Analytic Mechanics Lecture 2

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Instantaneous Velocity

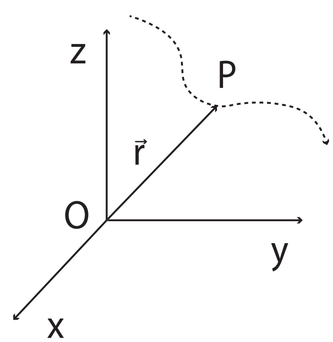

Figure 1. A particle P traversing along some path.

Our instantaneous velocity can be represented by

$$\frac{d\boldsymbol{r}}{dt} = \dot{\boldsymbol{r}} = \boldsymbol{v}$$

where $\boldsymbol{r}$ can be represented as

$$\boldsymbol{r} = x_i(t)\hat{x}_i(t)$$

Instantaneous Acceleration

To get the instantaneous acceleration, all we need to do is take a second time derivative of our position vector.

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{\dot{\boldsymbol{r}}}{dt} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2(x_i\hat{x}_i)}{dt^2}$$

Since both x_i and $\hat{x}_i$ can both be time dependent, we can see that this can get pretty messy.

Angular Velocity

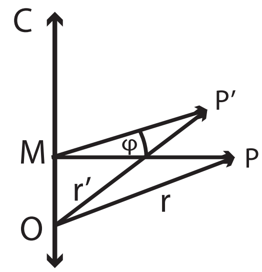

Figure 2. A particle at point P rotating around a central axis CO.

What we will want to know next is how to find the angular velocity of some type of point rotating around some axis of rotation.

$$\boldsymbol{r}(t+dt) = \boldsymbol{r}(t) + d\boldsymbol{r}$$

Assuming that the magnitude of $\boldsymbol{r}$ is unchanging, we can say that

$$r = |\boldsymbol{r}(t)|$$

We can try to derive our vector solution to our problem by playing around with the scalar forms of the quantities are after since they are easier to work with and see if we can guess our way to a solution.

$$|d\boldsymbol{r}| = ds = PMd\phi = r\sin(\theta)d\phi$$

We can say that OC is the instantaneous axis of rotation. In more complicated situations, this axis can start to move over time but in this situation we will assume it is fixed. A benefit of a fixed axis of rotation is that all points along it do not change over time. Using our friend, the time derivative, we can differentiate this relation we have for s to get our speed v.

$$v = \frac{ds}{dt} = \frac{d}{dt}(r\sin(\theta)d\phi = r\sin(\theta)\frac{d\phi}{dt} = r\sin(\theta)\dot{\phi}$$

$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} =$ instantaneous angular velocity. From this derivation of the speed, we can see that the result looks a lot like a cross product. The trick is to make sure we have the correct order of vectors to get our resultant velocity in the correct direction. The end result will then be

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$$

Kinematics in 2-D

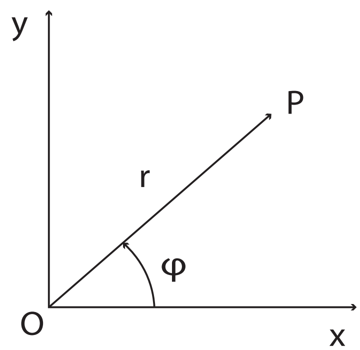

Figure 3. Here we are looking at a particle P in some 2-D space.

In this case we are going to want to use $r - \phi$ coordinates.

$$\boldsymbol{r} = r\hat{r}$$
$$\hat{r} = \hat{r}(\phi, t)$$

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d(r\hat{r})}{dt} = \left(\frac{dr}{dt}\right)\hat{r} + r\frac{d\hat{r}}{dt}$$

By looking into the trigonometry of the system, we can see that we the following two relations:

$$\hat{r} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$$
$$\hat{\phi} = -\sin(\phi)\hat{x} + \cos(\phi)\hat{y}$$

Differentiating these terms in respect to time t and angle ϕ, we discover the following relations.

$$\frac{d\hat{r}}{dt} = \dot{\phi}\hat{\phi}$$

$$\frac{d\hat{\phi}}{dt} = -p\dot{h}i\hat{r}$$

$$\frac{d\hat{r}}{d\phi} = p\hat{h}i$$

$$\frac{d\hat{\phi}}{d\phi} = -\hat{r}$$

These equations lead us to the equation for the velocity of our particle in this 2-D system.

$$\boldsymbol{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

The first term is the **radial** velocity and the second term is the **tangential** velocity of our particle.

Acceleration

Using this new equation we have for our particle's velocity, we can find an equation for our particle's acceleration by differentiating in respect to time.

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi}$$

Terms The radial part has two terms. The first term is its radial acceleration. The second term is the centripetal acceleration term. The Radial part also has two terms. The first term is the coriolis acceleration term. The second term is the tangential acceleration term.