Analytic Mechanics Homework Assignment 1

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Problem 1

(Taylor 1.19) Let $\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{a}$ be respectively the position, velocity, and acceleration of a particle. Show that

$$\frac{d}{dt}[\boldsymbol{a} \cdot (\boldsymbol{v} \times \boldsymbol{r})] = \dot{\boldsymbol{a}} \cdot (\boldsymbol{v} \times \boldsymbol{r})$$

Solution This problem can get a little messy so I am going to break it up into a couple of pieces. Let us say that $\boldsymbol{v} \times \boldsymbol{r} = \boldsymbol{\Gamma}$. This will allow us to break the problem down into a cross product part and a dot product part. We know rewrite the problem as

$$\frac{d}{dt}(\boldsymbol{a} \cdot \boldsymbol{\Gamma}) = \dot{\boldsymbol{a}} \cdot \boldsymbol{\Gamma}$$

Taking the time derivative, we get

$$\dot{\boldsymbol{a}} \cdot \boldsymbol{\Gamma} + \boldsymbol{a} \cdot \dot{\boldsymbol{\Gamma}} = \dot{\boldsymbol{a}} \cdot \boldsymbol{\Gamma}$$

We have two similar parts on each side, so after simplifying, we find for our proposed relation to be true, the following equation must be true.

$$\boldsymbol{a} \cdot \dot{\boldsymbol{\Gamma}} = 0$$

Let's try to find $\dot{\boldsymbol{\Gamma}}$ before this gets any more messy.

$$\frac{d}{dt}(\boldsymbol{\Gamma}) = \frac{d}{dt}(\hat{\boldsymbol{x}}(v_2 r_3 - r_2 v_3) - \hat{\boldsymbol{y}}(v_1 r_3 - v_3 r_1) + \hat{\boldsymbol{z}}(v_1 r_2 - v_2 r_1))$$

$$= \hat{\boldsymbol{x}}(a_2 r_3 + v_2 v_3 - v_3 v_2 - r_2 a_3) - \hat{\boldsymbol{y}}(a_1 r_3 + v_1 v_3 - a_3 r_1 - v_3 v_1) + \hat{\boldsymbol{z}}(a_1 r_2 + v_1 v_2 - v_1 v2 - a_2 r_1)$$

$$= \hat{\boldsymbol{x}}(a_2 r_3 - r_2 a_3) + \hat{\boldsymbol{y}}(a_3 r_1 - a_1 r_3) + \hat{\boldsymbol{z}}(a_1 r_2 - a_2 r_1)$$

We can now take the dot product and see if that can help clear this mess up.

$$a_1(a_2r_3 - r_2a_3) + a_2(a_3r_1 - a_1r_3) + a_3(a_1r_2 - a_2r_1)$$

$$= a_1a_2r_3 - r_2a_1a_3 + a_2a_3r_1 - a_1a_2r_3 + a_1a_3r_2 - a_3a_2r_1 = 0$$

Since that whole mess is 0, we can safely say $\frac{d}{dt}[\boldsymbol{a} \cdot (\boldsymbol{v} \times \boldsymbol{r})] = \dot{\boldsymbol{a}} \cdot (\boldsymbol{v} \times \boldsymbol{r})$.

Problem 2

Use the BAC-CAB rule to decompose any vector $\boldsymbol{D}$ into a piece that is parallel to $\hat{\boldsymbol{n}}$, an arbitrary unit vector, and another piece that is perpendicular to $\hat{\boldsymbol{n}}$.

Solution We begin by decomposing our vector $\boldsymbol{D}$ into its components parallel and perpendicular to some arbitrary unit vector $\hat{\boldsymbol{n}}$, with the perpendicular component lying within the $D - n$ plane.

We can get our parallel component with an easy dot product:

$$(\boldsymbol{D} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}$$

And we can find our perpendicular component using a series of cross products. The first cross product gives us a perpendicular vector, but it lies perpendicular to our desired plane, so we can use a second cross product to bring it into the plane.

$$\hat{\boldsymbol{n}} \times (\boldsymbol{D} \times \hat{\boldsymbol{n}})$$

Now that we have our components, we can add them together and hopefully find $\boldsymbol{D}$.

$$(\boldsymbol{D} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} + \hat{\boldsymbol{n}} \times (\boldsymbol{D} \times \hat{\boldsymbol{n}}) = (\boldsymbol{D} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} + \boldsymbol{D}(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}) - \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}} \cdot \boldsymbol{D}) = \boldsymbol{D}$$

This is exactly what we wanted! We used the BAC-CAB rule to break apart that triple vector product into to components that can be canceled out with the piece of $\boldsymbol{D}$ that runs parallel to $\hat{\boldsymbol{n}}$ to return our desired vector.

Problem 3

We want to find the velocity and acceleration of a particle P in three dimensions using spherical coordinates. The position vector of P in this case is

$$\boldsymbol{r} = r\hat{\boldsymbol{r}}$$

(a) Before differentiate the position vector with respect to time to obtain the velocity of P, express the position-dependent unit vectors $\hat{\boldsymbol{r}}, \hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$.

(b) Determine the velocity of P, $\boldsymbol{v}$, in terms of $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$.
(c) Determine the acceleration of P, $\boldsymbol{a}$, in terms of $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$.

Solution **a)** We can represent an arbitrary position vector in cartesian coordinates as such.

$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

Switching the magnitudes of each component into spherical coordinates gives us

$$\boldsymbol{r} = r\sin(\theta)\cos(\phi)\hat{\boldsymbol{x}} + r\sin(\theta)\sin(\phi)\hat{\boldsymbol{y}} + r\cos(\theta)\hat{\boldsymbol{z}}$$

We can find all of our unit vectors in spherical coordinates by differentiating that equation in respect to r, θ and ϕ and then normalize them.

$$\frac{\frac{d\boldsymbol{r}}{da}}{\left|\frac{d\boldsymbol{r}}{da}\right|} = \hat{\boldsymbol{a}}$$

Let's start off finding $\hat{\boldsymbol{r}}$.

$$\frac{d\boldsymbol{r}}{dr} = \sin(\theta)\cos(\phi)\hat{\boldsymbol{x}} + \sin(\theta)\sin(\phi)\hat{\boldsymbol{y}} + \cos(\theta)\hat{\boldsymbol{z}}$$

$$\left|\frac{d\boldsymbol{r}}{dr}\right| = \sqrt{\sin^2(\theta)\cos^2(\phi) + \sin^2(\theta)\sin^2(\phi) + \cos^2(\theta)} = \sqrt{\sin^2(\theta) + \cos^2(\theta)} = 1$$

So our radial unit vector is

$$\hat{\boldsymbol{r}} = \sin(\theta)\cos(\phi)\hat{\boldsymbol{x}} + \sin(\theta)\sin(\phi)\hat{\boldsymbol{y}} + \cos(\theta)\hat{\boldsymbol{z}}$$

Next we'll take a shot at finding $\hat{\boldsymbol{\theta}}$.

$$\frac{d\boldsymbol{r}}{d\theta} = r\cos(\theta)\cos(\phi)\hat{\boldsymbol{x}} + r\cos(\theta)\sin(\phi)\hat{\boldsymbol{y}} - r\sin(\theta)\hat{\boldsymbol{z}}$$

$$\left|\frac{d\boldsymbol{r}}{d\theta}\right| = r\sqrt{\cos^2(\theta)\cos^2(\phi) + \cos^2(\theta)\sin^2(\phi) + \sin^2(\theta)} = r\sqrt{\cos^2(\theta) + \sin^2(\theta)} = r$$

So our azimuthal unit vector is

$$\hat{\boldsymbol{\theta}} = \cos(\theta)\cos(\phi)\hat{\boldsymbol{x}} + \cos(\theta)\sin(\phi)\hat{\boldsymbol{y}} - \sin(\theta)\hat{\boldsymbol{z}}$$

Finally, we will hunt down $\hat{\boldsymbol{\phi}}$.

$$\frac{d\boldsymbol{r}}{d\phi} = -r\sin(\theta)\sin(\phi)\hat{\boldsymbol{x}} + r\sin(\theta)\cos(\phi)\hat{\boldsymbol{y}}$$

$$\left|\frac{d\boldsymbol{r}}{d\phi}\right| = r\sqrt{\sin^2(\theta)\sin^2(\phi) + \sin^2(\theta)\cos^2(\phi)} = r\sin(\theta)$$

So our final unit vectors are:

$$\hat{\boldsymbol{r}} = \sin(\theta)\cos(\phi)\hat{\boldsymbol{x}} + \sin(\theta)\sin(\phi)\hat{\boldsymbol{y}} + \cos(\theta)\hat{\boldsymbol{z}}$$

$$\hat{\boldsymbol{\theta}} = \cos(\theta)\cos(\phi)\hat{\boldsymbol{x}} + \cos(\theta)\sin(\phi)\hat{\boldsymbol{y}} - \sin(\theta)\hat{\boldsymbol{z}}$$

$$\hat{\boldsymbol{\phi}} = -\sin(\phi)\hat{\boldsymbol{x}} + \cos(\phi)\hat{\boldsymbol{y}}$$

b) We can find the velocity of P by differentiating our position vector in respect to t.

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\hat{\boldsymbol{r}}}$$

We can use the chain rule to break up that second term's derivative to make it easier to solve.

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\Big(\frac{d\hat{\boldsymbol{r}}}{dr}\frac{dr}{dt} + \frac{d\hat{\boldsymbol{r}}}{d\theta}\frac{d\theta}{dt} + \frac{d\hat{\boldsymbol{r}}}{d\phi}\frac{d\phi}{dt}\Big)$$

Let's differentiate our radial unit vector and plug those values in and see what happens.

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r((\cos(\theta)\cos(\phi)\hat{\boldsymbol{x}} + \cos(\theta)\sin(\phi)\hat{\boldsymbol{y}} - \sin(\theta)\hat{\boldsymbol{z}})\dot{\theta} + (-\sin(\theta)\sin(\phi)\hat{\boldsymbol{x}} + \sin(\theta)\cos(\phi)\hat{\boldsymbol{y}})\dot{\phi})$$

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin(\theta)\dot{\phi}\hat{\boldsymbol{\phi}}$$

c) The bad news is that finding acceleration will be a lot messier, but the good news is we just need to use that same technique we used in the previous section to find our solution. We are going to differentiate the velocity in pieces by their respective component to make it easier to keep track of what we are doing.

$$\boldsymbol{A} = \dot{r}\hat{\boldsymbol{r}}$$

$$\boldsymbol{B} = r\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$\boldsymbol{C} = r\sin(\theta)\dot{\phi}\hat{\boldsymbol{\phi}}$$

Let's start with the radial part $\boldsymbol{A}$.

$$\frac{d\boldsymbol{A}}{dt} = \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\hat{\boldsymbol{r}}} = \ddot{r}\hat{\boldsymbol{r}} + \dot{r}(\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{\phi}\sin(\theta)\hat{\boldsymbol{\phi}})$$

The next part we will look at is the azimuthal part.

$$\frac{d\boldsymbol{B}}{dt} = \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}} = \hat{\boldsymbol{\theta}}(\dot{r}\dot{\theta} + r\ddot{\theta}) + r\dot{\theta}\Big(\frac{d\hat{\boldsymbol{\theta}}}{dr}\frac{dr}{dt} + \frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} + \frac{d\hat{\boldsymbol{\theta}}}{d\phi}\frac{d\phi}{dt}\Big)$$

$$= \hat{\boldsymbol{\theta}}(\dot{r}\dot{\theta} + r\ddot{\theta}) + r\dot{\theta}(-\hat{\boldsymbol{r}}\dot{\theta} + \cos(\theta)\dot{\phi}\hat{\boldsymbol{\phi}}) = -r\dot{\theta}^2\hat{\boldsymbol{r}} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\phi}\cos(\theta)\hat{\boldsymbol{\phi}}$$

We just need one more piece, so we're onto the polar part.

$$\frac{d\boldsymbol{C}}{dt} = \dot{r}\sin(\theta)\dot{\phi}\hat{\boldsymbol{\phi}} + r\cos(\theta)\dot{\theta}\dot{\phi}\hat{\boldsymbol{\phi}} + r\sin(\theta)\ddot{\phi}\hat{\boldsymbol{\phi}} + r\sin(\theta)\dot{\phi}\dot{\hat{\boldsymbol{\phi}}}$$

$$= \hat{\boldsymbol{\phi}}(\dot{r}\sin(\theta)\dot{\phi} + r\cos(\theta)\dot{\theta}\dot{\phi} + r\sin(\theta)\ddot{\phi}) + r\sin(\theta)\dot{\phi}\Big(\frac{d\hat{\boldsymbol{\phi}}}{d\phi}\frac{d\phi}{dt}\Big)$$

$$= \hat{\boldsymbol{r}}(-\sin^2(\theta)r\dot{\phi}^2) + \hat{\boldsymbol{\theta}}(-r\sin(\theta)\cos(\theta)\dot{\phi}^2) + \hat{\boldsymbol{\phi}}(\dot{r}\sin(\theta)\dot{\phi} + r\cos(\theta)\dot{\theta}\dot{\phi} + r\sin(\theta)\ddot{\phi}$$

Putting these all together using the magical power of addition we get our expression for acceleration in spherical coordinates.

$$\boldsymbol{a} = \hat{\boldsymbol{r}}(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2(\theta)) + \hat{\boldsymbol{\theta}}(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin(\theta)\cos(\theta)\dot{\phi}^2) + \hat{\boldsymbol{\phi}}(2\dot{r}\dot{\phi}\sin(\theta) + 2r\dot{\theta}\dot{\phi}\cos(\theta) + r\sin(\theta)\ddot{\phi})$$

Problem 4

In a rainy day, a wheel of radius R is rolling without slipping along a muddy road with a speed V. Particles of mud attached to the rim of the wheel are being continuously thrown off from all points of the wheel. If $V^2 > Rg$, where g is the acceleration of gravity, show that the maximum height above the road H, attained by the mud will be

$$H = R + \frac{V^2}{2g} + \frac{R^2 g}{2V^2}$$

Solution We can say that the vertical height of the mud as it initially leaves the wheel as

$$y = R - R\cos(\theta)$$

With θ being the angle between the ground and a point on the wheel. We can differentiate this in respect to time to see how fast the mud is leaving the wheel in the vertical direction.

$$\frac{dy}{dt} = R\sin(\theta)\dot{\theta}$$

And since $V = R\dot{\theta}$.

$$\frac{dy}{dt} = V\sin(\theta)$$

Now that we have our vertical position for where the mud will launch from, we can treat the problem like a standard projectile problem.

$$V^2 = V_i^2 - 2gh$$

Plugging our information, we can solve for h.

$$h = \frac{V^2\sin^2(\theta)}{2g}$$

Our total height equation can the be defined as

$$H = R - R\cos(\theta) + \frac{V^2\sin^2(\theta)}{2g}$$

We will differentiate it with respect to the angle to maximize the function.

$$\frac{dH}{dt} = R\sin(\theta) + \frac{V^2\sin(\theta)\cos(\theta)}{g} = 0$$

$$R\sin(\theta) = \frac{-V^2\sin(\theta)\cos(\theta)}{g}$$

$$\cos(\theta) = \frac{-Rg}{V^2}$$

With $V^2 > Rg$, we know that the cosine will be defined so we can plug it into our height equation.

$$H = R + \frac{R^2 g}{V^2} + \frac{V^2(1-\cos^2(\theta))}{2g}$$

$$= R + \frac{R^2 g}{V^2} + \frac{V^2}{2g} - \frac{R^2 g^2}{2V^2 g}$$

$$H = R + \frac{R^2 g}{2V^2} + \frac{V^2}{2g}$$

Which is what we intended to show.

Problem 5

(Taylor 1.32) We are used to the fact that Newton's third law holds in mechanics. However, this is not absolutely the case in electromagnetism. To illustrate this point, consider the situation shown in Figure ??. The electric and magnetic fields at location $\boldsymbol{r_1}$ due to charge q_2 at $\boldsymbol{r_2}$ moving with constant velocity $\boldsymbol{v_2}$ (with $v_2 << c$) are

$$\boldsymbol{E}(\boldsymbol{r_1}) = \frac{1}{4\pi\epsilon_0}\frac{q_2}{r^2}\hat{\boldsymbol{r}} \quad , \quad \boldsymbol{B}(\boldsymbol{r_1}) = \frac{\mu_0}{4\pi}\frac{q_2}{r^2}\boldsymbol{v}_2 \times \hat{\boldsymbol{r}}$$

where $\boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2}$ is the vector pointing from $\boldsymbol{r_2}$ to $\boldsymbol{r_1}$. If $\boldsymbol{F}_{12}^{el}$ and $\boldsymbol{F}_{12}^{mag}$ denote the electric and magnetic forces on charge q_1 at $\boldsymbol{r_1}$ with velocity $\boldsymbol{v_1}$, show that $F_{12}^{mag} \leq (v_1 v_2/c^2)F_{12}^{el}$. This demonstrates that it is reasonable to ignore the magnetic force between two moving charges in the non-relativistic regime.

Solution Our electric force in this situation will be

$$\boldsymbol{F_{12}^{el}} = q_1\boldsymbol{E}(\boldsymbol{r_1})$$

We only really care about the magnitude of this which will be

$$|\boldsymbol{F_{12}^{el}}| = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

The magnetic force can be represented in a similar way

$$\boldsymbol{F_{12}^{mag}} = q_1\boldsymbol{v_1} \times \boldsymbol{B}(\boldsymbol{r_1})$$

Like before, we only really care about its maximum magnitude. This will have when $\boldsymbol{v_1} \perp \boldsymbol{B}$.

$$|\boldsymbol{F_{12}^{mag}}| = \frac{\mu}{4\pi}\frac{q_1 q_2 v_1 v_2}{r^2}$$

Now let's take the ratio between these two magnitudes and see what happens.

$$\frac{|\boldsymbol{F_{12}^{mag}}|}{|\boldsymbol{F_{12}^{el}}|} = \mu\epsilon_0 v_1 v_2 = \frac{v_1 v_2}{c^2}$$

Since c is the speed of light and is far greater than any speed in a non-relativistic regime, the magnitude of the magnetic force is so much smaller than the magnitude of the electric force, and can be ignored.

Problem 6

(Taylor 2.39) A cyclist coasts to a stop, he is subject to two forces, the quadratic force of air resistance, $f = -cv^2$ (with c being a constant), and a constant frictional force f_r of about 3 N. The former is dominant at high and medium speeds, the latter at low speed.

(a) Write down the equation of motion while the cyclist is coasting to a stop.

(b) Using the numerical values given in Taylor 2.26, determine how long the cyclist takes to slow from a speed of 20 m/s to 15 m/s, to 10 and 5 m/s, and to a full stop.

Solution **a)** Using Newton's second law, we know that

$$F = ma = m\dot{v} = -cv^2 - f$$

We are looking for an equation of time $t(v)$ so we will have to separate variables and integrate.

$$\frac{dv}{cv^2 + f} - \frac{-dt}{m}$$

$$\frac{dv}{v^2 + \frac{f}{c}} = \frac{-c}{m}dt$$

$$\sqrt{\frac{c}{m}}(\arctan\left(v\sqrt{\frac{c}{f}}\right) - \arctan\left(v\sqrt{\frac{c}{f}}\right)) = \frac{-ct}{m}$$

$$t = \frac{-m}{\sqrt{cf}}\arctan\left(v\sqrt{\frac{c}{f}}\right) + \frac{m}{\sqrt{cf}}\arctan\left(v_0\sqrt{\frac{c}{f}}\right)$$

b) We are now ready to plug our values in. $m = 80$ kg, $c = 0.20$ N/(m/s)^2, and at $t = 0$ the cyclist has an initial speed $v_0 = 20$ m/s.

Plugging in all of our values into the previous equation, we get the following.

$$t(15) \approx 363.32s$$

$$t(10) \approx 1054.66s$$

$$t(5) \approx 2768.06s$$

$$t(v = 0) \approx 8163.25s$$

Problem 7

(Taylor 2.55) A charged particle of mass m and charge q moves in uniform electric and magnetic fields, $\boldsymbol{E}$ in the y direction and $\boldsymbol{B}$ in the z direction respectively. Suppose the particle starts to move in the x direction with speed $v_x = v_{x0}$.

(a) Write down the equation of motion for the particle and show that the motion remains in the plane $z = 0$.

(b) Prove that there is a unique value of v_{x0}, called the drift speed v_{dr}, for which the particle moves undeflected through the fields. (This is the basis of velocity selectors, which select particles traveling at one chosen speed from a beam of particles with different speeds.)

(c) Solve the equations of motion to get the velocity of the particles as a function of t, for arbitrary values of v_{x0}.

(d) Find the position as a function of t, and use a computer program to show the trajectory for various values of v_{x0}.

Solution **a)** The force on this charged particle can be represented as

$$\begin{aligned}\boldsymbol{F} = m\dot{\boldsymbol{v}} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B} = qE\hat{\boldsymbol{x}} + (\dot{x}\hat{\boldsymbol{x}} + \dot{y}\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}) \times (B\hat{\boldsymbol{z}}) \\ = m\ddot{x}\hat{\boldsymbol{x}} + m\ddot{y}\hat{\boldsymbol{y}} + m\ddot{z}\hat{\boldsymbol{z}} = \hat{\boldsymbol{x}}(q\dot{y}B) + \hat{\boldsymbol{y}}(qE - q\dot{x}B)\end{aligned}$$

This gives us a few equations, one for each component.

$$\ddot{x} = \frac{q\dot{y}B}{m}$$
$$\ddot{y} = \frac{qE - q\dot{x}B}{m}$$
$$\ddot{z} = 0$$

The third equation tells that the velocity of the particle in the z direction is constant. Since the initial velocity is only in the x direction, we know that all movement of the particle is restricted to the x-y plane at $z = 0$.

b) In order for the particle to move undeflected while moving through both fields, the magnetic and electric forces must be equal.

$$F_e = F_m$$
$$qE = qv_{dr}B$$
$$v_{dr} = \frac{E}{B}$$

Our fields are fixed so v_{dr} is unique and is our desired drift speed.

c) We are going to begin this part by defining a couple of variables to help us out.

$$\dot{x} - v_{dr} = u$$
$$\dot{y} = \dot{w}$$

We are doing this because the y component has an extra term in it in our equations of motion and we want to work around it. Since v_{dr} is a constant, our derivatives will be the same. Our equations of motion that are relevant will then be

$$\ddot{x} = \dot{u} = \frac{quB}{m}$$
$$\ddot{y} = \dot{w} = \frac{-qwB}{m}$$

This gives us a couple of solutions to these differential equations.

$$u = A\cos\left(\frac{qBt}{m}\right)$$
$$w = -A\sin\left(\frac{qBt}{m}\right)$$

At $t = 0$ the velocity will be $v = \dot{x} = v_{0x}$. We can then solve for A.

$$\dot{x} = v_{0x} = v_{dr} + A\cos(0)$$
$$A = v_{0x} - v_{dr}$$

Plugging this into our velocity equations gives us what we have looking for.

$$\dot{x} = v_{dr} + (v_{0x} - v_{dr})\cos\left(\frac{qBt}{m}\right)$$
$$\dot{y} = (v_{dr} - v_{0x})\sin\left(\frac{qBt}{m}\right)$$
$$\dot{z} = 0$$

d) We are going to start integrating the equations we just found in respect to time. To make life easier on myself I came going to define

$$\omega = \frac{qB}{m}$$

Let's begin with find the position in the x direction.

$$\dot{x} = v_{dr} + (v_{0x} - v_{dr})\cos(\omega t)$$

$$x = (v_{dr}t' + \left(\frac{v_{0x} - v_{dr}}{\omega}\right)\sin(\omega t'))\Big|_0^t$$

$$= v_{dr} + \left(\frac{v_{0x} - v_{dr}}{\omega}\right)\sin(\omega t) = v_{dr}t + R\sin(\omega t)$$

With

$$R = \frac{v_{0x} - v_{dr}}{\omega}$$

Now let's find out the position in the y direction.

$$\dot{y} = (v_{dr} - v_{0x})\sin(\omega t)$$

$$y = (v_{0x} - v_{dr})\cos(\omega t')\Big|_0^t$$

$$= \left(\frac{v_{0x} - v_{dr}}{\omega}\right)(\cos(\omega t) - 1)$$

$$= R(\cos(\omega t) - 1)$$

We are going to let $v_{dr} = 1$ and $\omega = 1$ to make graphing easier. v_{0x} will be multiples of v_{dr} to make it easier to graph, as well.

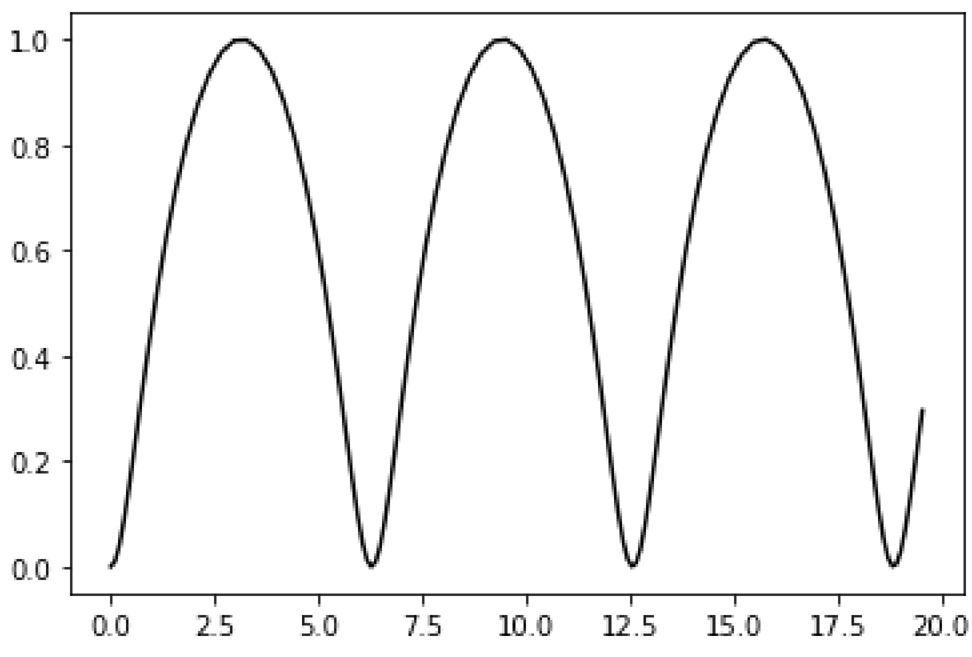

Figure 1: $v_{0x} = 0.5$

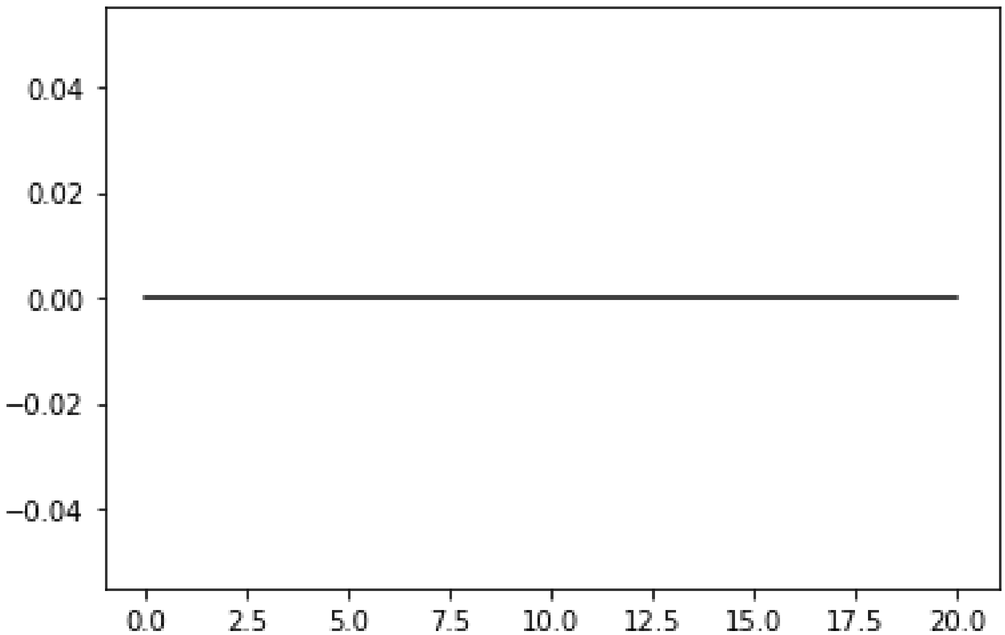

Figure 2: $v_{0x} = 1$

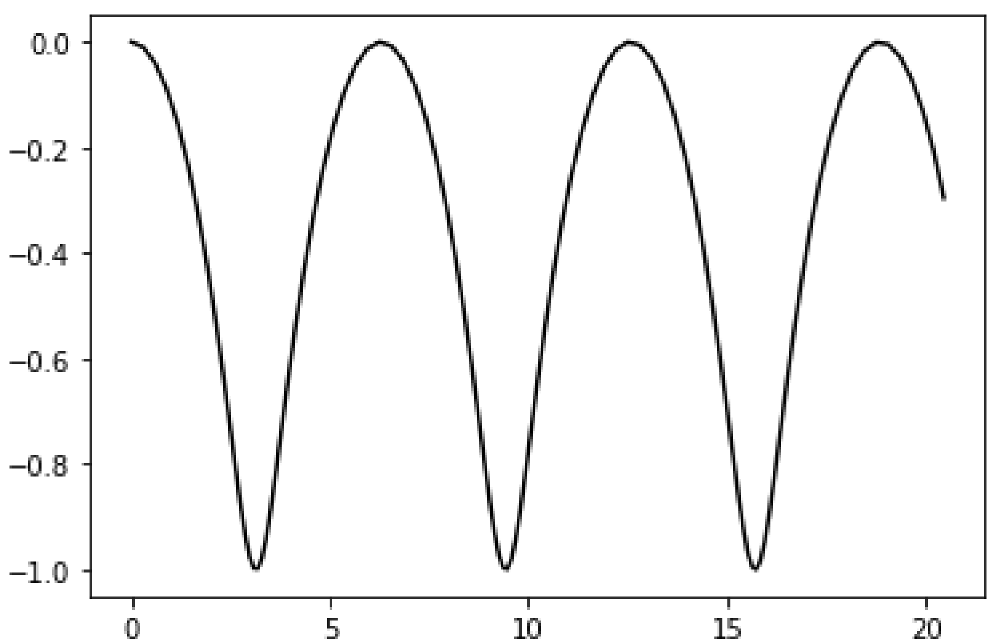

Figure 3: $v_{0x} = 1.5$

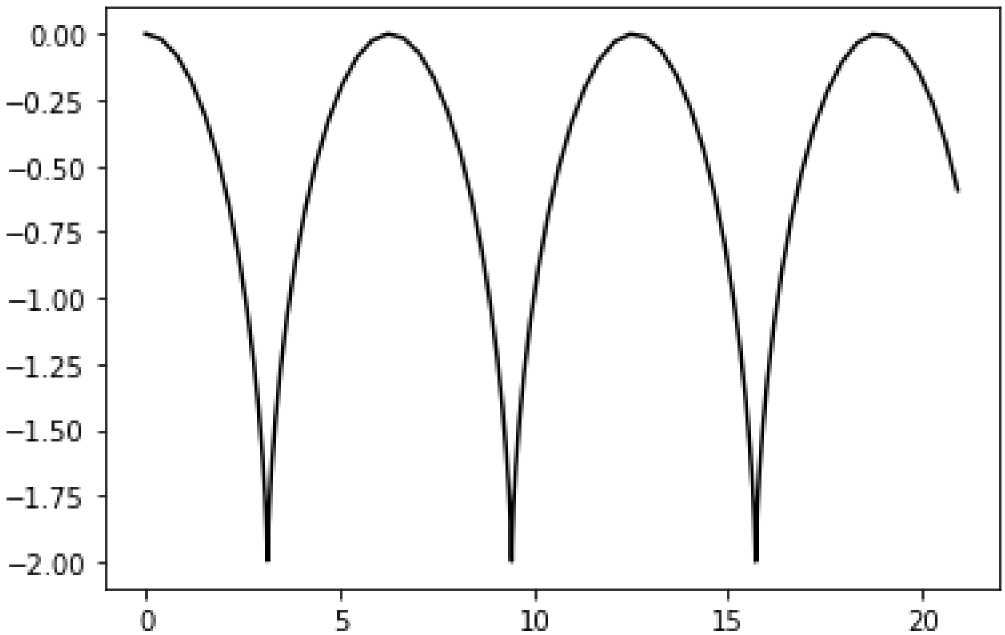

Figure 4: $v_{0x} = 2$

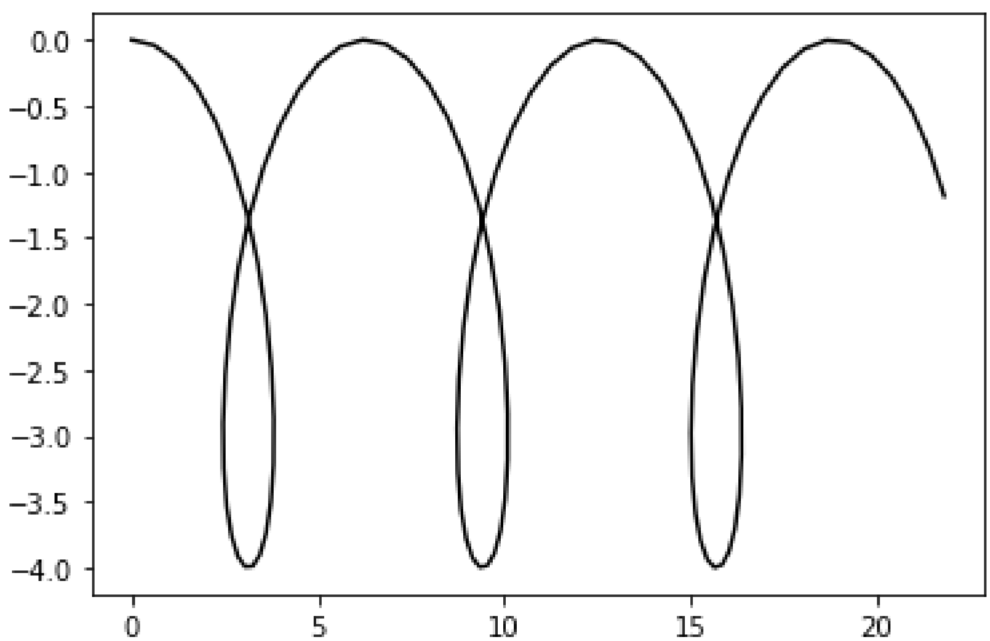

Figure 5: $v_{0x} = 3$

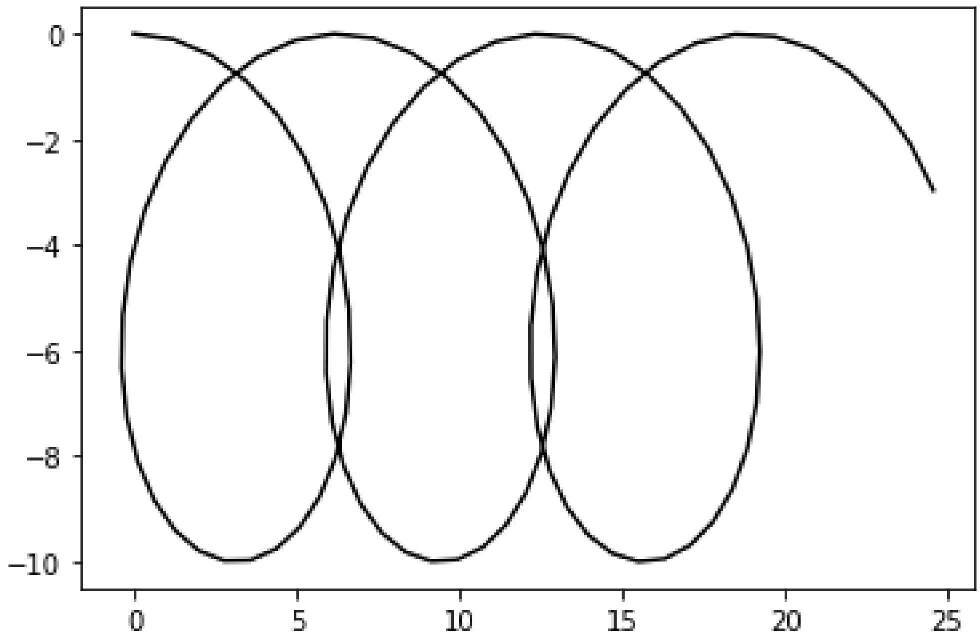

Figure 6: $v_{0x} = 6$

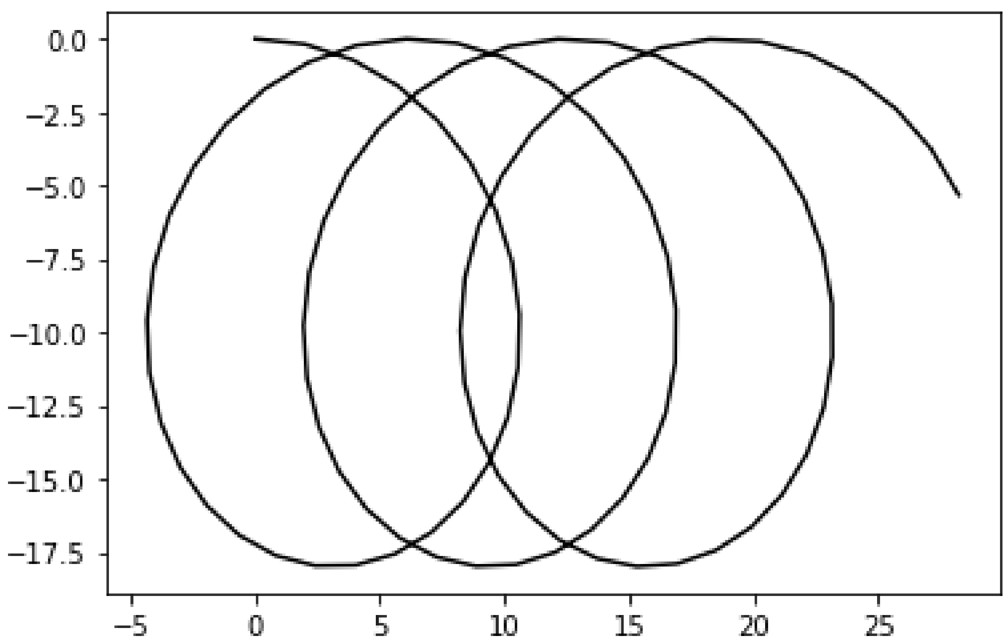

Figure 7: $v_{0x} = 10$

Problem 8

(Taylor 3.4) Two girls, each with mass m, are standing at one end of a stationary railroad flatcar with frictionless wheels and mass m_c. Either girl can run to the other end of the flatcar and jump off with the same speed u relative to the car.

(a) Use conservation of momentum to find the speed of the recoiling car if the two girls run and jump simultaneously.

(b) What is the speed of the recoiling car if the second girl starts running only after the first one has already jumped? Which procedure gives the greater speed to the car?

Solution a) The two girls are acting together, and can be considered one girl at twice the mass. Using conservation of momentum, that gives us

$$2m(U+v)+m_c v$$
$$v=\frac{-2mu}{2m+m_c}$$

b) We'll have to break the girls running separately into two events. When the first girl runs, we can pretend that the other girl and the cart have fused together and are one. We can then use conservation of momentum in the girl-girlcar system.

$$m(u+v_1)+(m+m_c)v_1=0$$
$$v_1=\frac{-mu}{m_c+2m}$$

Once the first girl jumps off the car, the total mass of the system changes, and the remaining girlcar is going at v_1. We now can use the conservation of momentum again to solve the speed of the car after the second girl runs off of it.

$$m(u+v_2)+m_c v_2=(m+m_c)v_1$$
$$v_2(m+m_c)=(m+m_c)v_1-mu$$
$$v_2=\frac{-mu}{m_c+2m}-\frac{mu}{m_c+m}$$

The first term is half the magnitude of the result in part a and the second term is larger than half of part a, so the second procedure gives the greater speed to the car.