \frac{v_{ex}m}{k}\ln\left(\frac{m}{m_0}\right) + v_{ex}t - \frac{1}{2}gt^2$$

Or to make the contribution of each piece more explicit

$$h(t) = v_{ex}t - \frac{v_{ex}m}{k}\ln\left(\frac{m_0}{m}\right) - \frac{1}{2}gt^2$$

d) Using $k = \frac{m_0}{t}$ we find that our height equation is

$$h(t) = v_{ex}t - \frac{v_{ex}mt}{m_0 - m}\ln\left(\frac{m_0}{m}\right) - \frac{1}{2}gt^2$$

Plugging in our give values this gives us a height of

$$h(2 \text{ minutes}) = 4 \times 10^4 \text{ m}$$

Problem 2

(Taylor 3.34) A juggler is juggling a uniform rod one end of which is coated in tar and burning. He is holding the rod by the opposite end and throws it up so that, at the moment of release, it is horizontal, its CM is traveling vertically up at speed v_0 and it is rotating with angular velocity ω_0. To catch it, he wants to arrange that when it returns to his hand it will have made an integer number of complete rotations. What should v_0 be, if the rod is to have made exactly n rotations when it returns to his hand?

Solution At first. let's see at what time the center of mass of the torch returns to the juggler.

$$h(t) = v_0 t - \frac{1}{2}gt^2$$

The torch returns to the juggler when the height is zero.

$$0 = v_0 t - \frac{1}{2}gt^2$$

$$v_0 t = \frac{1}{2}gt^2$$

$$t = \frac{2v_0}{g}$$

The angular momentum of the torch is constant since the only force acting on the rod is gravity, which acts on the center of mass of the torch. This means our angular velocity is constant. The frequency of rotations will be

$$f = \frac{\omega_0}{2\pi}$$

This will give us the number of each full rotation

$$n = ft = \left(\frac{\omega_0}{2\pi}\right)\left(\frac{2v_0}{g}\right) = \frac{\omega_0 v_0}{\pi g}$$

We can then solve this for v_0.

$$v_0 = \frac{ng\pi}{\omega_0}$$

Problem 3

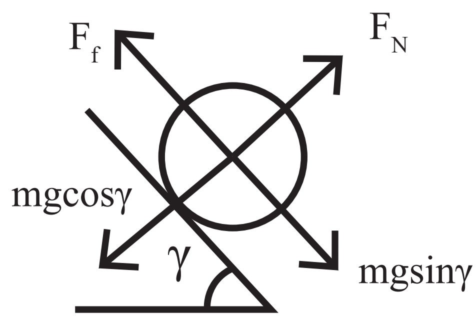
(Taylor 3.35) Consider a uniform solid disk of mass M and radius R, rolling without slipping down an incline which is at angle γ to the horizontal. The instantaneous point of contact between the disk and the incline is called P.

a) Draw a free-body diagram, showing all forces on the disk.

b) Find the linear acceleration $\dot{v}$ of the disk by applying the result $\dot{\boldsymbol{L}} = \boldsymbol{\Gamma}^{ext}$ for rotation about P. (Remember that $L = I\omega$ and the moment of inertia for rotation about a point on the circumference is $\frac{3}{2}MR^2$. The condition that the disk not slip is that $v = R\omega$ and hence $\dot{v} = R\dot{\omega}$.)

c) Derive the same result by applying $\dot{\boldsymbol{L}} = \boldsymbol{\Gamma}^{ext}$ to the rotation about the CM. (In this case you will find there is an extra unknown, the force of friction. You can eliminate this by applying Newton's second law to the motion of the CM. The moment of inertia for rotation about the CM is $\frac{1}{2}MR^2$.)

Solution a)



In this diagram, the gravitational force is broken into its components parallel to the incline and perpendicular to the incline.

b) Since the disk isn't slipping, we know the frictional force and the parallel component of the gravitational force are equal.

$$\boldsymbol{\tau} = \dot{\boldsymbol{L}} = \boldsymbol{r} \times \boldsymbol{F_{g\perp}}$$

Looking only at magnitudes we get

$$\tau = RMg\sin(\gamma)$$

Now, we also know

$$\dot{L} = I\dot{\omega} = \frac{3}{2}MR^2\dot{\omega} = RMg\sin(\gamma)$$

Solving for $\dot{\omega}$ we get

$$\dot{\omega} = \frac{2g\sin(\gamma)}{3R}$$

We can convert this to linear acceleration using

$$\dot{v} = R\dot{\omega} = \frac{2g\sin(\gamma)}{3}$$

c) The sum of the force acting on the CM is

$$\sum F = Mg\sin(\gamma) - M\dot{v}$$

This force will be the force acting on our disk. So let's plug it into our torque equation and see what we get.

$$\tau = R(Mg\sin(\gamma) - M\dot{v}) = I\dot{\omega} = \frac{1}{2}MR^2\dot{\omega}$$

$$RMg\sin(\gamma) = \frac{1}{2}MR\dot{v} + RM\dot{v} = \frac{3}{2}MR\dot{v}$$

$$\dot{v} = \frac{2g\sin(\gamma)}{3}$$

Which is what we were after.

Problem 4

(Taylor 4.4) A particle of mass m is moving on a frictionless, horizontal table and is attached to a massless string, whose other end passes through a hole in the table, where I am holding it. Initially the particle is moving in a circle of radius r_0 with angular velocity ω_0, but I now pull the string down through the hole until a length r remains between the hole and the particle.

a) What is the particle's angular velocity now?

b) Assuming that I pull the string so slowly that we can approximate the particle's path by a circle of slowly shrinking radius, calculate the work I did pulling the string.

c) Compare your answer to part b) with the particle's gain in kinetic energy.

Solution **a)** Angular momentum will be conserved. The angular momentum will be

$$L = I\omega = mr_0^2\omega_0 = mr^2\omega$$

Solving for ω we get

$$\omega = \omega_0\left(\frac{r_0}{r}\right)^2$$

b) The centripetal force will be causing the work which is described by

$$a = \frac{v^2}{r} = r\omega^2$$

Using the angular velocity we found in the previous part, our force creating the work will be

$$F = mr\omega^2 = mr\omega_0^2\left(\frac{r_0}{r}\right)^4 = \frac{m\omega_0^2 r_0^4}{r^3}$$

Let's integrate this force with respect to the changing radius.

$$W = \int_{r_0}^{r} F(-dr') = -\int_{r_0}^{r} \frac{m\omega_0^2 r_0^4}{r'^3} dr' = \frac{m\omega_0^2 r_0^4}{2}\left(\frac{1}{r^2}\right)\Big|_{r_0}^{r}$$

$$W = \frac{m\omega_0^2 r_0^4}{2}\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right)$$

c) The change in kinetic energy will be

$$\Delta T = \frac{1}{2}I\omega^2 - \frac{1}{2}I_0\omega_0^2 = \frac{1}{2}mr^2\omega_0^2\left(\frac{r_0}{r}\right)^4 - \frac{1}{2}mr_0^2\omega^2$$

$$= \frac{m\omega_0^2 r_0^4}{2}\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right)$$

Which is exactly what we found in the previous part.

Problem 5

(Taylor 4.8) Consider a small frictionless puck perched at the top of a fixed sphere of radius R. If the puck is given a tiny nudge so that it begins to slide down, through what vertical height will it descend before it leaves the surface of the sphere? [*Hint:* Use conservation of energy to find the puck's speed as a function of its height, then use Newton's second law to find the normal force of the sphere on the puck. At what value of this normal force does the puck leave the sphere?]

Solution As suggested, let's check out our energy conservation relations. Some of the gravitational potential energy of the puck at the tippy-top of the sphere will be transformed into kinetic energy as it falls off the sphere.

$$2Rmg = \frac{1}{2}mv^2 + mgR(1 + \cos(\theta))$$

$$\frac{1}{2}mv^2 = 2Rmg - mgR(1 + \cos(\theta))$$

$$v^2 = 4Rg - 2gR(1 + \cos(\theta))$$

That should be good enough for now. Let's take a crack at Newton's second law.

$$F = ma = -\frac{mv^2}{R} = F_N - mg\cos(\theta)$$

The normal force will be zero when the puck takes off, so after making that piece disappear, we can solve for the angle.

$$-mg\cos(\theta) = -\frac{mv^2}{R}$$

$$\cos(\theta) = \frac{v^2}{Rg}$$

Plugging in our speed from the energy relations gives us

$$\cos(\theta) = 4 - 2 - 2\cos(\theta)$$

$$3\cos(\theta) = 2$$

$$\cos(\theta) = \frac{2}{3}$$

Our vertical displacement can be determined with

$$h = R(1 - \cos(\theta)) = \frac{R}{3}$$

Problem 6

(Taylor 4.24) An infinitely long, uniform rod of mass μ per unit length is situated on the z axis.

a) Calculate the gravitational force $\boldsymbol{F}$ on a point mass m at a distance ρ from the z axis. (The gravitational force between two point masses is given in problem 4.21.)

b) Rewrite $\boldsymbol{F}$ in terms of the rectangular coordinates (x, y, z) of the point and verify that $\nabla \times \boldsymbol{F} = 0$.

c) Show that $\nabla \times \boldsymbol{F} = 0$ using the expression for $\nabla \times \boldsymbol{F}$ in cylindrical polar coordinates given inside the back cover.

d) Find the corresponding potential energy U.

Solution **a)** We start our journey with our equation for gravitational force.

$$F = -G\frac{mM}{r^2}$$

We need to swap out one of those masses with something reminiscent of our uniform rod. We want the closest distance to be directly from our reference point, we want it to increase as the angle from our reference point increase towards the top of the rod, and we want it to decrease as the angle from our reference point decreases towards the bottom of the rod. This gives us

$$M = \mu\cos(\theta)dz$$

as our missing puzzle piece. ρ will be our distance from some point in the xy plane towards the rod. Our force equation will then look like

$$dF = -G\frac{m\mu\cos(\theta)dz}{r^2}$$

That cosine can be related to r and ρ by the following relation

$$\cos(\theta) = \frac{\rho}{r}$$

Plugging this into our force equation gives us

$$dF = -G\frac{m\mu\rho dz}{r^3} = -G\frac{m\mu\rho dz}{(\rho^2+z^2)^{\frac{3}{2}}}$$

To integrate this we need to go back to our triangles and make some substitutions. Our substitutions will be

$$z = \rho\tan(u)$$

$$dz = \rho\sec^2(u)du$$

Plugging these in we get

$$dF = -G\frac{m\mu\rho^2\sec^2(u)du}{(\rho^2+\rho^2\tan^2(u))^{\frac{3}{2}}} = -G\frac{m\mu\sec^2(u)du}{\rho(1+\tan^2(u))^{\frac{3}{2}}}$$

$$= -G\frac{m\mu\sec^2(u)du}{\rho\sec^3(u)} = -G\frac{m\mu\cos(u)du}{\rho}$$

We are now ready to integrate. Since we made the trig substitution, our new bounds will be $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

$$F = -G\frac{m\mu}{\rho}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(u)du = -G\frac{m\mu}{\rho}(\sin(u))\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$F = -2G\frac{m\mu}{\rho}$$

This force will be in the ρ direction so in finality, the force is

$$\boldsymbol{F} = -2G\frac{m\mu}{\rho}\hat{\boldsymbol{\rho}}$$

b) This force expressed in cartesian coordinates will be

$$\boldsymbol{F} = -2G\frac{m\mu}{\sqrt{x^2+y^2}}(\frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}})$$

$$= -2Gm\mu(\frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}}{x^2+y^2})$$

Now we can find the curl of the force

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \hat{\boldsymbol{x}}\Big(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\Big) + \hat{\boldsymbol{y}}\Big(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\Big) + \hat{\boldsymbol{z}}\Big(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\Big)$$

There is no z dependence so all the z partial derivatives vanish and since there is no z component all terms with F_z vanish. This leaves us only with our third term, but that vanishes because $F_x(y)$ is $F_y(x)$ with variables swapped so those terms must cancel. This leaves us with

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \hat{\boldsymbol{x}}(0) + \hat{\boldsymbol{y}}(0) + \hat{\boldsymbol{z}}(0) = \boldsymbol{0}$$

c) The equation in the book for curl in cylindrical coordinates is

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \hat{\boldsymbol{\rho}}\Big(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\Big) + \hat{\boldsymbol{\phi}}\Big(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\Big) + \hat{\boldsymbol{z}}\frac{1}{\rho}\Big(\frac{\partial(\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi}\Big)$$

Like before we can reason these terms away. There is only a *rho* component so all other differentials in regards to other variables must vanish. Similarly, all terms that don't include F_ρ will also vanish. This leaves us with

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \hat{\boldsymbol{\rho}}(0) + \hat{\boldsymbol{\phi}}(0) + \hat{\boldsymbol{z}}(0) = \boldsymbol{0}$$

d) The potential energy can be found by integrating our force equation using a line integral of the form

$$U = -\int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r}$$

This integral will be easiest in cylindrical coordinates so let's get to it

$$U(\rho) = 2Gm\mu \int_{\rho_0}^{\rho} \frac{d\rho'}{\rho'}$$

There is no loss of energy moving perpendicular to the force so the other degrees of freedom don't need to be taken into account. This integral then evaluates to

$$U(\rho) = 2Gm\mu \ln\Big(\frac{\rho}{\rho_0}\Big)$$