Analytic Mechanics Lecture 3

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Rolling Disk Without Slipping

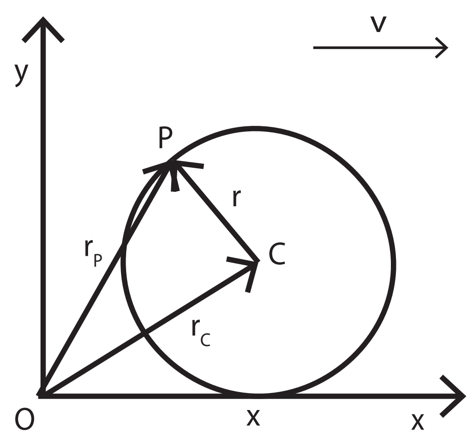

Figure 1

This disk will be rolling along a flat plane at a velocity v. We want to be able to determine the position and velocity of some point P on the disk as the disk traverses. The constraint that the disk is not slipping is very important for us or else we would not be able to solve this problem. We have defined $\boldsymbol{r_p}$ and $\boldsymbol{r_c}$ and $\boldsymbol{r}$ in such a way that we can isolate the horizontal movement of the disk from the rest of the motion of the disk.

$$\boldsymbol{r_P} = \boldsymbol{r_C} + \boldsymbol{r}$$

$$\boldsymbol{r_C} = x\hat{x} + r\hat{y} = vt\hat{x} + r\hat{y}$$

We will want to define our system in such a way that at $t = 0, x = 0$. This will make our starting position br away, vertically, from the origin.

$$x = vt$$

$$\dot{\boldsymbol{r_C}} = v\hat{x} = \boldsymbol{v}$$

$$\boldsymbol{r} = (-r\sin(\phi))\hat{x} + (-r\cos(\phi))\hat{y}$$

$$\dot{\boldsymbol{r}} = (-r\cos(\phi)\dot{\phi}\hat{x} + (r\sin(\phi)\dot{\phi}\hat{y}$$

Without slipping

$$x = r\phi$$

$$\dot{x} = v = r\dot{\phi}$$

We can finally use the pieces we have collected to solve for $\dot{\boldsymbol{r_P}}$.

$$\begin{aligned}\dot{\boldsymbol{r_P}} = \dot{\boldsymbol{r_C}} + \dot{\boldsymbol{r}} &= v\hat{x} + (-r\cos(\phi)\dot{\phi})\hat{x} + (r\sin(\phi)\dot{\phi})\hat{y} \\ &= v(1 - \cos(\phi))\hat{x} + v\sin(\phi)\hat{y}\end{aligned}$$

We can see that at $\phi = 0$, the point P has $\dot{\boldsymbol{r_P}} = 0$. We call this the momentary, or instantaneous, center of our system. At $\phi = \pi, \dot{\boldsymbol{r_P}} = 2v\hat{x}$ which shows us that the velocity of a point on this disk is changing over time, and is not constant.

Dynamics

Dynamics studies the motion of some system while taking into account the cause of motion. With this, we start using concepts like **Force** and **mass**.

Newton's Laws of Motion

1st Law

An object at rest will stay at rest, and an object moving will be under constant motion unless acted upon by some outside force.

$$\boldsymbol{r} = \boldsymbol{r_0} + \boldsymbol{v}t$$

2nd Law

Acceleration of an object is proportional to an applied force.

$$\boldsymbol{F} = m\boldsymbol{a}$$

3rd Law

A force acting on some object will experience a reacting force with the same magnitude but opposite direction.

$$\boldsymbol{F_A} = -\boldsymbol{F_B}$$

Equivalence Principle

Mass can be considered to inertia, or an object's resistance to change in motion.

$$\boldsymbol{F} = m_i\boldsymbol{a}$$

with m_i being the inertial mass of the object. This isn't good enough, however, because there is a gravitational mass that isn't necessarily equivalent.

Using Newton's Law of Gravitation

$$\boldsymbol{F_g} = -G\frac{m_g M}{r^2}\hat{\boldsymbol{r}}$$

We can also picture gravity in much the same way as we do with electromagnetism, in the form of a field.

$$\boldsymbol{g}(\boldsymbol{r}) = \frac{\boldsymbol{F_g}}{m_g} = -G\frac{M}{r^2}\hat{\boldsymbol{r}}$$

We want to find out is the relationship between m_i and m_g.

Galileo

We can look at some mass sitting on an inclined plane.

$$\boldsymbol{F} = m_i\boldsymbol{a} = m_g\boldsymbol{g}\sin(\theta)$$
$$\boldsymbol{a} = \left(\frac{m_g}{m_i}\right)\boldsymbol{g}\sin(\theta)$$

We can measure the acceleration and the angle, so all we have to do to find the ratio of the masses is to graph them against each other. The slope of the line will tell us the type of relationship the different types of mass have. If they are equal, we should see their ratio be equal to one.

Eötvös

Eötvös came up with a torsion scale that can be used to weight objects, as well as measure torque. A rod would have a weight hung at each end and will hang from a string somewhere in the midsection of the rod. The weights distance from the strong could then be varied. By comparing the distance of the weights from the central pivot, the mass of the weights can be determined. This scale also has two degrees of tilt and may be caused to rotate by the weights that hang on it.

The vertical tilt can be determined by

$$(m_{gA}g - m_{iA}a_V)l_A = (m_{gB}g - m_{iB}a_V)l_B$$

with

$$a_V = R\omega^2 \cos^2(\theta)$$

The horizontal tile can be determined by

$$\tau = m_{iA}a_H l_A - m_{iB}a_H l_B$$

with

$$a_H = R\omega^2 \sin(\theta)\cos(\theta)$$

This is withτ being the torque causing the scale to rotate along its hanging axis.

$$\tau \approx m_{gA}a_H l_A \Big(\Big(\frac{m_{iA}}{m_{gA}}\Big) - \Big(\frac{m_{iB}}{m_{gB}}\Big)\Big)$$

We can measure this torque to be about zero, which implies

$$\frac{m_{iA}}{m_{gA}} = \frac{m_{iB}}{m_{gB}}$$

This suggests that the mass is independent of composition.